ON ASYMPTOTIC PROPERTIES OF SUB-CRITICAL BRANCHING PROCESSES

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1. Introduction

Let Z_n be the number of individuals in the n-th generation of a discrete branching process, descended from a single ancestor, for which we put

$$F(s) = \sum_{j=0}^{\infty} s^j P[Z_1 = j], \quad 0 < F(0) < 1, \quad s \in [0, 1].$$

It is well known that the probability generating function of Z_n is $F_n(s)$, the n-th functional iterate of $F(s)$, and that if $m = EZ_1$ does not exceed unity, then $\lim n \to \infty \{F_n(s)\} = 1$, $0 \leqq s \leqq 1$ (Harris [1], Chapter 1). In particular, extinction is certain.

For a subcritical process (i.e. $m < 1$) results of Kolmogorov and Yaglom (see [1]) state that if $F''(1-) < \infty$

$$\lim_{n\to\infty} \frac{m^n}{1-F_n(0)} = \mu, \qquad 1 \leqq \mu < \infty \tag{1.1}$$

$$\lim_{n\to\infty} G_n(s) = G(s), \qquad s \in [0, 1] \tag{1.2}$$

exists, where

$$G_n(s) = \sum_{j=1}^{\infty} s^j P[Z_n = j | Z_n > 0] = \frac{F_n(s)-F_n(0)}{1-F_n(0)},$$

and $G(s)$ is a proper generating function, with the mean of the corresponding distribution $G'(1-) = \mu$, and the corresponding variance, σ^2, finite.

In two recent papers, Heathcote and Seneta [2], and Seneta [4], are concerned with bounds for ET, Var T and μ, where T is the time to extinction. In relation to this, we note that since

$$P[T > n] = 1-F_n(0) \sim \mu^{-1} \cdot m^n$$

as $n \to \infty$, *all* moments of the distribution of T exist. The second of the above-mentioned papers considers only the Poisson offspring distribution, for which $F(s) = \exp m(s-1)$. Here it is shown that the bounds are sufficiently good to yield the asymptotic expressions

$$ET \sim -\theta_m \log (1-m)$$

$$\sum_{k=0}^{\infty} kP[T > k] \sim \rho_m \frac{\pi^2}{6(1-m)}$$

$$\mu \sim \frac{1}{\psi_m(1-m)}$$

$$\frac{\mu^2}{\sigma^2} \sim \frac{1}{\psi_m - 1}$$

as $m \to 1-$, where $1 \lesseqqgtr \psi_m, \rho_m, \theta_m \lesseqqgtr 2$. The paper conjectures also that ψ_m, ρ_m, θ_m can in fact be replaced by the constant 2, in the above expressions.

In the present note we sharpen and generalize these results by considering a *class* of branching processes whose offspring distributions depend in a specific way on the mean. By this we mean that the following conditions are satisfied:

(i) $F(s) = F(m; s)$ is a p.g.f. for all m such that $1-\varepsilon < m < 1$, (i.e. in some left open neighbourhood of $m = 1$) and

$$F(m; s) \to F(*; s), \quad \text{as} \quad m \to 1-, \quad s \in [0, 1]$$

where $F(*; s)$ is a proper p.g.f.

(ii) $F''(*; 1) > 0$

(iii) $F'''(m; 1) < C = \text{const.}, m \in (1-\varepsilon, 1)$.

Note. Dashes shall always refer to differentiation with respect to s.

By utilizing some techniques from both [2] and [4], together with a general approach which is basically simpler, we show that for this class of branching processes:

$$ET \sim -\frac{2}{F''(*; 1)} \log (1-m); \tag{1.3}$$

$$\sum_{k=0}^{\infty} k^{\alpha} P[T > k] \sim \frac{2}{F''(*; 1)} \frac{\Gamma(\alpha+1)\zeta(\alpha+1)}{(1-m)^{\alpha}} \tag{1.4}$$

for integral $\alpha \geqq 1$;

$$\mu \sim \frac{F''(*; 1)}{2} \cdot \frac{1}{(1-m)}, \tag{1.5}$$

$$\frac{\mu^2}{\sigma^2} \sim 1; \tag{1.6}$$

as $m \to 1-$. In this simple situation, $\zeta(s)$, the Riemann zeta-function, is given by

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$$\zeta(\alpha+1) = \sum_{n=1}^{\infty} \frac{1}{n^{\alpha+1}}.$$

Thus, inter alia, the conjectures for the Poisson case are valid.

In a concluding section (§ 5) we discuss the effect of removing some of the above restrictions, particularly (ii), and give some examples. The correlation of results (1.5) and (1.6) to existing results is also discussed.

2. Bounds with mean fixed

Since we are not concerned with varying m in this section we shall omit explicit mention of it in the functional form. Our procedure, initially that of [2] viz. using first and second order mean-value theorems, differs from [2] in that we obtain bounds not for $\{1-F_n(0)\}$, but rather its reciprocal. Since $F_{k+n}(s) = F_n[F_k(s)]$, for integral n, $k \geqq 0$

$$1-F_{k+1}(0) = \{1-F_k(0)\}F'(\theta_k)$$

where $F_k(0) < \theta_k < 1$, we have by monotonicity of $F'(s)$ (and since $F_k(0) \uparrow 1$ as $k \to \infty$) that for $0 \leqq h \leqq k$

$$F'(F_h(0))\{1-F_k(0)\} \leqq 1-F_{k+1}(0) \leqq m\{1-F_k(0)\}. \tag{2.1}$$

Moreover, for $k \geqq 0$

$$F_{k+1}(0) = F(F_k(0)) = 1-\{1-F_k(0)\}m+ \frac{F''(\eta_k)}{2}\cdot\{1-F_k(0)\}^2$$

where $F_k(0) < \eta_k < 1$, so that putting $b_k = \{1-F_k(0)\}^{-1}$ we have

$$b_{k+1} = \frac{b_k}{m} + \frac{1}{m}\cdot\frac{F''(\eta_k)}{2}\cdot\frac{b_{k+1}}{b_k}. \tag{2.2}$$

Now, since $F''(\eta_k) \geqq F''(F_k(0)) \geqq F''(F_h(0))$ for $0 \leqq h \leqq k$, and using (2.1)

$$\frac{b_k}{m}+\frac{1}{m}\cdot\frac{F''(F_h(0))}{2m} \leqq b_{k+1} \leqq \frac{b_k}{m}+\frac{1}{m}\frac{F''(1)}{2}\cdot\frac{1}{F'(F_h(0))} \tag{2.3}$$

(we shall assume that $0 < F''(1) < \infty$). This inequality is the *crucial* one from which all subsequent results follow. Keeping h fixed, and iterating,

$$\frac{b_h}{m^n}+\frac{F''(F_h(0))}{2m^2}\cdot\sum_{i=0}^{n-1}\frac{1}{m^i} \leqq b_{h+n} \leqq \frac{b_h}{m^n}+\frac{F''(1)}{2mF'(F_h(0))}\cdot\sum_{i=0}^{n-1}\frac{1}{m^i}.$$

We then have

$$m^h b_h+\frac{F''(F_h(0))\cdot m^h(1-m^n)}{2m(1-m)} \leqq m^{h+n}\cdot b_{h+n} \leqq m^h b_h+\frac{F''(1)m^h(1-m^n)}{2F'(F_h(0))(1-m)}, \tag{$*$}$$

and letting $n \to \infty$

$$(2.4) \quad m^h b_h(1-m)+\frac{F''(F_h(0))m^h}{2m} \leqq (1-m)\mu \leqq m^h b_h(1-m)+\frac{F''(1)\cdot m^h}{2F'(F_h(0))}.$$

Also from (*) above,

$$\frac{2F'(F_h(0))(1-m)}{F''(1)}\left\{\frac{\theta m^n}{1-\theta m^n}\right\} \leqq \frac{1}{b_{h+n}} \leqq \frac{2m(1-m)}{F''(F_h(0))}\cdot\left\{\frac{\tau m^n}{1-\tau m^n}\right\}$$

where

$$1 > \theta = \frac{F''(1)}{2b_h F'(F_h(0))(1-m)+F''(1)} > 0;$$

$$1 > \tau = \frac{F''(F_h(0))}{2m(1-m)b_h+F''(F_h(0)} > 0$$

so that for integral $\alpha \geqq 0$

$$\sum_{k=0}^{h} k^\alpha\{1-F_k(0)\}+\frac{2F'(F_h(0))(1-m)}{F''(1)}\cdot\sum_{n=1}^{\infty}(h+n)^\alpha\left\{\frac{\theta m^n}{1-\theta m^n}\right\}$$

$$(2.5) \qquad \leqq \sum_{k=0}^{\infty} k^\alpha P[T>k]$$

$$\leqq \sum_{k=0}^{h} k^\alpha\{1-F_k(0)\}+\frac{2m(1-m)}{F''(F_h(0))}\cdot\sum_{n=1}^{\infty}(h+n)^\alpha\left\{\frac{\tau m^n}{1-\tau m^n}\right\}.$$

The two sets of inequalities (2.4) and (2.5) shall be sufficient to give us the required asymptotic results by suitable limiting considerations.

3. Limit results: preliminaries

A. We begin with some remarks on the sums occurring in the bounds (2.5), for $\alpha = 0, 1$. Since, from [4] (§ 3) for $0 < \rho, s < 1$

$$\frac{\log(1-\rho s)}{\log s} \leqq \sum_{j=1}^{\infty}\frac{\rho s^j}{1-\rho s^j} \leqq \frac{\rho s}{1-\rho s}+\frac{\log(1-\rho s)}{\log s}$$

if $\rho(=\rho(s))$ is such that, as $s \to 1-$

$$(1-\rho s) \sim c\cdot(1-s) \qquad (0 < c = \text{const.})$$

it follows that

$$(3.1) \qquad \lim_{s\to 1-}\left\{-\frac{(1-s)}{\log(1-s)}\cdot\sum_{j=1}^{\infty}\frac{\rho s^j}{1-\rho s^j}\right\} = 1.$$

Moreover, it was shown in [4] (§ 3) that

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$$\lim_{s\to 1-}\left\{\frac{6(\log s)^2}{\pi^2}\cdot\sum_{j=1}^{\infty}\frac{j\rho s^j}{1-\rho s^j}\right\}=1 \tag{3.2}$$

providing $1-\rho\sim k\cdot(1-s)$ as $s\to 1-$ where $0<k\cdot=\text{const}$. Since the procedure for sums of the form,

$$\sum_{j=1}^{\infty}\frac{j^\alpha\rho s^j}{1-\rho s^j}$$

is a slight extension, essentially, of the procedure to obtain (3.2), in the cited reference, we only outline it here. The remarks apply for integral $\alpha\geqq 1$, $0<\rho$, $s<1$.

(a) The function of a continuous variable $x>0$

$$\frac{x^\alpha\rho s^x}{1-\rho s^x}$$

has a unique maximum at $x=N^*$ which is the unique solution of

$$\alpha+x\log s-\alpha\rho s^x=0.$$

Hence N^* satisfies

$$0\leqq\alpha\rho s^{N^*}=\alpha+N^*\log s\Rightarrow N^*\leqq-\frac{\alpha}{\log s},$$

$$0=\alpha\left(1+\frac{N^*}{\alpha}\log s-\rho s^{N^*}\right)\geqq\alpha(1+N^*\log s-\rho s^{N^*})\Rightarrow N^*\gtrapprox\frac{\sqrt{2k}}{\sqrt{-\log s}}$$

if $1-\rho\sim k\cdot(1-s)$, as $s\to 1-$, where $k>0$ is independent of s. To see the validity of the last asymptotic inequality[1], we have

$$0\geqq 1+N^*\log s-\rho e^{-(-N^*\log s)}$$

and since for $x\geqq 0$,

$$e^{-x}\leqq 1-x+\frac{x^2}{2}$$

$$0\geqq 1+N^*\log s-\rho\left\{1+N^*\log s+\frac{(N^*\log s)^2}{2}\right\},$$

i.e.

$$0\geqq(1-\rho)+(1-\rho)N^*\log s-\rho\,\frac{(N^*\log s)^2}{2}.$$

This is just a quadratic inequality for $N^*>0$, whose solution, as $s\to 1-$, (if $1-\rho\sim k\cdot(1-s)$) is given by

[1] The symbol '$\gtrapprox$' is to be interpreted as '... is not less than a quantity asymptotically equal to ...'.

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$$N^* \gtrapprox \frac{\sqrt{2k}}{\sqrt{-\log s}}.$$

Hence, if $1-\rho \sim k \cdot (1-s)$, we have

$$\frac{(N^*)^\alpha \rho s^{N^*}}{1-\rho s^{N^*}} \gtrapprox \frac{\text{const.}}{(-\log s)^{\alpha+(\frac{1}{2})}} \qquad (0 < \text{const.} < \infty)$$

as $s \to 1-$.

(b) From a double use of the Cauchy integral test

$$\sum_{j=0}^{\infty} \frac{j^\alpha \rho s^j}{1-\rho s^j} = \int_0^\infty \frac{x^\alpha \rho s^x}{1-\rho s^x} \cdot dx + \varepsilon(s)$$

where

$$|\varepsilon(s)| \leqq \text{const.} \frac{(N^*)^\alpha \rho s^{N^*}}{1-\rho s^{N^*}}.$$

(c) $$\int_0^\infty \frac{x^\alpha \rho s^x}{1-\rho s^x} \cdot dx = -\frac{1}{(\log s)^{\alpha+1}} \int_0^\rho \frac{(\log y - \log \rho)^\alpha}{1-y} \cdot dy.$$

(d) $$\int_0^1 \frac{(\log y)^\alpha}{1-y} \cdot dy = (-1)^\alpha \Gamma(\alpha+1)\zeta(\alpha+1),$$

for $\alpha \geqq 0$ integral.

An obvious combination of these results shows that for $\alpha \geqq 1$ and integral, if $1-\rho \sim k \cdot (1-s)$ as $s \to 1-$,

$$\sum_{j=1}^{\infty} \frac{j^\alpha \rho s^j}{1-\rho s^j} \sim \frac{\Gamma(\alpha+1)\zeta(\alpha+1)}{(-\log s)^{\alpha+1}}. \tag{3.3}$$

B. Secondly, we remark that under our conditions (i), (ii) and (iii) of § 1, as $m \to 1-$

(a) $F_k(m; s) \to F_k(*; s), \qquad s \in [0, 1]$
(b) $F'(m; F_h(m; 0)) \to F'(*; F_h(*; 0))$
(c) $F''(m; F_h(m; 0)) \to F''(*; F_h(*; 0))$
(d) $F'(*; 1) = 1$
(e) $F''(m; 1) \to F''(*; 1), \qquad 0 < F''(*; 1) < \infty.$

To prove (a) consider the inequality

$$\begin{aligned}|F_k(m; s) - F_k(*; s)| \leqq\ & |F_{k-1}(m; F(m; s)) - F_{k-1}(m; F(*; s))| \\ &+ |F_{k-1}(m; F(*; s)) - F_{k-1}(*; F(*; s))|\end{aligned}$$

and notice that the first part tends to zero since by the mean value theorem

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$$|F_{k-1}(m;\, F(m;\, s)) - F_{k-1}(m;\, F(*;\, s))| = F'_{k-1}(m;\, \delta_m)|F(m;\, s) - F(*;\, s)|$$

where $0 \leqq \delta_m \leqq 1$ so that $F'_{k-1}(m;\, \delta_m) (\leqq F'_{k-1}(m;\, 1) = m^{k-1})$ is bounded as $m \to 1-$, and $F(m;\, s) \to F(*;\, s)$ by (i). The second part of the right hand side approaches zero by induction on k, and (i). Propositions (b) and (c) are proved by analogous arguments: consider (c):

$$|F''(m;\, F_h(m;\, 0)) - F''(*;\, F_h(*;\, 0))| \leqq |F''(m;\, F_h(m;\, 0)) - F''(m;\, F_h(*;\, 0))| + |F''(m;\, F_h(*;\, 0)) - F''(*;\, F_h(*;\, 0))|.$$

Here let us focus attention first on

$$|F''(m;\, F_h(*;\, 0)) - F''(*;\, F_h(*;\, 0))|,$$

in which we notice that

$$j(j-1)P[Z_1 = j][F_h(*;\, 0)]^j \leqq j(j-1)[F_h(*;\, 0)]^j,$$

where the right hand side is independent of m, and so

$$\begin{aligned} F''(m;\, F_h(*;\, 0)) &= \sum_{j=0}^{\infty} j(j-1)P[Z_1 = j][F_h(*;\, 0)]^{j-2} \\ &\leqq \sum_{j=0}^{\infty} j(j-1)[F_h(*;\, 0)]^{j-2} \\ &= 2[1-F_h(*;\, 0)]^{-3} < \infty \end{aligned}$$

since $0 < F_h(*;\, 0) < 1$ (see below). Thus by dominated convergence of the series for $F''(m;\, F_h(*;\, 0))$, and since the assumption (i) implies coefficient convergence in $F(m;\, s)$ to $F(*;\, s)$, it follows that as $m \to 1-$

$$|F''(m;\, F_h(*;\, 0)) - F''(*;\, F_h(*;\, 0))| \to 0.$$

On the other hand,

$$|F''(m;\, F_h(m;\, 0)) - F''(m;\, F_h(*;\, 0))| = F'''(m;\, \theta_m)|F_h(m;\, 0) - F_h(*;\, 0)|$$

where $0 \leqq \theta_m \leqq 1$; and so as $m \to 1-$ (since $F'''(m;\, 1)$ is bounded, by (iii), and from (a) above) we get the requisite tendency to zero.

Propositions (d) and (e) follow since condition (i) implies convergence in distribution as $m \to 1-$, and condition (iii) is equivalent to uniform boundedness of the third moment as $m \to 1-$. Hence by a well known corollary of the moment convergence theorem, we have convergence of the first and second moments to those of the limit distribution, which are necessarily finite. Condition (ii) completes assertion (e).

In concluding this section, we note in particular that

$$0 < F(*;\, 0) < 1$$

this being implied by (d) and (e), and since from (d) also $F'(*;\, 1) = 1$, the branching process defined by $F(*; s)$ is 'critical', and extinction is therefore certain, i.e.

$$F_h(*; 0) \uparrow 1$$

as $h \to \infty$. (This was of course used above also in the proof of (b) and (c), to obtain uniform convergence).

4. Limit results

We are now in a position to combine the results of § 2 and § 3 to deduce first (1.3) and (1.4), and then (1.5) and (1.6).

To obtain (1.3), consider (2.5), with $\alpha = 0$ and identify θ and τ successively with $\rho \equiv \rho(s)$ of § 3; and s with m. (Note also, that since $F''(m; 1) \to F''(*; 1)$ as $m \to 1-$, $\infty > F''(m; 1) > 0$ for m sufficiently close to unity.) First notice that we have putting $\rho = \theta$, $s = m$

(4.1)

$$1-\rho s = 1-\theta m = (1-m)\,\frac{2\{1-F_h(m; 0)\}^{-1}F'(m; F_h(m; 0))+F''(m; 1)}{2\{1-F_h(m; 0)\}^{-1}F'(m; F_h(m; 0))\,(1-m)+F''(m; 1)}$$
$$\sim C\cdot(1-m) \qquad (0 < C \equiv C(h) < \infty)$$

as $m \to 1-$. This result, which amounts to saying that the part in square brackets approaches $C \equiv C(h)$ as $m \to 1-$, is a direct consequence of the propositions of § 3.B and the assumptions (i)—(iii). So also

$$1-\tau m \sim K\cdot(1-m) \qquad (0 < K \equiv K(h) < \infty) \tag{4.2}$$

as $m \to 1-$.

The remarks (4.1) and (4.2) make it possible to apply (3.1) to the bounds of (2.5) with $\alpha = 0$, which are of the form required by (3.1) after division throughout by $-\log(1-m)$. Letting $m \to 1-$ we obtain, therefore, using also the results of § 3.B,

$$\frac{2F'(*; F_h(*; 0))}{F''(*; 1)} \leqq \liminf_{m\to 1-}\left\{\frac{-ET}{\log(1-m)}\right\}$$
$$\leqq \limsup_{m\to 1-}\left\{\frac{-ET}{\log(1-m)}\right\}$$
$$\leqq \frac{2}{F''(*; F_h(*; 0))}.$$

N.B. Until this point, h has been fixed, but arbitrary.

Now, $F(*; s)$ is a proper generating function with $F'(*; 1) = 1$, $F''(*; 1) > 0$. Hence from the well known extinction property in this case, as pointed out $F_h(*; 0) \uparrow 1$ as $h \to \infty$. Since, in the above expression, h may

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be made arbitrarily large, and we have the necessary dominated convergence,

$$\lim_{m\to 1-}\left\{\frac{-ET}{\log(1-m)}\right\} = \frac{2}{F''(*;1)}$$

which is (1.3) as required.

To prove (1.4), we need to consider (2.5) with integral $\alpha \geqq 1$; we do this in a little more detail than in the case $\alpha = 0$, since the situation is slightly more complex. Consider the right hand inequality of (2.5):

$$\begin{aligned} &\sum_{k=0}^{\infty} k^{\alpha} P[T>k] \\ (4.3) \qquad &\leqq \sum_{k=0}^{h} k^{\alpha}\{1-F_h(0)\} + \frac{2m(1-m)}{F''(F_h(0))}\cdot\sum_{n=1}^{\infty}(h+n)^{\alpha}\left\{\frac{\tau m^n}{1-\tau m^n}\right\}\end{aligned}$$

where h is arbitrary and fixed. We notice first that $1-\tau \sim K'\cdot(1-m)$ as $m\to 1-$ where $0 < K' = K'(h) < \infty$. Identifying τ and m with ρ and s of § 3.A then have from (3.3) (since $\alpha \geqq 1$) that as $m \to 1-$

$$\sum_{j=1}^{\infty}\frac{j^{\alpha}\tau m^j}{1-\tau m^j} \sim \frac{\Gamma(\alpha+1)\zeta(\alpha+1)}{(-\log m)^{\alpha+1}}.$$

Thus multiplying (4.3) by $(1-m)^{\alpha}/\Gamma(\alpha+1)\zeta(\alpha+1)$ we have as $m\to 1-$

$$\begin{aligned}&\limsup_{m\to 1-}\left\{\frac{(1-m)^{\alpha}}{\Gamma(\alpha+1)\zeta(\alpha+1)}\cdot\sum_{k=0}^{\infty}k^{\alpha}P[T>k]\right\} \\ &\quad\leqq \lim_{m\to 1-}\left\{\frac{2m(1-m)^{\alpha+1}}{\Gamma(\alpha+1)\zeta(\alpha+1)F''(m;F_h(m;0))}\cdot\sum_{n=1}^{\infty}n^{\alpha}\left\{\frac{\tau m^n}{1-\tau m^n}\right\}\right\} \\ &\quad= \frac{2}{F''(*;F_h(*;0))}\end{aligned}$$

since the remaining contributing terms of the right hand side of (4.3) are $O\{-(1-m)\log(1-m)\}$ as $m\to 1-$.

The left hand inequality of (2.5) may be treated in the same way, since $1-\theta \sim C'\cdot(1-m)(0<C'=C'(h)<\infty)$ as $m\to 1-$, so that we get eventually

$$\begin{aligned}\frac{2F'(*;F_h(*;0))}{F''(*;1)} &\leqq \liminf_{m\to 1-}\left\{\frac{(1-m)^{\alpha}}{\Gamma(\alpha+1)\zeta(\alpha+1)}\cdot\sum_{k=0}^{\infty}k^{\alpha}P[T>k]\right\} \\ &\leqq \limsup_{m\to 1-}\left\{\frac{(1-m)^{\alpha}}{\Gamma(\alpha+1)\zeta(\alpha+1)}\cdot\sum_{k=0}^{\infty}k^{\alpha}P[T>k]\right\} \\ &\leqq \frac{2}{F''(*;F_h(*;0))}.\end{aligned}$$

Thus once more letting $h \to \infty$, we get the required result (1.4):

$$\lim_{m\to 1-} \left\{\frac{(1-m)^\alpha}{\Gamma(\alpha+1)\zeta(\alpha+1)} \cdot \sum_{k=0}^{\infty} k^\alpha P[T > k]\right\} = \frac{2}{F''(*; 1)}.$$

To obtain (1.5) and (1.6) we return to (2.4) where, first letting $m \to 1-$, and then letting $h \to \infty$ yields

$$\lim_{m\to 1-} (1-m)\mu = \frac{F''(*; 1)}{2}.$$

Since (see [4])

$$\sigma^2 = \frac{v^2\mu}{m(1-m)} - \mu^2$$

where

$$v^2 = \text{Var } Z_1 = F''(m; 1)+F'(m; 1)-\{F'(m; 1)\}^2$$

it is easily shown that

$$\lim_{m\to 1-} \frac{\mu^2}{\sigma^2} = 1$$

as required.

5. Supplementary remarks

Some remarks on the class of distributions defined by conditions (i), (ii) and (iii) are in order. The condition (i) is one which renders the procedure $m \to 1-$ meaningful; (iii) ensures the convergence of first and second moments, and is also used to prove assertion $B(c)$ of § 3. Neither of these is open to obvious relaxation, as their role is relatively clear cut.

On the other hand condition (ii) is obviously necessary to give the correct asymptotic behaviour in formulae (1.3)—(1.6), and a relaxation of this condition is of interest in that we are concerned as to how this changes the behaviour as $m \to 1-$. First we notice that $F''(*; 1) = 0$, in view of (i) of § 1 and $B(d)$ of § 3 implies $F(*; s) = s$; in fact, as pointed out, $F''(*; 0) > 0$ renders $F(*; s)$ a *sensible p.g.f. for a branching process*, and since $F'(*; 1) = 1$, enables us to say $F_h(*; 0) \to 1$ as $h \to \infty$, a most important step in our arguments.

Nevertheless, when $F''(*; 1) = 0$, some deductions are possible, if we make some further assertion. We shall only consider one such in general viz. $F''(m; 1) > 0$ for all m sufficiently close to one. A careful consideration of the bounds in § 2 reveals that in this case (as expected from (1.3) and (1.5))

$$\lim_{m\to 1-} (1-m)\mu = 0 \tag{5.1}$$

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$$\lim_{m\to 1-}\left\{\frac{ET}{-\log(1-m)}\right\} = \infty. \tag{5.2}$$

An example of such a distribution is given by the probability generating function of bilinear fractional form

$$F(m;s) = 1-m^2+\frac{m^3 s}{1-(1-m)s}, \qquad 0<m<1$$

which defines a modified geometric distribution. In this case we can calculate μ and σ^2 (see [4]) and obtain ET asymptotically:

$$\mu = \frac{1+m}{m} \to 2$$

$$\sigma^2 = \frac{(1+m)}{m^2} \to 2$$

$$ET \sim \frac{\log 2}{(1-m)}$$

as $m \to 1-$, which agrees with (5.1) and (5.2). Note also $\mu^2/\sigma^2 \to 2$ as $m \to 1-$.

The extremely pathological case not covered by any of the above is the two-point offspring distribution

$$F(m;s) = (1-m)+ms, \qquad 0<m<1$$

since $F''(m;1) = 0$ all $m \in (0,1)$. In this case

$$\frac{m^k}{1-F_k(m;0)} = 1$$

for all $k \geqq 0$, and

$$ET = \sum_{k=0}^{\infty}\{1-F_k(m;0)\} = \sum_{k=0}^{\infty} m^k = [1-m]^{-1}$$

which seems to behave analogously to the case just discussed.

In conclusion, we point out the relation of some of the present results, to relevant ones in the literature.

It was pointed out in [4], § 6, that a diffusion approximation result of Feller, as "$m \to 1-$" suggested the validity of (1.5) and (1.6) in a wide class of cases for which '$m \to 1-$' had a meaning. Another result of more immediate relevance in relation to this is the apparent assertion of Nagaev and Muhamedhanova [3] that if we put for our branching process (under conditions closely resembling (i), (ii) and (iii))

$$S_n(y) = P\left[\left\{Z_n\left(\frac{1-F_n(0)}{m^n}\right)\right\} < y \,\middle|\, Z_n \overset{!}{>} 0\right]$$

then

$$S_n(y) \to \begin{cases} 1-e^{-y} & y \geqq 0 \\ 0 & y < 0 \end{cases}$$

as $n \to \infty$, and $m \to 1+$ or $m \to 1-$, which certainly suggests that, as $m \to 1-$

$$\frac{\sigma^2}{\mu^2} \to 1,$$

from considerations (1.1) and (1.2). (See also [5].)

Finally, the procedure in our main discussion via inequality (2.3), was suggested to the author by (the body of) the proof of Lemma 1 in [3] where the expression (2.2) occurs. There is no other overlap in actual content: in fact the proofs of Nagaev and Muhamedhanova seem to concentrate equally on the case $m > 1$, and thus do not consider time to extinction at all.

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