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ON A CHARACTERIZATION OF COLLECTIONWISE NORMALITY

BY

O. T. ALAS

The purpose of this note is to give some characterizations of collectionwise normality by means of the normality of certain product topological spaces; they are a consequence of a theorem proved by M. Katětov, but they appear to be new.

Let (E, τ) be a Hausdorff normal space.

THEOREM (*M. Katětov* [3]). *(E, τ) is countably paracompact and collectionwise normal if and only if for any locally finite family of closed subsets of E, $(F_i)_{i\in I}$, there is a locally finite family of open subsets of E, $(X_i)_{i\in I}$, such that $F_i \subset X_i$, for any $i \in I$.*

NOTATION. For any set Y let $A(Y)$ denote the Alexandrov's (one point) compactification of the discrete space of support Y; thus $A(Y)=Y\cup\{w\}$, where w does not belong to Y.

REMARK. It is not difficult to prove that (E, τ) is countably paracompact if and only if the topological product space $E\times A(Y)$ is normal, where Y is an infinite countable set.

THEOREM 1. *If (E, τ) is countably paracompact and collectionwise normal, then the topological product space $E\times A(Y)$ is normal, for any set Y.*

Proof. Suppose that Y is a nonempty set and let F and G be two nonempty disjoint closed subsets of $E\times A(Y)$. Thus, we can write

$$F = \bigcup \{F_y \times \{y\} \mid y \in Y\} \cup F_w \times \{w\}$$

and

$$G = \bigcup \{G_y \times \{y\} \mid y \in Y\} \cup G_w \times \{w\}.$$

For any $z \in A(Y)$, the sets F_z and G_z are disjoint closed subsets of E; it follows that there are disjoint open subsets of E, V_z and W_z, such that $F_z \subset V_z$ and $G_z \subset W_z$. Thus, $V_w\times A(Y)$ and $W_w \times A(Y)$ are disjoint open sets and $\bigcup\{V_y\times\{y\} \mid y\in Y\}$ and $\bigcup\{W_y\times\{y\}\mid y\in Y\}$ are disjoint open sets.

The family $(F_y - V_w)_{y\in Y}$ is locally finite; by Katětov's theorem there is a locally finite family of open subsets of E, $(X_y)_{y\in Y}$, such that $F_y - V_w \subset X_y$, for any $y\in Y$. Now, for any $t\in G_w$, there is an open set U_t, such that $t\in U_t \subset W_w$ and the set

$$P_t = \{y\in Y \mid X_y \cap U_t \neq \varnothing\}$$

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is finite. It follows that

$$G_w \times \{w\} \subset \bigcup \{U_t \times (A(Y) - P_t) \mid t \in G_w\}$$

and

$$F \subset \bigcup \{X_y \times \{y\} \mid y \in Y\} \cup V_w \times A(Y).$$

In the same way we prove that $F_w \times \{w\}$ and G have disjoint neighborhoods. Thus, F and G have disjoint neighborhoods. The proof is completed.

THEOREM 2. *(E, τ) is countably paracompact and collectionwise normal if and only if the topological product $E \times A(E)$ is normal.*

Proof. By virtue of Theorem 1, it is sufficient to prove that if the topological product $E \times A(E)$ is normal, then (E, τ) is countably paracompact and collectionwise normal. Suppose E is infinite. Let $(F_i)_{i \in I}$ be a locally finite family of closed subsets of E. Since the cardinal number of I is less than or equal to the cardinal number of E, we can, and will suppose $I \subset E$. The closed sets $E \times \{w\}$ and $\bigcup \{F_i \times \{i\} \mid i \in I\}$ are disjoint; it follows that there are two disjoint open sets X and M such that

$$E \times \{w\} \subset M \quad \text{and} \quad \bigcup \{F_i \times \{i\} \mid i \in I\} \subset X.$$

For any $i \in I$ put $X_i = \{x \in E \mid (x, i) \in X\}$; thus, $F_i \subset X_i$ for any $i \in I$ and the family $(X_i)_{i \in I}$ is locally finite, since $X \cap M = \varnothing$. The proof is completed.

THEOREM 3. *The topological product $E \times A(E)$ is normal if and only if $E \times A(E)$ is collectionwise normal.*

Proof. Suppose that $E \times A(E)$ is normal, thus $E \times A(E)$ is countably paracompact. Let $(F_i)_{i \in I}$ be a discrete family of closed subsets of $E \times A(E)$; thus, we can write

$$F_i = \bigcup \{F_i^y \times \{y\} \mid y \in E\} \cup F_i^w \times \{w\}, \quad \text{for any } i \in I.$$

$(F_i^w)_{i \in I}$ is a discrete family of closed subsets of E. Thus, there is a discrete family of open subsets of E, $(L_i)_{i \in I}$, such that $F_i^w \subset L_i$ for any $i \in I$. For any $i \in I$ put

$$U_i = \bigcup \{F_i^y \mid y \in E\} - L_i.$$

The family $(U_i)_{i \in I}$ is locally finite; it follows that there is a locally finite family of open subsets of E, $(M_i)_{i \in I}$, such that $U_i \subset M_i$ for any $i \in I$. For any $t \in F_i^w$ there is an open neighborhood V_t, with $V_t \subset L_i$, and a finite subset of E, θ_t, such that $V_t \times (A(E) - \theta_t)$ intercepts at most one set F_k with $k \in I$. For any $i \in I$ put

$$G_i = \bigcup \{V_t \times (A(E) - \theta_t) \mid t \in F_i^w\} \cup M_i \times E \cup L_i \times E.$$

The family $(G_i)_{i \in I}$ is locally finite and for any $i \in I$ the open set G_i contains F_i. By virtue of the normality of $E \times A(E)$ the proof is completed.

THEOREM 4. *The following conditions are equivalent*:

(1) *the topological product* $E \times A(E)$ *is normal*;

(2) *the topological product* $E \times A(E)$ *is collectionwise normal*;

(3) (E, τ) *is countably paracompact and collectionwise normal*;

(4) *the topological product* $E \times A(Y)$ *is normal for any set* Y;

(5) *each pair of disjoint closed subsets of* $E \times A(E)$, *one of which is* $E \times \{w\}$, *is completely separated.*

REMARK. For other characterizations of collectionwise normality see, for instance, J. Nagata [4].

As a matter of fact, it is possible to prove the following theorem: *Suppose* (E, τ) *countably paracompact and let* Y *be an infinite set. The topological product* $E \times A(Y)$ *is normal if and only if for any discrete family of closed subsets of* E, $(F_i)_{i \in I}$, *with the cardinal of* I *less than or equal to the cardinal of* Y, *there is a discrete family of open subsets of* E, $(X_i)_{i \in I}$, *such that* $F_i \subset X_i$ *for any* $i \in I$. The example of a normal space which is not collectionwise normal proposed by Bing [1] is such that its product by $A(Y)$, where the cardinal of Y is $\aleph_1$, is not normal.

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UNIVERSITY OF SÃO PAULO,
SÃO PAULO, BRAZIL

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