# On the highest space in which a non-ruled surface of given order can lie

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It is well known[1] that a non-ruled (*i.e.* not consisting of an infinity of lines) surface of order $n$ lies in a space of not more than $n$ dimensions ($n \neq 4$), and that for $n > 9$, the maximum dimension actually attained (here denoted by $R$) is certainly less than $n$.

The precise value of $R$ does not seem to have been stated explicitly: the purpose of this note is to shew that

$$R = I\left(\tfrac{3}{4}n\right) + 2, \qquad n \neq 1, 2, 3, 9;$$

where $I(x)$ denotes the integral part of $x$.

This conclusion is deduced from del Re's theorem[2] that for a non-ruled surface whose prime sections are of genus $p$ ($\geqq 2$), the maximum order is $4p + 4$.

Consider a surface of order $n$ in space of $r$ dimensions, whose prime sections are of genus $p$ ($\geqq 2$). The series cut on any prime section by all the other prime sections (characteristic series of prime sections) is evidently of order $n$ and freedom $r - 1$: suppose that it is contained in a complete series of freedom $r_0 - 1$ (so that $r \leqq r_0$), and that its index of speciality is $i$.

From the Riemann-Roch theorem $n - r_0 + 1 = p - i$.
From del Re's result quoted above $p \geqq \frac{1}{4}n - 1$.
Eliminating $p$ $\qquad r_0 \leqq \frac{3}{4}n + 2 + i$.
In general $i = 0$, *i.e.* the series is not special, so that

$$r \leqq r_0 \leqq \tfrac{3}{4}n + 2.$$

If the series is special, the prime sections are not hyperelliptic (for the characteristic series of a prime section is cut thereon by primes, and is therefore certainly simple). Hence from Clifford's theorem

$$r_0 - 1 \leqq \tfrac{1}{2}n,$$

so that

$$r \leqq r_0 \leqq \tfrac{1}{2}n + 1 < \tfrac{3}{4}n + 2.$$

Thus in no case for which $p \geqq 2$ can $r$ exceed $I\left(\tfrac{3}{4}n\right) + 2$.

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[1] Del Pezzo, *Rend. R. Acc. Napoli*, **24** (1885), 215, and **25** (1886), 208.

[2] *Rend. R. Acc. Napoli* (3), **30** (1924), 80.

The value $r = I(\frac{3}{4}n) + 2$ is attained[1] by the surface represented upon a plane by curves of order $m$ having one fixed $\overline{m-2}$-fold point and $4m-4-n$ fixed simple points, $m$ being the least integer $\geqq \frac{1}{4}n + 1$. This confirms incidentally del Pezzo's surmise that the maximum $R$ is attained by a *rational* surface.

It follows that for $p \geqq 2$, $R = I(\frac{3}{4}n) + 2$.

The cases $p=0, 1$ (and therefore $n \leqq 9$) are well-known and yield the same value for $R$, except when $n = 1, 2$ (when non-ruled surfaces are impossible); $n = 3$, $(R = 3)$; $n = 9$, $(R = 9)$.

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[1] The prime sections are evidently hyperelliptic. The same surfaces, when $\frac{1}{4}n$ is an integer, are in fact those having the maximum order for a given genus of section.

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