Bull. Aust. Math. Soc. **85** (2012), 280–294
doi:10.1017/S0004972711003261

UNIQUENESS OF MEROMORPHIC FUNCTIONS SHARING ONE VALUE

CHUN WU✉**, CHUNLAI MU and JIANGTAO LI**

(Received 17 May 2011)

Abstract

In this paper, we study the uniqueness of meromorphic functions concerning differential polynomials sharing nonzero finite values, and obtain some results which improve the results of Yang and Hua, Xu and Qiu, Fang and Hong, and Dyavanal, among others.

2010 *Mathematics subject classification*: primary 30D35.

Keywords and phrases: uniqueness, meromorphic functions, differential polynomials.

1. Introduction and main results

Let $f(z)$ be a nonconstant meromorphic function in the whole complex plane. We will use the standard notation of Nevanlinna's value distribution theory such as $T(r, f)$, $N(r, f)$, $\overline{N}(r, f)$ and $m(r, f)$, as found in Hayman [4], Yang [9] and Yi and Yang [11]. We denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$, possibly outside a set with finite measure.

For any constant a we define

$$\Theta(a, f) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}(r, \frac{1}{f-a})}{T(r, f)},$$

where $\overline{N}(r, 1/(f - a))$ is the counting function which counts zeros of $f - a$ in $|z| \le r$, ignoring multiplicities.

Let a be a finite complex number and k be a positive integer. We denote by $N_{k)}(r, 1/(f - a))$ the counting function for zeros of $f - a$ with multiplicity at most k, and by $\overline{N}_{k)}(r, 1/(f - a))$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r, 1/(f - a))$ be the counting function for zeros of $f - a$ with multiplicity at least k and $\overline{N}_{(k}(r, 1/(f - a))$ the corresponding one for which multiplicity is not counted.

This work is supported in part by NSF of China (11071266) and in part by NSF project CQ CSTC (2010BB9218).

Set

$$N_k\Big(r, \frac{1}{f-a}\Big) = \overline{N}\Big(r, \frac{1}{f-a}\Big) + \overline{N}_{(2}\Big(r, \frac{1}{f-a}\Big) + \cdots + \overline{N}_{(k}\Big(r, \frac{1}{f-a}\Big).$$

We define

$$\delta_k(a, f) = 1 - \overline{\lim_{r\to\infty}} \frac{N_k(r, \frac{1}{f-a})}{T(r, f)}.$$

Let f and g be two nonconstant meromorphic functions. Let a be a finite complex number. We say that f, g share the value a CM (counting multiplicities) if f, g have the same a-points with the same multiplicities, and we say that f, g share the value a IM (ignoring multiplicities) if we do not consider the multiplicities. We denote by $\overline{N}_L(r, 1/(f-a))$ the counting function for a-points of both f and g where f has larger multiplicity than g, with multiplicity not counted. Similarly, we have the notation $\overline{N}_L(r, 1/(g-a))$.

It is assumed that the reader is familiar with the notation of Nevanlinna theory that can be found in [4, 9, 11].

On uniqueness problems for entire functions sharing one value, Yang and Hua [10] obtained the following result in 1997.

Theorem A. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, $n \ge 11$ an integer and $a \in C - \{0\}$. If $f^n f'$ and $g^n g'$ share the value a CM, then either $f = tg$ for a constant t with $t^{n+1} = 1$ or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1c_2)^{n+1}c^2 = -a^2$.*

In 2000, Xu and Qiu [7] proved the following result, which generalised Theorem A.

Theorem B. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, $n \ge 12$ an integer. If $f^n f'$ and $g^n g'$ share the value 1 IM, then either $f = tg$ for a constant t with $t^{n+1} = 1$ or $f(z) = c_1 e^{cz}$ and $g(z) = c_2 e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1c_2)^{n+1}c^2 = -1$.*

In 2001, Fang and Hong [2] proved the following result.

Theorem C. *Let $f(z)$ and $g(z)$ be two transcendental entire functions, $n \ge 11$ a positive integer. If $f^n(z)(f(z)-1)f'(z)$ and $g^n(z)(g(z)-1)g'(z)$ share the value 1 CM, then $f(z) \equiv g(z)$.*

In 2004, Lin and Yi [6] proved the following three theorems.

Theorem D. *Let f and g be two transcendental entire functions, $n \ge 7$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then $f(z) \equiv g(z)$.*

Theorem E. *Let f and g be two distinct nonconstant meromorphic functions, $n \ge 12$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then*

$$g = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}, \qquad f = \frac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})},$$

where h is a nonconstant meromorphic function.

https://doi.org/10.1017/S0004972711003261 Published online by Cambridge University Press

THEOREM F. *Let f and g be two nonconstant meromorphic functions, $n \geq 13$ an integer. If $f^n(f-1)^2 f'$ and $g^n(g-1)^2 g'$ share the value 1 CM, then $f(z) \equiv g(z)$.*

In 2011, Dyavanal [1] proved the following theorems.

THEOREM G. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let $n \geq 2$ be an integer satisfying $(n+1)s \geq 12$. If $f^n f'$ and $g^n g'$ share the value 1 CM, then either $f = tg$ for a constant t with $t^{n+1} = 1$ or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$.*

THEOREM H. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let n be an integer satisfying $(n-2)s \geq 10$. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then*

$$g = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}, \quad f = \frac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})},$$

where h is a nonconstant meromorphic function.

Dyavanal raised the open question in his paper whether the differential polynomials can be replaced by differential polynomials of the forms $(f^n)^{(k)}$ and $(f^n(f-1))^{(k)}$ and whether a CM shared value can be replaced by an IM shared value in Theorem G and H. In this paper, we solve these problems and obtain the following results.

THEOREM 1.1. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions whose zeros and poles are of multiplicities at least l, where l is a positive integer. Let $n \geq 2k+1$ be an integer satisfying $nl > 7k + 12$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share the value 1 IM, then either $f = tg$ for a constant t with $t^n = 1$ or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where c, c_1, c_2 are constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$.*

THEOREM 1.2. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, whose zeros and poles are of multiplicities at least $l < (7k/2) + 7$, where l is a positive integer. Let n be an integer satisfying $(n+1)l > 7k + 17$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share the value 1 IM, $\Theta(\infty, f) > 2/n$, then $f \equiv g$.*

REMARK. The following example shows that Theorem 1.2 is sharp. Let

$$f(z) = \frac{h(z)(1-h^n(z))}{1-h^{n+1}(z)}, \quad g(z) = \frac{1-h^n(z)}{1-h^{n+1}(z)}, \tag{1.1}$$

where n is a positive integer and $h(z)$ is a nonconstant meromorphic function. We deduce from (1.1) that $f^n(f-1) = g^n(g-1)$; thus f and g satisfy the conditions of Theorem 1.2, but $f \not\equiv g$. Note that

$$T(r, f) = T(r, gh) = nT(r, h) + S(r, f).$$

By the second fundamental theorem, we deduce that

$$\overline{N}(r, f) = \sum_{i=1}^{n} \overline{N}\left(r, \frac{1}{h - a_i}\right) \geq (n-2)T(r, h) + S(r, f),$$

where $a_i \neq 1$ $(i = 1, 2, \ldots, n)$ are distinct roots of the equation $h^{n+1} = 1$. Therefore,

$$\Theta(\infty, f) = 1 - \varlimsup_{r\to\infty} \frac{\overline{N}(r, f)}{T(r, f)} \leq \frac{2}{n}.$$

Thus Theorem 1.2 is the best possible in some sense, at least for the case $\Theta(\infty, f) > 2/n$.

Theorem 1.3. *Let $f(z)$ and $g(z)$ be two transcendental entire functions whose zeros are of multiplicities at least l, where l is a positive integer. Let n be an integer satisfying $nl > 4k + 6$. If $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share the value 1 IM, then either $f = tg$ for a constant t with $t^n = 1$ or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where c, c_1, c_2 are constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$.*

Theorem 1.4. *Let f and g be two transcendental entire functions whose zeros are of multiplicities at least l, where l is a positive integer. Let n be an integer satisfying $(n+1)l \geq 4k + 11$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share the value 1 IM, then $f(z) \equiv g(z)$.*

2. Some lemmas

Lemma 2.1 (See [9, 11]). *Let $f(z)$ be a nonconstant meromorphic function, k a positive integer and c a nonzero finite complex number. Then*

$$\begin{aligned} T(r, f) &\leq \overline{N}(r, f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)} - c}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f) \\ &\leq \overline{N}(r, f) + N_{k+1}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - c}\right) - N_0\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f), \end{aligned}$$

where $N_0(r, 1/f^{(k+1)})$ is the counting function which only counts those points such that $f^{(k+1)} = 0$ but $f(f^{(k)} - c) \neq 0$.

Lemma 2.2 (See [12]). *Let $f(z)$ be a nonconstant meromorphic function, and let k be a positive integer. Suppose that $f^{(k)} \not\equiv 0$. Then*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f).$$

Lemma 2.3 (See [5]). *Let $f(z)$ be a nonconstant meromorphic function and s, k be two positive integers. Then*

$$N_s\left(r, \frac{1}{f^{(k)}}\right) \leq k\overline{N}(r, f) + N_{s+k}\left(r, \frac{1}{f}\right) + S(r, f).$$

Clearly, $\overline{N}(r, 1/f^{(k)}) = N_1(r, 1/f^{(k)})$.

https://doi.org/10.1017/S0004972711003261 Published online by Cambridge University Press

Lemma 2.4. *Let $f(z)$ and $g(z)$ be two nonconstant transcendental meromorphic functions and k be a positive integer. If $f^{(k)}$ and $g^{(k)}$ share the value 1 IM and if*

$$\begin{aligned}\Delta = (k+3)\Theta(\infty, g) + (2k+3)\Theta(\infty, f) + \delta_{k+2}(0, g) + \delta_{k+1}(0, g)\\ + 2\delta_{k+1}(0, f) + \Theta(0, f) > 3k + 10,\end{aligned}$$

then either $f^{(k)}g^{(k)} \equiv 1$ or $f \equiv g$.

Proof. Let

$$\Phi(z) = \frac{f^{(k+2)}}{f^{(k+1)}} - 2\frac{f^{(k+1)}}{f^{(k)}-1} - \frac{g^{(k+2)}}{g^{(k+1)}} + 2\frac{g^{(k+1)}}{g^{(k)}-1}. \tag{2.1}$$

Clearly $m(r, \Phi) = S(r, f) + S(r, g)$. We consider the cases $\Phi(z) \not\equiv 0$ and $\Phi(z) \equiv 0$.

Let $\Phi(z) \not\equiv 0$. Then if z_0 is a common simple 1-point of $f^{(k)}$ and $g^{(k)}$, substituting their Taylor series at z_0 into (2.1), we see that z_0 is a zero of $\Phi(z)$. Thus

$$\begin{aligned} N_{1)}\Big(r, \frac{1}{f^{(k)}-1}\Big) &= N_{1)}\Big(r, \frac{1}{g^{(k)}-1}\Big)\\ &\le \overline{N}\Big(r, \frac{1}{\Phi}\Big)\\ &\le T(r, \Phi) + O(1)\\ &\le N(r, \Phi) + S(r, f) + S(r, g). \end{aligned} \tag{2.2}$$

Here, $N_{1)}(r, 1/f^{(k)} - 1)$ is the counting function which only counts those points such that$f^{(k)} - 1 = 0$ but $f^{(k+1)} \neq 0$.

Our assumptions are that $\Phi(z)$ has poles, all simple, only at zeros of $f^{(k+1)}$ and $g^{(k+1)}$ and poles of f and g, and 1-points of f whose multiplicities are not equal to the multiplicities of the corresponding 1-points of g. Thus, we deduce from (2.1) that

$$\begin{aligned} N(r, \Phi) \le \overline{N}(r, f) + \overline{N}(r, g) + \overline{N}_{(k+2}\Big(r, \frac{1}{f}\Big) + \overline{N}_{(k+2}\Big(r, \frac{1}{g}\Big) + N_0\Big(r, \frac{1}{f^{(k+1)}}\Big)\\ + N_0\Big(r, \frac{1}{g^{(k+1)}}\Big) + \overline{N}_L\Big(r, \frac{1}{f^{(k)}-1}\Big) + \overline{N}_L\Big(r, \frac{1}{g^{(k)}-1}\Big). \end{aligned}$$

Here, $N_0(r, 1/f^{(k+1)})$ has the same meaning as in Lemma 2.1. From Lemma 2.1,

$$T(r, g) \le \overline{N}(r, g) + N_{k+1}\Big(r, \frac{1}{g}\Big) + \overline{N}\Big(r, \frac{1}{g^{(k)}-1}\Big) - N_0\Big(r, \frac{1}{g^{(k+1)}}\Big) + S(r, g) \tag{2.3}$$

since

$$\overline{N}\Big(r, \frac{1}{g^{(k)}-1}\Big) = N_{1)}\Big(r, \frac{1}{g^{(k)}-1}\Big) + \overline{N}_{(2}\Big(r, \frac{1}{f^{(k)}-1}\Big). \tag{2.4}$$

Thus we deduce from (2.2)–(2.4) that

$$\begin{aligned} T(r,g) \le{}& 2\overline{N}(r,g) + \overline{N}(r,f) + N_{k+1}\Big(r,\frac{1}{g}\Big) + \overline{N}_{(k+2}\Big(r,\frac{1}{f}\Big) \\ &+ \overline{N}_{(k+2}\Big(r,\frac{1}{g}\Big) + N_0\Big(r,\frac{1}{f^{(k+1)}}\Big) \\ &+ \overline{N}_{(2}\Big(r,\frac{1}{f^{(k)}-1}\Big) + \overline{N}_L\Big(r,\frac{1}{f^{(k)}-1}\Big) + \overline{N}_L\Big(r,\frac{1}{g^{(k)}-1}\Big) \\ &+ S(r,f) + S(r,g). \end{aligned} \tag{2.5}$$

From the definition of $N_0(r, 1/f^{(k+1)})$, we see that

$$\begin{aligned} &N_0\Big(r,\frac{1}{f^{(k+1)}}\Big) + \overline{N}_{(2}\Big(r,\frac{1}{f^{(k)}-1}\Big) + N_{(2}\Big(r,\frac{1}{f^{(k)}}\Big) - \overline{N}_{(2}\Big(r,\frac{1}{f^{(k)}}\Big) \\ &\quad\le N\Big(r,\frac{1}{f^{(k+1)}}\Big). \end{aligned}$$

This and Lemma 2.2 give

$$\begin{aligned} &N_0\Big(r,\frac{1}{f^{(k+1)}}\Big) + \overline{N}_{(2}\Big(r,\frac{1}{f^{(k)}-1}\Big) \\ &\quad\le N\Big(r,\frac{1}{f^{(k+1)}}\Big) - N_{(2}\Big(r,\frac{1}{f^{(k)}}\Big) + \overline{N}_{(2}\Big(r,\frac{1}{f^{(k)}}\Big) \\ &\quad\le N\Big(r,\frac{1}{f^{(k)}}\Big) - N_{(2}\Big(r,\frac{1}{f^{(k)}}\Big) + \overline{N}_{(2}\Big(r,\frac{1}{f^{(k)}}\Big) + \overline{N}(r,f) + S(r,f) \\ &\quad\le \overline{N}\Big(r,\tfrac{1}{f^{(k)}}\Big) + \overline{N}(r,f) + S(r,f). \end{aligned}$$

Substituting the above inequality into (2.5),

$$\begin{aligned} T(r,g) \le{}& 2\overline{N}(r,g) + \overline{N}(r,f) + N_{k+1}\Big(r,\frac{1}{g}\Big) + \overline{N}_{(k+2}\Big(r,\frac{1}{f}\Big) + \overline{N}_{(k+2}\Big(r,\frac{1}{g}\Big) + \overline{N}\Big(r,\frac{1}{f^{(k)}}\Big) \\ &+ \overline{N}(r,f) + \overline{N}_L\Big(r,\frac{1}{f^{(k)}-1}\Big) + \overline{N}_L\Big(r,\frac{1}{g^{(k)}-1}\Big) + S(r,f) + S(r,g) \\ \le{}& 2\overline{N}(r,g) + 2\overline{N}(r,f) + N_{k+2}\Big(r,\frac{1}{g}\Big) + \overline{N}\Big(r,\frac{1}{f}\Big) + \overline{N}\Big(r,\frac{1}{f^{(k)}}\Big) \\ &+ \overline{N}_L\Big(r,\frac{1}{f^{(k)}-1}\Big) + \overline{N}_L\Big(r,\frac{1}{g^{(k)}-1}\Big) + S(r,f) + S(r,g). \end{aligned} \tag{2.6}$$

According to Lemma 2.3,

$$\overline{N}\Big(r,\frac{1}{f^{(k)}}\Big) = N_1\Big(r,\frac{1}{f^{(k)}}\Big) \le N_{k+1}\Big(r,\frac{1}{f}\Big) + k\overline{N}(r,f) + S(r,f). \tag{2.7}$$

https://doi.org/10.1017/S0004972711003261 Published online by Cambridge University Press

Therefore,

$$\begin{aligned}\overline{N}_L\Big(r, \frac{1}{f^{(k)}-1}\Big) &\le N\Big(r, \frac{1}{f^{(k)}-1}\Big) - \overline{N}\Big(r, \frac{1}{f^{(k)}-1}\Big)\\ &\le N\Big(r, \frac{f^{(k)}}{f^{(k+1)}}\Big) \le N\Big(r, \frac{f^{(k+1)}}{f^{(k)}}\Big) + S(r, f)\\ &\le \overline{N}\Big(r, \frac{1}{f^{(k)}}\Big) + \overline{N}(r, f) + S(r, f)\\ &\le N_{k+1}\Big(r, \frac{1}{f}\Big) + (k+1)\overline{N}(r, f) + S(r, f).\end{aligned}$$

Similarly,

$$\overline{N}_L\Big(r, \frac{1}{g^{(k)}-1}\Big) \le N_{k+1}\Big(r, \frac{1}{g}\Big) + (k+1)\overline{N}(r, g) + S(r, g).$$

Combining this and (2.6),

$$\begin{aligned}T(r, g) \le (k+3)\overline{N}(r, g) + (2k+3)\overline{N}(r, f) + N_{k+2}\Big(r, \frac{1}{g}\Big)\\ + N_{k+1}\Big(r, \frac{1}{g}\Big) + 2N_{k+1}\Big(r, \frac{1}{f}\Big) + \overline{N}\Big(r, \frac{1}{f}\Big) + S(r, f) + S(r, g).\end{aligned}$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r, f) \le T(r, g)$ for $r \in I$. Hence

$$\begin{aligned}T(r, g) \le ((k+3)(1-\Theta(\infty, g)) + (2k+3)(1-\Theta(\infty, f)) + (1-\delta_{k+2}(0, g))\\ + (1-\delta_{k+1}(0, g)) + 2(1-\delta_{k+1}(0, f)) + (1-\Theta(0, f)) + \varepsilon)T(r, g) + S(r, g)\end{aligned}$$

for $r \in I$ and $0 < \varepsilon < \Delta - (3k+10)$. That is, $(\Delta - (3k+10) - \varepsilon)T(r, g) \le S(r, g)$, or $\Delta - (3k+10) \le 0$, or $\Delta \le 3k+10$, which contradicts our hypothesis that $\Delta > 3k+10$.

Hence, $\Phi(z) \equiv 0$. Therefore, by (2.1),

$$\frac{f^{(k+2)}}{f^{(k+1)}} - \frac{2f^{(k+1)}}{f^{(k)}-1} \equiv \frac{g^{(k+2)}}{g^{(k+1)}} - \frac{2g^{(k+1)}}{g^{(k)}-1}.$$

Integrating both sides of this equation, we obtain

$$\frac{1}{f^{(k)}-1} = \frac{bg^{(k)}+a-b}{g^{(k)}-1}, \tag{2.8}$$

where $a \ne 0$ and b are constants.

We now consider three cases.

Case 1. $b \ne 0$ and $a = b$.

(i) If $b = -1$, then, from (2.8), $f^{(k)}g^{(k)} \equiv 1$.

https://doi.org/10.1017/S0004972711003261 Published online by Cambridge University Press

(ii) If $b \neq -1$, then, from (2.8),

$$\frac{1}{f^{(k)}} = \frac{bg^{(k)}}{(1+b)g^{(k)} - 1}. \tag{2.9}$$

We can write

$$\overline{N}\left(r, \frac{1}{g^{(k)} - \frac{1}{1+b}}\right) \leq \overline{N}\left(r, \frac{g^{(k)}}{g^{(k)} - \frac{1}{1+b}}\right). \tag{2.10}$$

From (2.9) and (2.10),

$$\overline{N}\left(r, \frac{1}{g^{(k)} - \frac{1}{1+b}}\right) \leq \overline{N}\left(r, \frac{1}{f^{(k)}}\right). \tag{2.11}$$

Therefore, from (2.7) and (2.11),

$$\overline{N}\left(r, \frac{1}{g^{(k)} - \frac{1}{1+b}}\right) \leq k\overline{N}(r, f) + N_{k+1}\left(r, \frac{1}{f}\right) + S(r, f). \tag{2.12}$$

From (2.12) and Lemma 2.1,

$$\begin{aligned} T(r, g) &\leq \overline{N}(r, g) + N_{k+1}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g^{(k)} - \frac{1}{1+b}}\right) - N_0\left(r, \frac{1}{g^{(k+1)}}\right) + S(r, g) \\ &\leq \overline{N}(r, g) + N_{k+1}\left(r, \frac{1}{g}\right) + k\overline{N}(r, f) + N_{k+1}\left(r, \frac{1}{f}\right) + S(r, f) + S(r, g) \\ &\leq (k+3)\overline{N}(r, g) + (2k+3)\overline{N}(r, f) + N_{k+2}\left(r, \frac{1}{g}\right) \\ &\quad + N_{k+1}\left(r, \frac{1}{g}\right) + 2N_{k+1}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f}\right) + S(r, f) + S(r, g). \end{aligned}$$

Thus we obtain

$$(\Delta - (3k+10))T(r, g) \leq S(r, g),$$

a contradiction.

Case 2. $b \neq 0$ and $a \neq b$.

From (2.8),

$$f^{(k)} - \left(1 + \frac{1}{b}\right) = \frac{-a}{b^2\left(g^{(k)} + \frac{a-b}{b}\right)}.$$

This implies that

$$\overline{N}\left(r, \frac{1}{g^{(k)} + \frac{a-b}{b}}\right) = \overline{N}\left(r, f^{(k)} - \left(1 + \frac{1}{b}\right)\right) = \overline{N}(r, f^{(k)}) = \overline{N}(r, f). \tag{2.13}$$

https://doi.org/10.1017/S0004972711003261 Published online by Cambridge University Press

From Lemma 2.1 and from (2.13),

$$\begin{aligned}T(r,g) &\le \overline{N}(r,g) + N_{k+1}\Big(r,\frac{1}{g}\Big) + \overline{N}\Big(r,\frac{1}{g^{(k)}+\frac{a-b}{b}}\Big) + S(r,g)\\ &\le \overline{N}(r,g) + N_{k+1}\Big(r,\frac{1}{g}\Big) + \overline{N}(r,f) + S(r,g)\\ &\le (k+3)\overline{N}(r,g) + (2k+3)\overline{N}(r,f) + N_{k+2}\Big(r,\frac{1}{g}\Big)\\ &\qquad + N_{k+1}\Big(r,\frac{1}{g}\Big) + 2N_{k+1}\Big(r,\frac{1}{f}\Big) + \overline{N}\Big(r,\frac{1}{f}\Big) + S(r,f) + S(r,g).\end{aligned}$$

Using the argument as in Case 1, we get a contradiction.

Case 3. $b = 0$.

From (2.8),

$$f^{(k)} = \frac{1}{a}g^{(k)} + 1 - \frac{1}{a}, \tag{2.14}$$

$$f = \frac{1}{a}g + p(z), \tag{2.15}$$

where $p(z)$ is a polynomial with degree at most k. If $p(z) \not\equiv 0$, then, by the second fundamental theorem for small functions,

$$\begin{aligned}T(r,g) &\le \overline{N}(r,g) + \overline{N}\Big(r,\frac{1}{g}\Big) + \overline{N}\Big(r,\frac{1}{g+ap(z)}\Big) + S(r,f)\\ &\le \overline{N}(r,g) + \overline{N}\Big(r,\frac{1}{g}\Big) + \overline{N}\Big(r,\frac{1}{f}\Big) + S(r,f).\end{aligned}$$

Using the same argument as in Case 2, we get a contradiction. Therefore, $p(z) \equiv 0$ and, from (2.14) and (2.15), we obtain $a = 1$ and so $f \equiv g$.

This proves the lemma. □

Lemma 2.5. *Let $f(z)$ and $g(z)$ be two nonconstant transcendental entire functions, and k a positive integer. If $f^{(k)}$ and $g^{(k)}$ share the value 1 IM and if $\Delta = \delta_{k+2}(0,g) + \delta_{k+1}(0,g) + 2\delta_{k+1}(0,f) + \Theta(0,f) > 4$, then either $f^{(k)}g^{(k)} \equiv 1$ or $f \equiv g$.*

Proof. Since f and g are entire functions, $\overline{N}(r,f) = 0$ and $\overline{N}(r,g) = 0$. Proceeding as in the proof of Lemma 2.4, we obtain the desired result. □

Lemma 2.6 (See [3]). *Let f be a nonconstant entire function and $k \ge 2$ be an integer. If $ff^{(k)} \ne 0$, then $f = e^{az+b}$, where $a \ne 0$ and b are two constants.*

Lemma 2.7 (See [8]). *Let $f(z)$ be a nonconstant meromorphic function. Let k be a positive integer, and let c be a nonzero finite complex number. Then*

$$T(r, a_nf^n + a_{n-1}f^{n-1} + \cdots + a_0) = nT(r,f) + S(r,f).$$

https://doi.org/10.1017/S0004972711003261 Published online by Cambridge University Press

3. Proof of theorems

3.1. Proof of Theorem 1.1. Let $F = f^n$ and $G = g^n$. Consider

$$\overline{N}\Big(r, \frac{1}{F}\Big) = \overline{N}\Big(r, \frac{1}{f^n}\Big) \le \frac{1}{ln} N\Big(r, \frac{1}{F}\Big) \le \frac{1}{ln}(T(r, F) + O(1)).$$

Therefore,

$$\Theta(0, F) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}(r, \frac{1}{F})}{T(r, F)} \ge 1 - \frac{1}{ln},$$
$$\delta_{k+2}(0, G) = 1 - \overline{\lim_{r\to\infty}} \frac{N_{k+2}(r, \frac{1}{G})}{T(r, G)} \ge 1 - \overline{\lim_{r\to\infty}} \frac{(k+2)\overline{N}(r, \frac{1}{G})}{T(r, G)} \ge 1 - \frac{k+2}{ln}.$$

Similarly,

$$\delta_{k+1}(0, G) \ge 1 - \frac{k+1}{ln}, \quad \delta_{k+1}(0, F) \ge 1 - \frac{k+1}{ln},$$
$$\Theta(\infty, F) \ge 1 - \frac{1}{ln}, \quad \Theta(\infty, G) \ge 1 - \frac{1}{ln}.$$

Therefore,

$$\begin{aligned}\Delta &= (k+3)\Theta(\infty, G) + (2k+3)\Theta(\infty, F) + \delta_{k+2}(0, G) + \delta_{k+1}(0, G)\\ &\quad + 2\delta_{k+1}(0, F) + \Theta(0, F) \ge (3k+11) - \frac{7k+12}{ln}.\end{aligned}$$

Since $nl > 7k + 12$, we obtain $\Delta > 3k + 10$. So, by Lemma 2.4, either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$.

Case 1. $F^{(k)}G^{(k)} \equiv 1$, that is,

$$(f^n)^{(k)}(g^n)^{(k)} \equiv 1. \tag{3.1}$$

We first prove that $f \ne 0, \infty$ and $g \ne 0, \infty$. Suppose that $f(g)$ has a zero z_0 (with order $p \ge l$). Then z_0 is a pole of $g(f)$ (with order $q \ge l$). By (3.1),

$$np - k = nq + k.$$

That is, $n(p - q) = 2k$, which is impossible since $n \ge 2k + 1$. Therefore, $f \ne 0$ and $g \ne 0$. Similarly, we can prove that $f \ne \infty$ and $g \ne \infty$. So f and g are entire and $f \ne 0, \infty$ and $g \ne 0, \infty$ holds. From this and (3.1) we obtain $(f^n)^{(k)} \ne 0$ and $(g^n)^{(k)} \ne 0$. In view of this and (3.1) using Lemma 2.6, we obtain that $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where c, c_1, c_2 are constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, when $k \ge 2$.

Next we consider $(f^n)^{(k)}(g^n)^{(k)} \equiv 1$ for the case $k = 1$, that is,

$$n^2 f^{n-1} f' g^{n-1} g' \equiv 1. \tag{3.2}$$

https://doi.org/10.1017/S0004972711003261 Published online by Cambridge University Press

From the above, there exist two entire functions $\alpha(z)$ and $\beta(z)$ such that $f = e^{\alpha(z)}$, $g = e^{\beta(z)}$. From this and (3.2),

$$n^2\alpha'\beta' e^{n(\alpha+\beta)} \equiv 1. \tag{3.3}$$

Thus, α' and β' have no zeros and we may set

$$\alpha' = e^{\delta(z)}, \quad \beta' = e^{\gamma(z)}, \tag{3.4}$$

where δ and γ are entire functions. By (3.3) and (3.4),

$$n^2 e^{n(\alpha+\beta)+\delta+\gamma} \equiv 1.$$

Differentiating this yields, in view of (3.4),

$$n(e^{\delta} + e^{\gamma}) + \delta' + \gamma' \equiv 0, \tag{3.5}$$

that is, $ne^{\delta} + \delta' \equiv -ne^{\gamma} - \gamma'$.

Since δ and γ are entire, we obtain

$$T(r, \delta') = m(r, \delta') = m\left(r, \frac{(e^{\delta})'}{e^{\delta}}\right) = S(r, e^{\delta}),$$
$$T(r, \gamma') = m(r, \gamma') = m\left(r, \frac{(e^{\gamma})'}{e^{\gamma}}\right) = S(r, e^{\gamma}).$$

Thus $T(r, e^{\delta}) = T(r, e^{\gamma}) + S(r, e^{\delta}) + S(r, e^{\gamma})$, which implies that $S(r, e^{\delta}) = S(r, e^{\gamma}) := S(r)$.

Let $\sigma \equiv -(\delta' + \gamma')$. Then $T(r, \sigma) = S(r)$. If $\sigma \not\equiv 0$, then we rewrite (3.5) as

$$\frac{e^{\delta}}{\sigma} + \frac{e^{\gamma}}{\sigma} \equiv \frac{1}{n}.$$

From this and the second fundamental theorem,

$$\begin{aligned} T(r, e^{\delta}) &\leq T\left(r, \frac{e^{\gamma}}{\sigma}\right) + S(r) \\ &\leq \overline{N}\left(r, \frac{e^{\gamma}}{\sigma}\right) + \overline{N}\left(r, \frac{1}{\frac{e^{\gamma}}{\sigma}}\right) + \overline{N}\left(r, \frac{1}{\frac{e^{\gamma}}{\sigma} - \frac{1}{n}}\right) + S(r) \\ &\leq S(r), \end{aligned}$$

which is a contradiction. This shows that $\sigma \equiv -(\delta' + \gamma') \equiv 0$. From (3.5) we have $\alpha' + \beta' = e^{\delta} + e^{\gamma} = -\sigma/n \equiv 0$, which implies that $\delta = \gamma + (2\rho + 1)\pi i$ for some integer ρ. This, together with $\delta' + \gamma' \equiv 0$, implies that $\delta + \gamma = t$. Taking $\delta = t_1$, we get $\gamma = t_2$, where t, t_1, t_2 are constants satisfying $t_1 + t_2 = t$. Therefore, α' and β' are constants. From this, $\alpha' + \beta' \equiv 0$, $f = e^{\alpha(z)}$ and $g = e^{\beta(z)}$, we can also obtain the above results.

Case 2. $F \equiv G$.

This gives $f^n \equiv g^n$. Hence, $f = tg$ for a constant t with $t^n = 1$. This proves the theorem.

3.2. Proof of Theorem 1.2. Let

$$F = f^n(f-1), \quad G = g^n(g-1).$$

By Lemma 2.7,

$$\begin{aligned}\Theta(\infty, F) &= 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}(r,F)}{T(r,F)} = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}(r, f^n(f-1))}{T(r,F)} \\ &\geq 1 - \overline{\lim_{r\to\infty}} \frac{\frac{1}{l}T(r,f)}{(n+1)T(r,f)} \geq 1 - \frac{1}{l(n+1)},\end{aligned}$$

and similarly, $\Theta(\infty, G) \geq 1 - 1/(l(n+1))$. Also,

$$\begin{aligned}\Theta(0, F) &= 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}(r, \frac{1}{F})}{T(r,F)} = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}\big(r, \frac{1}{f^n(f-1)}\big)}{T(r,F)} \\ &\geq 1 - \overline{\lim_{r\to\infty}} \frac{\frac{2}{l}T(r,f)}{(n+1)T(r,f)} \geq 1 - \frac{2}{l(n+1)},\end{aligned}$$

and similarly, $\Theta(0, G) \geq 1 - 2/(l(n+1))$. Again,

$$\delta_{k+1}(0,F) = 1 - \overline{\lim_{r\to\infty}} \frac{N_{k+1}(r, \frac{1}{F})}{T(r,F)} \geq 1 - \overline{\lim_{r\to\infty}} \frac{(k+2)T(r,f)}{l(n+1)T(r,f)} \geq 1 - \frac{k+2}{l(n+1)},$$

and similarly, $\delta_{k+1}(0,G) \geq 1 - (k+2)/(l(n+1))$, $\delta_{k+2}(0,G) \geq 1 - (k+3)/(l(n+1))$.

Therefore,

$$\begin{aligned}\Delta &= (k+3)\Theta(\infty, G) + (2k+3)\Theta(\infty, F) \\ &\quad + \delta_{k+2}(0,G) + \delta_{k+1}(0,G) + 2\delta_{k+1}(0,F) + \Theta(0,F) \\ &\geq (3k+6)\Big(1 - \frac{1}{l(n+1)}\Big) + \Big(1 - \frac{k+3}{l(n+1)}\Big) + \Big(1 - \frac{k+2}{l(n+1)}\Big) \\ &\quad + \Big(2 - 2\frac{k+2}{l(n+1)}\Big) + \Big(1 - \frac{2}{l(n+1)}\Big).\end{aligned}$$

Since $(n+1)l > 7k+17$, we get $\Delta > 3k+10$, and then, by Lemma 2.4, we obtain $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$.

Suppose that $F^{(k)}G^{(k)} \equiv 1$, that is, $(f^n(f-1))^{(k)}(g^n(g-1))^{(k)} \equiv 1$. Then $f(g) \neq 0$, $f(g) \neq \infty$.

Let $f = e^{\alpha}$, where α is a nonconstant entire function. Then by induction we get

$$\begin{aligned}(f^n)^{(k)} &= (e^{n\alpha})^{(k)} = p_1(\alpha', \alpha'', \ldots, \alpha^{(k)})e^{n\alpha}, \\ (f^{n+1})^{(k)} &= (e^{(n+1)\alpha})^{(k)} = p_2(\alpha', \alpha'', \ldots, \alpha^{(k)})e^{(n+1)\alpha},\end{aligned}$$

where $p_1(\alpha', \alpha'', \ldots, \alpha^{(k)})$ and $p_2(\alpha', \alpha'', \ldots, \alpha^{(k)})$ are differential polynomials. Obviously,

$$p_1(\alpha', \alpha'', \ldots, \alpha^{(k)}) \not\equiv 0, \quad p_2(\alpha', \alpha'', \cdots, \alpha^{(k)}) \not\equiv 0.$$

https://doi.org/10.1017/S0004972711003261 Published online by Cambridge University Press

Noting that g is an entire function, we obtain from $(f^n(f-1))^{(k)}(g^n(g-1))^{(k)} \equiv 1$ that $(f^n(f-1))^{(k)} \neq 0$. Thus,

$$p_2(\alpha', \alpha'', \dots, \alpha^{(k)})e^{\alpha(z)} - p_1(\alpha', \alpha'', \dots, \alpha^{(k)}) \neq 0.$$

Since α is an entire function, $T(r, \alpha') = m(r, \alpha') = S(r, f)$. Thus,

$$T(r, \alpha^{(j)}) \leq T(r, \alpha') + S(r, f) = S(r, f) \tag{3.6}$$

for $j = 1, 2, \dots, k$. Hence,

$$T(r, p_1) = S(r, f), \quad T(r, p_2) = S(r, f). \tag{3.7}$$

Thus, by (3.6) and (3.7),

$$\begin{aligned} T(r, f) &\leq T(r, p_2 e^{\alpha}) + S(r, f) \\ &\leq \overline{N}\Big(r, \frac{1}{p_2 e^{\alpha}}\Big) + \overline{N}\Big(r, \frac{1}{p_2 e^{\alpha} - p_1}\Big) + S(r, f) \\ &\leq T\Big(r, \frac{1}{p_2}\Big) + S(r, f) = S(r, f), \end{aligned}$$

which is a contradiction.

Hence $F \equiv G$, that is, $(f^n(f-1))^{(k)} \equiv (g^n(g-1))^{(k)}$, or $f^n(f-1) = g^n(g-1) + p(z)$, where $p(z)$ is a polynomial of degree at most $k-1$. It follows that $T(r, f) = T(r, g) + S(r, f)$.

If $p(z) \not\equiv 0$, by the second fundamental theorem,

$$\begin{aligned} T(r, g^n(g-1)) \leq \overline{N}\Big(r, \frac{1}{g^n(g-1)}\Big) &+ \overline{N}\Big(r, \frac{1}{g^n(g-1) + p(z)}\Big) \\ &+ \overline{N}(r, g^n(g-1)) + S(r, g), \end{aligned}$$

that is,

$$\begin{aligned} (n+1)T(r, g) &\leq \overline{N}\Big(r, \frac{1}{g}\Big) + \overline{N}\Big(r, \frac{1}{g-1}\Big) + \overline{N}\Big(r, \frac{1}{f}\Big) \\ &\qquad + \overline{N}\Big(r, \frac{1}{f-1}\Big) + \overline{N}(r, g) + S(r, g) \\ &\leq \Big(\frac{3}{l} + 2\Big)T(r, g) + S(r, g), \end{aligned}$$

which contradicts the assumption that $l < (7k/2) + 7$ under the condition that $(n+1)l > 7k + 7$.

Hence $p(z) \equiv 0$, that is,

$$f^n(f-1) = g^n(g-1). \tag{3.8}$$

Let $h = f/g$ be a constant. Suppose that $f \not\equiv g$. Then from (3.8) it follows that $h \neq 1$, $h^n \neq 1$, $h^{n+1} \neq 1$ and $g = (1 - h^n)/(1 - h^{n+1})$ is a constant, a contradiction. So we suppose that h is not a constant. Since $f \not\equiv g$, we have $h \not\equiv 1$. From (3.8), $g = (1 - h^n)/(1 - h^{n+1})$ and $f = h(1 - h^n)/(1 - h^{n+1})$. Hence,

$$T(r, f) = nT(r, h) + S(r, f).$$

By the second fundamental theorem of Nevanlinna,

$$\overline{N}(r, f) = \sum_{i=1}^{n} \overline{N}\Big(r, \frac{1}{h - a_i}\Big) \geq (n - 2)T(r, h) + S(r, f)$$

where $a_i \neq 1$ $(i = 1, 2, \ldots, n)$ are distinct roots of the equation $h^{n+1} = 1$. Then

$$\Theta(\infty, f) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}(r, f)}{T(r, f)} \leq \frac{2}{n}.$$

which contradicts the assumption that $\Theta(\infty, f) > 2/n$. Thus $f \equiv g$.

This completes the proof of Theorem 1.2.

3.3. Proof of Theorem 1.3. Since f and g are entire functions, $N(r, f) = N(r, g) = 0$. Proceeding as in the proof of Theorem 1.1 and applying Lemma 2.5, we obtain Theorem 1.3.

3.4. Proof of Theorem 1.4. Since f and g are entire functions, we have $N(r, f) = N(r, g) = 0$. Proceeding as in the proof of Theorem 1.2 and applying Lemma 2.5, we can easily prove Theorem 1.4.

References

[1] R. S. Dyavanal, 'Uniqueness and value-sharing of differential polynomials of meromorphic functions', *J. Math. Anal. Appl.* **374** (2011), 335–345.

[2] M. L. Fang and W. Hong, 'A unicity theorem for entire functions concerning differential polynomials', *Indian J. Pure Appl. Math.* **32**(9) (2001), 1343–1348.

[3] G. Frank, 'Eine Vermutung von Hayman über Nullstellen meromorpher Funktionen', *Math. Z.* **149** (1976), 29–36.

[4] W. K. Hayman, *Meromorphic Functions* (Clarendon Press, Oxford, 1964).

[5] I. Lahiri and A. Sarkar, 'Uniqueness of a meromorphic function and its derivative', *J. Inequal. Pure Appl. Math.* **5**(1) (2004), Article 20.

[6] W. C. Lin and H. X. Yi, 'Uniqueness theorems for meromorphic function', *Indian J. Pure Appl. Math.* **32**(9) (2004), 121–132.

[7] Y. Xu and H. L. Qu, 'Entire functions sharing one value IM', *Indian J. Pure Appl. Math.* **31** (2000), 849–855.

[8] C. C. Yang, 'On deficiencies of differential polynomials II', *Math. Z.* **125** (1972), 107–112.

[9] L. Yang, *Value Distribution Theory* (Springer/Science Press, Berlin/Beijing, 1993).

[10] C. C. Yang and X. Hua, 'Uniqueness and value sharing of meromorphic functions', *Ann. Acad. Sci. Fenn. Math.* **22** (1997), 395–406.

https://doi.org/10.1017/S0004972711003261 Published online by Cambridge University Press

[11] C. C. Yang and H. X. Yi, *Uniqueness Theory of Meromorphic Functions*, Mathematics and its Applications, 557 (Kluwer Academic Publishers, Dordrecht, 2003).

[12] H. X. Yi and C. C. Yang, *Uniqueness Theory of Meromorphic Functions* (Science Press, Beijing, 1995).

CHUN WU, College of Mathematics and Statistics, Chongqing University,
Chongqing 401331, PR China
and
College of Mathematics Science, Chongqing Normal University,
Chongqing 401331, PR China
e-mail: wuchun1997@163.com

CHUNLAI MU, College of Mathematics and Statistics, Chongqing University,
Chongqing 401331, PR China

JIANGTAO LI, College of Mathematics and Statistics, Chongqing University,
Chongqing 401331, PR China