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ON THE POWERS OF SOME TRANSCENDENTAL NUMBERS

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We construct a transcendental number α whose powers $\alpha^{n!}$, $n = 1, 2, 3, \ldots$, modulo 1 are everywhere dense in the interval $[0, 1]$. Similarly, for any sequence of positive numbers $\boldsymbol{\delta} = (\delta_n)_{n=1}^{\infty}$, we find a transcendental number $\alpha = \alpha(\boldsymbol{\delta})$ such that the inequality $\{\alpha^n\} < \delta_n$ holds for infinitely many $n \in \mathbb{N}$, no matter how fast the sequence $\boldsymbol{\delta}$ converges to zero. Finally, for any sequence of real numbers $(r_n)_{n=1}^{\infty}$ and any sequence of positive numbers $(\delta_n)_{n=1}^{\infty}$, we construct an increasing sequence of positive integers $(q_n)_{n=1}^{\infty}$ and a number $\alpha > 1$ such that $\|\alpha^{q_n} - r_n\| < \delta_n$ for each $n \geqslant 1$.

1. INTRODUCTION

Throughout this paper, we shall denote by $\{x\}$, $[x]$ and $\|x\|$ the fractional part of a real number x, the integral part of x, and the distance from x to the nearest integer, respectively. Clearly, $x = [x] + \{x\}$ and $\|x\| = \min\big(\{x\}, 1 - \{x\}\big)$. By $\mathbb{N}$ and $\mathbb{Q}$ we denote the set of positive integers and the set of rational numbers, respectively.

Let $\alpha > 1$ be a real number. Koksma [7] proved that for almost all $\alpha > 1$ the fractional parts $\{\alpha^n\}_{n=1}^{\infty}$ are uniformly distributed in the interval $[0, 1]$. However, for most specific α, the distribution of the sequence $\{\alpha^n\}_{n=1}^{\infty}$ is an open question. The "exceptional" α in this respect (in the sense that for them the distribution of the sequence $\{\alpha^n\}_{n=1}^{\infty}$ in $[0, 1]$ is quite well-known) are Pisot and Salem numbers. See, for instance, Salem's book [**14**] and some recent papers on this kind of problems [**3, 5, 6, 9, 17**]. In general, the problem of the distribution of the fractional parts $\{\alpha^n\}_{n=1}^{\infty}$ goes back to Weyl [**16**]. Later, some unsolved problems about the distribution of the powers of the number $\alpha = 3/2$ were raised by Vijayaraghavan [**15**] and Mahler [**11**]. The current status of these problems is described in a recent review of Adhikari and Rath [**1**].

Since we shall be concerned with Pisot numbers later on, let us recall that a real algebraic integer $\alpha > 1$ is called a *Pisot number* if its conjugates over $\mathbb{Q}$, except for α itself, all lie in the open unit disc $|z| < 1$. For each Pisot number α, we have $\|\alpha^n\| \to 0$ as $n \to \infty$ (see also [**5, 6, 9**] for some related problems). In contrast, for a Salem number α, by a result of Pisot and Salem [**13**], the sequence $\{\alpha^n\}_{n=1}^{\infty}$ is everywhere dense in $[0, 1]$,

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but not uniformly distributed in $[0,1]$. Hence, for every α which is an mth root of a Salem number with some $m \in \mathbb{N}$, the sequence $\{\alpha^n\}_{n=1}^{\infty}$ is also everywhere dense in $[0,1]$.

However, if $\alpha > 1$ is an algebraic number which is neither a Pisot number nor a root of a Salem number, then the distribution of the sequence $\{\alpha^n\}_{n=1}^{\infty}$ is not known. Moreover, if α is a transcendental number, say, $\alpha = e, \pi, \log 3$ or similar, then it is not even known whether the sequence $\{\alpha^n\}_{n=1}^{\infty}$ has just one or more than one limit point. One of the results of Pisot [**12**] implies, for example, that there are arbitrarily large numbers α for which $\{\alpha^n\} \in [1/2 - 1/\alpha, 1/2 + 1/\alpha]$ for every $n \geqslant n_0$. It is clear that such an α cannot be a Pisot number or a Salem number. So, generally speaking, the sequence $\{\alpha^n\}_{n=1}^{\infty}$ need not even be dense in $[0,1]$ for some α that are not Pisot numbers. It is quite tempting to conjecture that if $\alpha > 1$ is an algebraic number, but not a Pisot number, then the sequence $\{\alpha^n\}_{n=1}^{\infty}$ is dense in $[0,1]$. However, such a problem is far beyond reach even for $\alpha = \sqrt{2}$. Curiously, but except for an unpublished manuscript of Lerma [**8**] who gives a (quite complicated) construction of some $\alpha > 1$ whose powers are uniformly distributed in $[0,1]$ it seems like that there is no method known which would allow the explicit construction of a transcendental number α whose powers modulo 1 are everywhere dense in $[0,1]$, although, by the above mentioned result of Koksma, almost all transcendental numbers have this property. We thus begin with the following construction of α by a recurrent sequence similar to [**2**]. For such α, the sequence $\{\alpha^n\}_{n=1}^{\infty}$ is everywhere dense, because its subsequence $\{\alpha^{n!}\}_{n=1}^{\infty}$ is everywhere dense.

THEOREM 1. *Let $(r_n)_{n=1}^{\infty}$ be a sequence of real numbers in $[0,1)$ which is everywhere dense in $[0,1]$ such that $r_n = 0$ for infinitely many indices n. Suppose that $x_1 := 1$ and $x_n := 1 + \left[(x_{n-1} + r_{n-1})^n - r_n\right]$ for $n \geqslant 2$. Then the limit $\alpha := \lim_{n\to\infty}(x_n + r_n)^{1/n!} > 1$ exists, it is a transcendental number, and the sequence $\{\alpha^{n!}\}_{n=1}^{\infty}$ is everywhere dense in $[0,1]$.*

We can take, for instance, r_n to be the nth term of the sequence of blocks of Farey fractions that are separated by one zero

$$1/2, 0, 1/3, 2/3, 0, 1/4, 3/4, 0, 1/5, 2/5, 3/5, 4/5, 0, 1/6, 5/6, 0, \ldots.$$

The problem of the distribution of the sequence $\{\alpha^n\}_{n=1}^{\infty}$ in $[0,1]$ is related to a purely diophantine problem of how close the elements of this sequence are to 0 and 1. Recently, Corvaja and Zannier [**4**] generalised an old result of Mahler [**10**] and proved that if $\alpha > 1$ is an algebraic number such that, for some positive $\delta < 1$, the inequality $\|\alpha^n\| < (1-\delta)^n$ has infinitely many solutions in positive integers n then α^m is a Pisot number for some $m \in \mathbb{N}$. Earlier, Mahler proved this result for rational numbers α using a version of Roth's theorem. In principle, using some properties of Pisot numbers, one can derive our next theorem from [**4**]. However, since the condition on δ_n is much stronger than the one considered in [**4**], we shall give a simple direct proof without using the results of [**4**].

Theorem 2. *Let α be a real number and let $(\delta_n)_{n=1}^{\infty}$ be a sequence of positive numbers satisfying $\lim_{n\to\infty} \delta_n^{1/n} = 0$. If the inequality $\|\alpha^n\| < \delta_n$ has infinitely many solutions in $n \in \mathbb{N}$ then either α is a transcendental number or α^m is an integer for some $m \in \mathbb{N}$.*

In addition, it is shown in [4] that there exists a transcendental number $\alpha > 1$ such that $\|\alpha^n\| < 2^{-n}$ for infinitely many $n \in \mathbb{N}$. In this direction, for any sequence $\boldsymbol{\delta} = (\delta_n)_{n=1}^{\infty}$ of positive numbers, we construct a transcendental number $\alpha = \alpha(\boldsymbol{\delta})$ such that the inequality $\|\alpha^n\| < \delta_n$ holds for infinitely many $n \in \mathbb{N}$, no matter how fast the the sequence $\boldsymbol{\delta}$ converges to 0.

Theorem 3. *Let $\boldsymbol{\delta} = (\delta_n)_{n=1}^{\infty}$ be a sequence of positive numbers. Set $x_1 := 1$ and $x_n := x_{n-1}^{u_n} + 1$ for $n \geqslant 2$, where $u_1 = 1, u_2, u_3, \dots$ are some positive integers depending on $\boldsymbol{\delta}$ (see the proof how). Then the limit $\alpha := \lim_{n\to\infty} x_n^{1/(u_1 u_2 \dots u_n)} > 1$ exists, it is a transcendental number, and the inequality $\{\alpha^n\} < \delta_n$ holds for infinitely many $n \in \mathbb{N}$.*

In fact, not only zero but also any given sequence can be "copied" by some powers of α modulo 1 with any prescribed accuracy. In our final theorem, we do not bother about the arithmetical nature of the limit α. (One can easily ensure that the number α in Theorem 4 below is transcendental, for example, by adding infinitely many "extra terms" $r_n = 0$ and by increasing the "gaps" between consecutive q_n's if necessary.) Also, we replace $1+[x]$ by the ceiling function $\lceil x \rceil$ and construct the approximants to α directly rather than via integer parts of their powers as in Theorems 1 and 3. More precisely, we show that, for any sequence of real numbers $(r_n)_{n=1}^{\infty}$, there is a number $\alpha > 1$ whose powers α^{q_n}, where q_n are some positive integers, tend to the numbers r_n (with respect to the metric $\|\cdot\|$) with any prescribed rate.

Theorem 4. *Let $\boldsymbol{\delta} = (\delta_n)_{n=1}^{\infty}$ be a sequence of positive numbers, and let $(r_n)_{n=1}^{\infty}$ be a sequence of real numbers. Suppose that $y_0 \geqslant 2$ and $y_n := (\lceil y_{n-1}^{q_n} \rceil + r_n)^{1/q_n}$ for $n \geqslant 1$, where $q_1 < q_2 < q_3 < \dots$ are any positive integers satisfying $q_{n+1} \geqslant q_n + \log_2(1/\delta_n) + 3$ for $n \geqslant 1$. Then the limit $\alpha := \lim_{n\to\infty} y_n \geqslant 2$ exists, and, for this α, the inequality $\|\alpha^{q_n} - r_n\| < \delta_n$ holds for each $n \in \mathbb{N}$.*

In particular, Theorem 4 implies that, for any sequence of real numbers $(r_n)_{n=1}^{\infty}$ and any sequence of positive integers $q_1 < q_2 < q_3 < \dots$ satisfying $\lim_{n\to\infty}(q_{n+1} - q_n) = \infty$, there is an $\alpha > 2$ such that $\lim_{n\to\infty} \|\alpha^{q_n} - r_n\| = 0$. Also, setting $\delta_n = \varepsilon$ for $n \in \mathbb{N}$, taking $q_n = mn$ for $n \in \mathbb{N}$ with some fixed $m \geqslant \log_2(1/\varepsilon) + 3$, and writing α for α^m, we deduce the following corollary:

Corollary 5. *Let $(r_n)_{n=1}^{\infty}$ be a sequence of real numbers. Then, for any $\varepsilon > 0$, there is an $\alpha > 1$ such that $\|\alpha^n - r_n\| < \varepsilon$ for each $n \in \mathbb{N}$.*

The construction itself and all of the proofs in this paper are similar to those in [2]. In the next section, we first give a self-contained proof of Theorem 2 and then derive from it an auxiliary lemma. The proofs of Theorems 1 and 3 given in Section 3 are based on the lemma. In Section 4 we shall prove Theorem 4.

2. On the approximation of the powers of a number

Proof of Theorem 2: If $|\alpha| < 1$ then $\|\alpha^n\| = |\alpha|^n$ for each $n \geqslant n_1(\alpha)$, so $|\alpha| = \|\alpha^n\|^{1/n} < \delta_n^{1/n}$ has infinitely many solutions in $n \in \mathbb{N}$ only if $\alpha = 0$. For $\alpha = \pm 1$, the claim is also trivial. So, without loss of generality, we can assume that $|\alpha| > 1$.

Let I be the infinite set of indices n for which $\|\alpha^n\| < \delta_n$. Suppose that α is an algebraic number, say, of degree d with conjugates $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_d$ over $\mathbb{Q}$. Let also $a_d \in \mathbb{N}$ be the leading coefficient of the minimal polynomial of α over $\mathbb{Q}$. Put $x_n := [\alpha^n + 1/2]$. Consider the product $P_n := a_d^n \prod_{j=1}^{d} (\alpha_j^n - x_n)$. It is a rational integer.

If $P_n = 0$, then $\alpha_j^n = x_n$ for some index j. By considering any automorphism of the normal extension $\mathbb{Q}(\alpha_1, \dots, \alpha_d)/\mathbb{Q}$ which maps $\alpha_j \mapsto \alpha$ and using the fact that x_n is an integer, we obtain that $\alpha^n = x_n$. This implies that α^m is an integer for some $m \in \mathbb{N}$. If $P_n \neq 0$, then $|P_n| \geqslant 1$. For each $n \in I$, we have $|\alpha^n - x_n| < \delta_n$. Hence

$$a_d^n \delta_n \prod_{j=2}^{d} |\alpha_j^n - x_n| > |P_n| \geqslant 1.$$

Putting $c := \max_{1 \leqslant j \leqslant d} |\alpha_j|$ and using $|x_n| \leqslant |\alpha|^n + 1/2 < c^n + 1$, we obtain that

$$1 < a_d^n \delta_n \big(|x_n| + c^n\big)^{d-1} \leqslant a_d^n \delta_n (2c^n + 1)^{d-1} \leqslant \delta_n b^n,$$

where b is a positive constant depending on α only (and not on n). Hence $1/b < \delta_n^{1/n}$ for every $n \in I$. This is a contradiction with $\lim_{n\to\infty} \delta_n^{1/n} = 0$, which implies that α is a transcendental number. ∎

Lemma 6. *Let $(r_n)_{n=1}^{\infty}$ be an arbitrary sequence of real numbers in $[0, 1)$ satisfying $r_n = 0$ for infinitely many indices n. Suppose that $x_1 := 1$ and*

$$x_n := 1 + \big[(x_{n-1} + r_{n-1})^{nv_n} - r_n\big]$$

for $n \geqslant 2$, where $v_1 = 1, v_2, v_3, \dots$ are positive integers. Then

$$\alpha := \lim_{n\to\infty} (x_n + r_n)^{1/(n! v_1 v_2 \dots v_n)}$$

is a transcendental number greater than 1 and

$$x_n + r_n < \alpha^{n! v_1 \dots v_n} < x_n + r_n + (x_n + r_n)^{-n v_{n+1}}$$

for each $n \geqslant 2$.

Proof: Observe that the sequence $(x_n + r_n)^{1/(n! v_1 \dots v_n)}$ is increasing. Indeed, by the definition of x_n,

$$x_n + r_n = 1 + \big[(x_{n-1} + r_{n-1})^{nv_n} - r_n\big] + r_n > (x_{n-1} + r_{n-1})^{nv_n}.$$

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Next, we shall show that the sequence $(x_n + r_n + (x_n + r_n)^{-nv_{n+1}})^{1/(n!v_1\dots v_n)}$ is decreasing. To prove this, we need to show that

$$x_n + r_n + (x_n + r_n)^{-nv_{n+1}} < \left(x_{n-1} + r_{n-1} + (x_{n-1} + r_{n-1})^{-(n-1)v_n}\right)^{nv_n}.$$

Indeed, using $x_n + r_n \leqslant 1 + (x_{n-1} + r_{n-1})^{nv_n}$ and $v_n \geqslant 1$, we deduce that, for each $n \geqslant 3$,

$$\begin{aligned}(x_{n-1} + r_{n-1} &+ (x_{n-1} + r_{n-1})^{-(n-1)v_n})^{nv_n} \\ &\geqslant (x_{n-1} + r_{n-1})^{nv_n} + nv_n(x_{n-1} + r_{n-1})^{nv_n - 1-(n-1)v_n} \\ &\geqslant (x_{n-1} + r_{n-1})^{nv_n} + nv_n \geqslant (x_{n-1} + r_{n-1})^{nv_n} + 3 \\ &\geqslant x_n + r_n + 2 > x_n + r_n + (x_n + r_n)^{-nv_{n+1}}.\end{aligned}$$

It follows that the sequences $x_n^{1/(n!v_1\dots v_n)}$, $n = 1, 2, \dots$, (which is increasing) and $(x_n + r_n + (x_n + r_n)^{-nv_{n+1}})^{1/(n!v_1\dots v_n)}$, $n = 2, 3, \dots$, (which is decreasing) tend to certain limits, say, α and γ, respectively, as n tends to infinity. Obviously, $\alpha \leqslant \gamma$, so

$$x_n + r_n < \alpha^{n!v_1\dots v_n} \leqslant \gamma^{n!v_1\dots v_n} < x_n + r_n + (x_n + r_n)^{-nv_{n+1}}$$

for each $n \geqslant 2$. Note that, since the right hand side is at most $x_n + r_n + 1$, we have $\alpha = \gamma$ (although we shall not need it). It is clear that $\alpha > 1$.

Next, we shall prove that the number α is transcendental. Let I be the infinite set of indices n for which $r_n = 0$. Denote $V_n := n!v_1 \dots v_n$. We have $x_n < \alpha^{V_n} < x_n + x_n^{-nv_{n+1}} \leqslant x_n + x_n^{-n} \leqslant x_n + 1$. Fix $\beta \in (1, \alpha)$. Then $\alpha^{V_n} - 1 > \beta^{V_n}$ for each sufficiently large n. Hence $\|\alpha^{V_n}\| < x_n^{-n} < (\alpha^{V_n} - 1)^{-n} < \beta^{-nV_n}$ for each sufficiently large $n \in I$. By Theorem 2, either α is a transcendental number or $\alpha^m \in \mathbb{N}$ for some $m \in \mathbb{N}$. However, if α^m is an integer, then α^{V_n} must be an integer too for every $n \geqslant m$, because $V_n = n!v_1 \dots v_n$ is divisible by m. This is, however, not the case, because $\alpha^{V_n} \in (x_n, x_n + 1)$ for $n \geqslant 2$. Consequently, α is a transcendental number. □

3. Proofs of Theorems 1 and 3

Proof of Theorem 1: Let us apply the lemma for $v_1 = v_2 = v_3 = \dots = 1$. The lemma implies that $\alpha := \lim_{n\to\infty} x_n^{1/n!}$ is a transcendental number greater than 1 and $x_n + r_n < \alpha^{n!} < x_n + r_n + x_n^{-n}$ for $n \geqslant 2$.

Fix $y \in (0, 1)$. In order to prove that y is a limit point of the sequence $\{\alpha^{n!}\}_{n=1}^{\infty}$ it is sufficient to show that, for any positive number ε satisfying $\varepsilon < 1 - y$, there is an $n \in \mathbb{N}$ such that $\{\alpha^{n!}\} \in (y, y + \varepsilon)$. Indeed, the interval $(y, y + \varepsilon/2)$ contains infinitely many r_n's. Let I be the set of corresponding n's. We claim that $\{\alpha^{n!}\} \in (y, y + \varepsilon)$ for all sufficiently large $n \in I$. For this, it is sufficient to show that

$$x_n + y < \alpha^{n!} < x_n + y + \varepsilon.$$

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Indeed, adding two inequalities $y < r_n$ and $x_n + r_n < \alpha^{n!}$, we immediately get the first inequality $x_n + y < \alpha^{n!}$. The second inequality, namely, $\alpha^{n!} < x_n + y + \varepsilon$ would follow from the inequalities $r_n < y + \varepsilon/2$ (which holds by the definition of I) and $\alpha^{n!} < x_n + r_n + \varepsilon/2$. From $\alpha^{n!} < x_n + r_n + x_n^{-n}$, we see that the required inequality holds if $x_n^n > 2/\varepsilon$. This is indeed the case, because $x_n > \alpha^{n!} - r_n - 1$, so $x_n \to \infty$ as $n \to \infty$. Finally, since the sequence $\{\alpha^{n!}\}_{n=1}^{\infty}$ is everywhere dense in $(0, 1)$, it is everywhere dense in $[0, 1]$. □

Proof of Theorem 3: This time, we shall apply the lemma with $r_1 = r_2 = r_3 = \dots = 0$ and with $u_n = nv_n$. Here, v_n, $n = 1, 2, \dots$, are some positive integers to be chosen later. Then the lemma implies that $\alpha := \lim_{n\to\infty} \alpha^{1/(n!v_1\dots v_n)}$ is a transcendental number and

$$x_n < \alpha^{n!v_1\dots v_n} < x_n + x_n^{-nv_{n+1}}.$$

Fix any $\beta \in (1, \alpha)$. For each n large enough, say $n \geqslant n_1$, we have $x_n > \alpha^{n!v_1\dots v_n} - 1 > \beta^{n!v_1\dots v_n}$. Hence $\log x_n > n!v_1 \dots v_n \log \beta$. The inequality $\{\alpha^N\} < \delta_N$ holds for every number $N = n!v_1 \dots v_n$ provided that $x_n^{-nv_{n+1}} < \delta_N$, that is, $nv_{n+1} \log x_n > \log(1/\delta_N)$. So we can simply put $v_1 = \dots = v_{n_1} = 1$ and, for each $n \geqslant n_1$, take any positive integer v_{n+1} greater than $\log(1/\delta_{n!v_1\dots v_n})/(n!v_1 \dots v_n n \log \beta)$, which is always possible. □

In particular, let us consider the sequence $x_1 := 1$ and $x_{n+1} := x_n^2 + 1$ for each $n \geqslant 1$. As above, the sequence $x_n^{1/2^n}$, $n = 1, 2, \dots$, is increasing, whereas the sequence $\big(x_n + 1/(2x_n)\big)^{1/2^n}$, $n = 1, 2, \dots$, is decreasing. They both thus tend to the same limit ξ. Since the inequality

$$\{\xi^{2^n}\} < 1/(2x_n) < 1/\big(2(\xi^{2^n} - 1)\big) < (1/\xi)^{2^n}$$

holds for all sufficiently large n, the theorem of Corvaja and Zannier [4] implies that either the number ξ is transcendental or there is an $m \in \mathbb{N}$ such that ξ^m is a Pisot number. The second possibility seems very unlikely. We thus conclude this section with the following transcendence type problem: *prove that the number ξ is transcendental.*

4. Proof of Theorem 4

Without loss of generality we may assume that $r_n \in [0, 1)$ for each $n \geqslant 1$. Also, we can assume that $\delta_n \leqslant 1/2$, so $q_{n+1} - q_n \geqslant 4$. Since

$$y_n = (\lceil y_{n-1}^{q_n} \rceil + r_n)^{1/q_n} \geqslant (y_{n-1}^{q_n} + r_n)^{1/q_n} \geqslant y_{n-1},$$

the sequence $(y_n)_{n=1}^{\infty}$ is non-decreasing. Also, $y_n^{q_n} - r_n$ is an integer, so that $\{y_n^{q_n}\} = r_n$ for every $n \in \mathbb{N}$.

From $\lceil y_{n-1}^{q_n} \rceil < y_{n-1}^{q_n} + 1$ and $r_n < 1$, we have

$$y_n/y_{n-1} < (1 + 2y_{n-1}^{-q_n})^{1/q_n} < 1 + 2/(q_n y_{n-1}^{q_n}).$$

Hence $y_n - y_{n-1} < 2/(q_n y_{n-1}^{q_n - 1})$. Adding n such inequalities (for $y_n - y_{n-1}$, for $y_{n-1} - y_{n-2}$, ..., for $y_1 - y_0$) and using $y_j \geqslant y_0$ for $j = 1, 2, \dots, n-1$, we obtain that $y_n - y_0$ is bounded from above by $2/\big(q_1 y_0^{q_1 - 2}(y_0 - 1)\big)$, so the limit $\alpha := \lim_{n\to\infty} y_n$ exists. Obviously, it is greater than or equal to $y_0 \geqslant 2$.

Next, we shall estimate the quotient $(y_{k+1}/y_k)^{q_n}$ for $k \geqslant n$. Since $q_n/q_{k+1} < 1$ and $y_k \geqslant 2$, we have

$$(y_{k+1}/y_k)^{q_n} < (1 + 2y_k^{-q_{k+1}})^{q_n/q_{k+1}} < 1 + 2q_n/(q_{k+1} y_k^{q_{k+1}}) < 1 + 2/y_k^{q_{k+1}} \leqslant 1 + y_k^{-q_{k+1}+1}.$$

It follows that, for every fixed $n \in \mathbb{N}$,

$$(\alpha/y_n)^{q_n} = \prod_{k=n}^{\infty} (y_{k+1}/y_k)^{q_n} < \prod_{k=n}^{\infty} (1 + y_k^{-q_{k+1}+1}).$$

In order to estimate the product $\prod_{k=n}^{\infty} (1 + \tau_k)$, where $\tau_k := y_k^{-q_{k+1}+1}$, we shall first bound it as $\exp\left(\sum_{k=n}^{\infty} \tau_k\right)$ and then use the inequality $\exp(\tau) < 1 + 2\tau$, because the sum $\tau = \sum_{k=n}^{\infty} \tau_k$ turns out to be bounded by 1. Indeed, using the inequality $y_k \geqslant y_n \geqslant 2$, we obtain that

$$\tau = \sum_{k=n}^{\infty} y_k^{-q_{k+1}+1} \leqslant \frac{1}{y_n^{q_{n+1}-2}(y_n - 1)} \leqslant y_n^{-q_{n+1}+2}$$

(which is at most 1), hence $(\alpha/y_n)^{q_n} < 1 + 2/y_n^{q_{n+1}-2} \leqslant 1 + 1/y_n^{q_{n+1}-3}$. Therefore $0 \leqslant \alpha^{q_n} - y_n^{q_n} < 1/y_n^{q_{n+1}-q_n-3} \leqslant 1/2^{q_{n+1}-q_n-3}$. Using $\{y_n^{q_n}\} = r_n$, we conclude that $\|\alpha^{q_n} - r_n\| < 2^{-q_{n+1}+q_n+3}$ for each $n \in \mathbb{N}$. The right hand side of this inequality does not exceed δ_n provided that $q_{n+1} \geqslant q_n + \log_2(1/\delta_n) + 3$. This completes the proof of Theorem 4.

If the sequence $(q_n)_{n=1}^{\infty}$ is not growing very fast, then the arithmetical nature of the limit obtained by this kind of iterations seems to be quite mysterious even in the simplest case $r_1 = r_2 = r_3 = \dots = 0$ and $q_n = n$. For instance, let us start with $y_1 \in (1, \sqrt{2}]$, and consider the sequence $(y_n)_{n=1}^{\infty}$ obtained by the following iterations

$$y_n := \lceil y_{n-1}^n \rceil^{1/n}$$

for $n \geqslant 2$. Then $y_2 = 2^{1/2}, y_3 = 3^{1/3}, y_4 = 5^{1/4}, y_5 = 8^{1/5}, y_6 = 13^{1/6}, \dots$. By the same argument as above, the limit $\zeta := \lim_{n\to\infty} y_n$ exists: *prove that ζ is a transcendental number.*

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