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Bernoulli diffeomorphisms with $n-1$ non-zero exponents

M. BRIN†
From the Department of Mathematics, University of Maryland, USA, and the Hebrew University of Jerusalem, Israel

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To the memory of V. M. Alekseyev

Abstract. For every manifold of dimension $n \geq 5$ a diffeomorphism f which has $n-1$ non-zero characteristic exponents almost everywhere is constructed. The diffeomorphism preserves the Lebesgue measure and is Bernoulli with respect to this measure. To produce this example a diffeomorphism of the 2-disk is extended by means of an Anosov flow, and this skew product is embedded in $\mathbb{R}^n$.

1. *Introduction*

(1.1) A diffeomorphism f of a compact Riemannian manifold M is called a Bernoulli diffeomorphism if f preserves a positive smooth probability measure μ and if the automorphism (M, μ, f) is conjugate by a measure preserving invertible mapping to a Bernoulli shift [**4**].

The main mechanism of strong ergodic properties (for example, the K-property, the Bernoulli property) in smooth dynamical systems, which is known up to now, is hyperbolicity. The relation between ergodic and hyperbolic properties of diffeomorphisms and smooth flows was studied by Pesin [**8, 9, 10**]. Hyperbolic properties of smooth dynamical systems can be expressed in terms of the so-called characteristic exponents [**8**]. Pesin described the Pinsker partition in terms of the stable and unstable foliation [**9**] and also showed that, if the characteristic exponents are non-zero almost everywhere, then almost every ergodic component has positive measure and, under some extra conditions, the system is Bernoullian [**10**]. Two natural questions emerge: (1) Does every compact manifold carry a Bernoulli diffeomorphism? (2) Does every compact manifold carry a diffeomorphism with non-zero exponents? (Exponents being non-zero means at almost every point all exponents are non-zero.) The first question was answered positively in [**4**] and [**2**]. The second question is still open, though the answer seems to be 'yes'. The problem is that Pesin's conditions for a diffeomorphism to be Bernoullian are essentially sufficient and not necessary. The diffeomorphism constructed in [**2**] has only 2 non-zero exponents and $n-2$ zero exponents (n is the dimension); and still it is Bernoullian. Pesin's entropy formula shows that, for any diffeomorphism preserving

† Address for correspondence: Dr M. Brin, Department of Mathematics, University of Maryland, College Park, Md 20742, USA.

a smooth measure with positive entropy, there is a set of positive measure where at least two of the exponents are non-zero. Thus, the example constructed in [**2**] is optimal in the sense that the diffeomorphism has the least possible number of non-zero exponents which is necessary to provide the Bernoulli property.

(1.2) In this paper we construct a *Bernoulli diffeomorphism* with $n-1$ non-zero exponents on any n-dimensional compact manifold, $n \geq 5$. In dimension 2 there is an example of a Bernoulli diffeomorphism with non-zero exponents due to Katok and Grines [**4**]. The main element of their proof is the construction of a Bernoulli diffeomorphism with non-zero exponents on the 2-disk. By suspending this diffeomorphism, one can obtain a Bernoulli flow with non-zero exponents on any 3-manifold, each diffeomorphism of this flow being Bernoullian and having 2 non-zero exponents. Another way to obtain such an example in dimension 3 can be found in [**2**]. For topological reasons the construction given below does not work in dimension 4, although the statement obviously seems to be true.

(1.3). The construction consists of several steps. In the first step we consider the Bernoulli diffeomorphism g of the 2-disk D^2 described in [**4**], an Anosov flow h^t on an $(n-2)$-dimensional manifold N and their skew product $f(x, y) = (gx, h^{\alpha(x)}y)$, $\alpha(x)$ being a skewing function which is equal to 0 in a neighbourhood of the boundary ∂D^2. We choose α in such a way that f is a K- and Bernoulli automorphism (actually it is true for almost every α). In the second step the phase space $M^n = D^2 \times N^{n-2}$ is embedded in $\mathbb{R}^n$ and a continuous mapping ϕ from M^n onto the n-dimensional ball B^n is constructed which has the following properties: (i) ϕ sends the Lebesgue measure on M^n into the Lebesgue measure on B^n; (ii) ϕ is a diffeomorphic C^∞-embedding on int $D^2 \times N^{n-2}$; (iii) $\phi(\partial M^n)$ is a finite union of submanifolds in B^n of positive codimension. Thus $f_B = \phi \circ f \circ \phi^{-1}$ is a Bernoulli diffeomorphism of B^n with $n-1$ non-zero exponents. A similar argument shows that f_B can be transferred from B^n to any compact n-dimensional manifold.

2. *Construction of a skew product over D^2 with $n-1$ non-zero exponents*

Let $h^t: N^{n-2} \to N^{n-2}$ be a C^∞ Anosov flow which preserves the Lebesgue measure ν and is a K- and Bernoulli flow (the spectrum contains no discrete component and the measurable hull of each foliation is trivial [**1**]). Let $g: D^2 \to D^2$ be the Bernoulli diffeomorphism constructed in [**4**]. This C^∞-diffeomorphism preserves the Lebesgue measure λ on D^2, has non-zero exponents and stops at the boundary ∂D^2 with all derivatives. Any smooth function $g: D^2 \to \mathbb{R}$ determines a diffeomorphism f of the direct product $M = D^2 \times N$ given by the formula

$$f(x, y) = (g(x), h^{\alpha(x)}y). \tag{2.1}$$

(2.1) PROPOSITION. *Let α be a smooth non-negative function $\alpha: D^2 \to \mathbb{R}$, $\alpha \not\equiv 0$. Then the diffeomorphism $f: M \to M$ given by* (2.1) *has $n-1$ non-zero exponents* (a.e. *with respect to the invariant product measure $\mu = \lambda \times \nu$*).

Proof. Our construction is similar to the one from [**5**]. Let W_g^s, W_g^u, W_h^s and W_h^u be the stable and unstable foliations of g and h and denote by $\pi: M \to D^2$ the natural projection. Let $x \in D^2$ be a point where $W_g^s(x)$ and $W_g^u(x)$ exist. For every point $(x, y) \in D^2 \times N$ consider the following two sets

$$W_f^s(x, y) = \bigcup_{x_s \in W_g^s(x)} (x_s, W_h^s(h^{t_s(x_s)} y)), \tag{2.2}$$

$$W_f^u(x, y) = \bigcup_{x_u \in W_g^u(x)} (x_u, W_h^u(h^{t_u(x_u)} y)), \tag{2.3}$$

where

$$t_s(x_s) = \sum_{n=0}^{\infty} (\alpha(g^n(x_s)) - \alpha(g^n(x))), \tag{2.4}$$

$$t_u(x_u) = \sum_{n=1}^{\infty} (\alpha(g^{-n}(x_u)) - \alpha(g^{-n}(x))). \tag{2.5}$$

It is easy to see that, if $W_g^s(x)$ is exponentially contracted by g and if $L^s(x) = \lim_{n\to\infty} (1/n) \sum_{i=1}^{n} \alpha(g^n x) > 0$, then the submanifold $W_f^s(x, y)$ is exponentially contracted by f. The same is true for $W_f^u(x, y)$ if we reverse the time. Since g is ergodic, the limit $L^s(x)$ (and $L^u(x)$) exists λ-almost everywhere. Thus, for μ almost every point there exist the stable and unstable manifolds. The sum of their dimensions is equal to $n-1$. Therefore, f has $n-1$ non-zero exponents; the zero exponent corresponds to the direction of the flow h^t (it acts globally on M). The proposition is proved. □

(2.2) PROPOSITION. *If $\alpha(x) \geq 0$ and $\alpha(x) \not\equiv 0$, then f is a K-automorphism of the Lebesgue space (M, μ).*

Proof. Pesin showed in [**9**] that the Pinsker partition of a diffeomorphism preserving a smooth measure is equal to both the measurable hull of the stable and unstable foliations. Suppose A is a set consisting of entire leaves $W_f^s(x, y)$. Since every leaf $W_f^s(x, y)$ intersects every vertical fibre $\pi^{-1}(x_s)$ $(x_s \in W_g^s(x))$ by exactly one stable leaf of h^t, then A intersects with every fibre $\pi^{-1}(x)$ by a set consisting of entire stable leaves of h^t (this set may be empty). If A has positive measure, then for x belonging to a set of positive measure, the intersection $\pi^{-1}(x) \cap A$ has positive measure and, therefore, must have full measure, since h^t is not a suspension and the foliation W_h^s is ergodic. It follows that A has full measure, because the stable foliation W_g^s is also ergodic (g is a K-automorphism of (D^2, λ)). The proposition is proved. □

(2.3) PROPOSITION. *If α is such that f is a K-automorphism, then f is Bernoullian.*

The proof repeats Pesin's argument (see [**10**] and [**11**]) for flows with non-zero exponents. The diffeomorphism we consider has three invariant foliations: the stable foliation W_f^s, the unstable foliation W_f^u, and the neutral foliation W_f^0 formed by the trajectories of h in each vertical fibre. Pesin deals with a flow F^t which also has three invariant foliations W_F^s, W_F^u and the neutral foliation W_F^0 formed by the trajectories of F. To prove the existence of a very weak Bernoulli generator he considers the so-called parallelepipeds, i.e. small measurable sets consisting of pieces of weak stable manifolds W_F^{0s} and of unstable manifolds W_F^u. To carry on the proof one needs the following two properties: (a) the distance between the images of any two points belonging to the same local weak stable manifold does not increase, and (b) the jacobian of the succession mapping between any two weak stable manifolds inside a parallelepiped, which is carried out by means of unstable manifolds, tends to 1 when the size of the parallelepiped tends to 0 (see [**11**], § 3). Property (a) is valid in our case because the foliations W_f^s and W_f^0 are certainly integrable by construction, and if x and y belong to the same neutral leaf, the distance between their images does not change. Property (b) is also valid because the corresponding succession mapping is an isometry in the W_f^0-direction. Thus, Pesin's proof goes through.

(2.4) *Remark.* Actually this argument works in a more general case. Let f be a diffeomorphism which preserves a smooth measure and has three invariant foliations, two of them being hyperbolic (may be non-uniformly) and the third one being isometric in the sense that the distances on each leaf are preserved by f. Then, provided f is a K-automorphism, it is Bernoullian.

3. *Construction of a Bernoulli diffeomorphism on the n-dimensional ball B^n*

(a) *The flow h^t.* Let A be a hyperbolic $(n-3)\times(n-3)$ matrix with determinant 1 and integer entries. Consider the corresponding Anosov diffeomorphism of the $(n-3)$-torus $\mathbb{T}^{n-3}$ and its suspension $\tilde{h}^t$ with function 1. The flow $\tilde{h}^t$ is not a K-flow (since it has a discrete component in the spectrum), but we can perturb $\tilde{h}^t$ in the class of smooth flows preserving the Lebesgue measure to obtain an Anosov flow h^t which is a K-flow. The phase space N of h^t (as well as of $\tilde{h}^t$) is diffeomorphic to the product $\mathbb{T}^{n-3}\times[0, 1]$, where the tori $\mathbb{T}^{n-3}\times 0$ and $\mathbb{T}^{n-3}\times 1$ are identified by the action of A. From now on we assume that A has the following form:

$$A=\begin{pmatrix} A_1 & & & \bigcirc \\ & A_2 & & \\ & & \ddots & \\ \bigcirc & & & A_k \end{pmatrix}, \qquad k=\left[\frac{n-3}{2}\right], \qquad A_i=\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{if } i<k, \tag{3.1}$$

$$A_k=\begin{cases} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, & \text{if } n \text{ is odd} \\[2ex] \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}=\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}, & \text{if } n \text{ is even.} \end{cases} \tag{3.2}$$

(3.1) PROPOSITION. *The matrix A defined by* (3.1) *and* (3.2) *induces an Anosov diffeomorphism of $\mathbb{T}^{n-3}$.*

Proof. It is obvious that $\det A=1$ and that the eigenvalues of $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ are real and different from 1. Therefore, it is sufficient to show that the eigenvalues of $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

are real and different from 1. Consider the characteristic polynomial $P(\lambda) = \lambda^3 - 5\lambda^2 + 6\lambda - 1$. Observe that $P(0) = -1$, $P(1) = 1$, $P(2) = -1$, $P(4) = 7$. The proposition is proved. □

(*b*) *Embedding in* $\mathbb{R}^n$

(3.2) PROPOSITION. *The phase space* N^{n-2} *of the flow* h^t *corresponding to the matrix* A *defined by* (3.1) *and* (3.2) *can be embedded in* $\mathbb{R}^n$.

Proof. Topologically the phase spaces of h^t and $\tilde{h}^t$ are the same (see § 3(*a*)). Thus, it suffices to show that the product $\mathbb{T}^{n-3} \times [0, 1]$ with the top and the bottom tori identified by A can be embedded in $\mathbb{R}^n$. The embedding is constructed step by step using the block structure of A. The matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is the natural simplest element of A and $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. We will construct a deformation $\mathbb{T}_t^{n-3}$ of the torus $\mathbb{T}^{n-3}$ in $\mathbb{R}^{n-1}$ which transforms the torus into its image under A. The deformation consists of k steps each of them dealing only with the corresponding block of A and with the corresponding 2- or 3-dimensional torus.

(3.3) LEMMA. *There exists a deformation* $\mathbb{T}_t^2$ *of the* 2*-torus* $\mathbb{T}^3$ *in* $\mathbb{R}^4$ *which transforms* $\mathbb{T}_0^2$ *into its image under* $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Proof. Let $B^3 = \{(x_1, x_2, x_3) | x_1^2 + x_2^2 + x_3^2 < R\}$ be a large ball in $\mathbb{R}^3$. The product $S^1 \times B^3$ can be embedded in a natural way into $\mathbb{R}^4$. Consider first the strip $D_0 = S^1 \times \{(x_1, 0, 0) | 0 < x_1 < 1\} \subset S^1 \times B^3$ and denote by I_ϕ the segment $\phi \times \{(x_1, 0, 0) | 0 < x_1 < 1\}$, $\phi \in S^1$. Let D_1 be the strip $\{(\phi, r\cos\phi, r\sin\phi, 0) | 0 < r < 1\}$. A deformation $D_t = \{I_\phi(t)\}$ of D_0 into D_1 in $S^1 \times B^3$ such that $I(t) \subset \phi \times B^3$ for every $0 \le t \le 1$ is shown in figure 1. For $t = 0$ we have the strip D_0. For a small t let $I_\phi(t)$

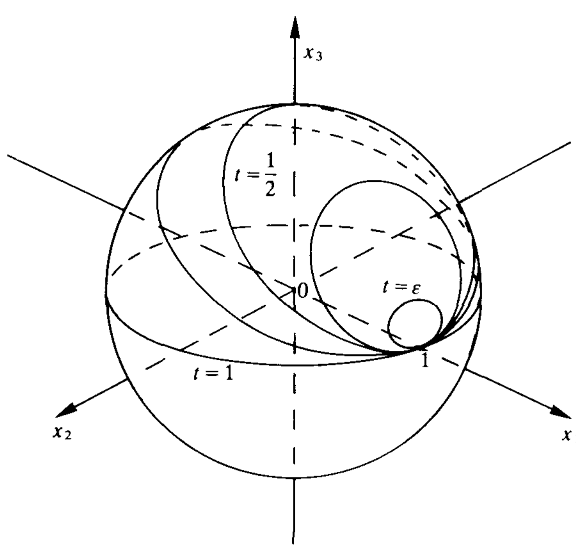

FIGURE 1

describe a small circle near $I_\phi(0)$ when ϕ changes from 0 to 2π. For larger ts the circles are larger, and the last one corresponding to $t-1$ coincides with the equator in the plane (x_1, x_2). Observe that the strip D_2 is twisted in the plane (x_1, x_2). Since the codimension is 2 and the manifold is orientable, the normal bundle is trivial [**6**] and there exist two non-zero vector fields $v_1(t)$ and $v_2(t)$ normal to D_t. Let $\bar{v}(t, \phi)$ be the restriction of $v_1(t)$ to the middle line of D_t. The pair $I_\phi(t)$ and $\bar{v}(t, \phi)$ defines an oriented circle $S^1_{\phi,t}$ in $\phi \times B^3$. We obtain a 2-torus for each t, the last one being the image of the first one under the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The lemma is proved. □

Of course, the same statement is true for the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Thus, there exists a deformation $\mathbb{T}^2_t$ in $\mathbb{R}^2$ which transforms $\mathbb{T}^2_0$ into its image under the action of $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Consider now the manifold $F = \{t \times T^2_t\}$, $0 \le t \le 1$, (lying in $\mathbb{R}^5 = \mathbb{R} \times \mathbb{R}^4$. It has codimension 2; therefore, the normal bundle is trivial [**6**] and the manifold $F \times S^1 = \{t \times T^2_t \times S^1\}$ is also embedded in $\mathbb{R} \times \mathbb{R}^4$. Any torus $\mathbb{T}^m$ can be embedded in $\mathbb{R}^{m+1}$. Hence, there is an embedding of $F \times \mathbb{T}^{n-5} = (F \times S^1) \times \mathbb{T}^{n-6}$ into $\mathbb{R}^5 \times \mathbb{R}^{n-5} = \mathbb{R}^n$. Thus, there exists a deformation $\mathbb{T}^{n-3}_t$, $i-1 \le t \le i$, of $\mathbb{T}^{n-3}$ in $\mathbb{R}^{n-1}$ which transforms the torus $\mathbb{T}^{n-3}$ into its image under the action of the matrix

$$\begin{pmatrix} E_{2i-2} & & & \bigcirc \\ & 2 & 1 & \\ & 1 & 1 & \\ \bigcirc & & & E_{n-3-2i} \end{pmatrix},$$

where E_m is the unit matrix of order m (the last step may be slightly different). The composition $\mathbb{T}^{n-3}_t$, $0 \le t \le k$, of these deformations corresponds to the action of A. If we assume now that t belongs to a circle S^1 of length k and $\mathbb{T}^{n-3}_t$ belongs to $t \times B^{n-1}$ (where B^{n-1} is a large ball in $\mathbb{R}^{n-1}$), then we have an embedding of the phase space N^{n-2} in $S^1 \times B^{n-1}$ which, in its turn, is embedded in $\mathbb{R}^n$. The proposition is proved. □

(*c*) *Construction of a Bernoulli diffeomorphism of B^n with $n-1$ non-zero exponents*
It follows from the previous section that N^{n-2} can be embedded in $\mathbb{R}^n$. Since N^{n-2} is orientable, the normal bundle is trivial [**6**] and there exists an embedding ψ of the product $D^2 \times N^{n-2}$ in $\mathbb{R}^n$. Let us assume that $\psi(D^2 \times N^{n-2}) \subset B^n$ and consider a smooth triangulation $\sigma_1, \cdots, \sigma_m$ of $B^n \backslash \psi(D^2 \times N^{n-2})$. Simple induction shows (see [**4**]) that there exists a smooth mapping $\psi_0 : D^2 \times N^{n-2} \to B^n$ such that its image is B^n and the restriction $\psi_0|_{\mathrm{int}(D^2 \times N^{n-2})}$ is a diffeomorphic embedding.

(3.4) PROPOSITION. *There exists a Bernoulli diffeomorphism of B^n preserving the Lebesgue measure and having $n-1$ non-zero characteristic exponents.*

Proof. Consider the following skew product acting on $D^2 \times N^{n-2}$

$$f(x, y) = (g(x), h^{\alpha(x)}y),$$

where $g: D^2 \to D^2$ preserves the Lebesgue measure and stops at the boundary with all derivatives, and $\alpha: D^2 \to \mathbb{R}$ is a non-negative function which is positive at the centre and equals 0 in a neighbourhood of the boundary ∂D^2. According to § 2, f preserves the Lebesgue measure, has $n-1$ non-zero characteristic exponents and is Bernoullian. Since g stops at ∂D^2 with all derivatives so does f at $\partial(D^2 \times N^{n-2})$, and the topological construction in § 3(a), (b) implies that the diffeomorphism $f_1 = \psi_0 \circ f \circ \psi_0^{-1}$ acting on B^n has all required properties, except that it does not preserve the Lebesgue measure. However, the invariant measure for f_1 has smooth positive density and an argument from [**4**] (see proposition 1.2) shows that there exists a diffeomorphism $\psi_1: B_n \to B^n$ such that $f_2 = \psi_1 \circ f \circ \psi_1^{-1}$ preserves the Lebesgue measure. The proposition is proved. □

(3.5) THEOREM. *For every smooth compact manifold Q^n, $n \geq 5$, there is a diffeomorphism $\gamma: Q^n \to Q^n$ which preserves the Riemannian volume, has $n-1$ non-zero characteristic exponents and is Bernoullian.*

Proof. Katok showed in [**4**] (see proposition 1.2) that there is a continuous mapping $F: B^n \to Q^n$ such that $F|_{\mathrm{int} B^n}$ is a diffeomorphic embedding, $F(B^n) = Q^n$, the image of the boundary has measure 0 and the image of the Lebesgue measure on B^n is the Riemannian volume on Q^n. It follows that $\gamma = F \circ f_2 \circ F^{-1}$ has all required properties. The theorem is proved. □

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