ON SOME PROPERTIES OF BINARY RELATIONS

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Some important notions in the theory of binary relations such as the relative product of two relations and the converse of a relation are defined in Whitehead and Russell's "Principia mathematica" ([1]). McKinsey ([2]) and Tarski ([3]) gave their systems of postulates for the calculus of relations. Ore studied on equivalence relations ([4]) and Riguet on closures (fermeture) of relations ([5] and [6]), and they obtained remarkable results on the structure of these relations. The purpose of this paper is to examine the relative properties of some relations to each other, whose notions are closely related to the notions given by Riguet.

We denote simply by RS the relative product of the relations R and S, by R^{-1} the converse of the relation R, and by $R[X]$ the set obtained by operating R on the set X, i.e. $R``X$ of "Principia mathematica." The notions of the equivalence relation and of the function slightly deviate from the ordinary ones, because a relation is called here an equivalence relation or a function if it is an equivalence relation or a function on their natural domain. In §1, definitions of these notations and notions and some elementary properties are shortly described as prelimaries.

1. Elementary properties

Let V be the universal class. A class R is said to be a relation when and only when $R \subseteq V \times V$. I will write down definitions of fundamental notions and elementary properties without proof in the following:

(1)　Def.　$R[X] = \{ y \mid (Ex)(yRx \text{ and } x \in X) \}$.

(2)　$(x)(R[\{x\}] = S[\{x\}]) \implies R = S$,

i.e. a relation is perfectly determined by its operator character.

(3)　$R \subseteq S \implies R[X] \subseteq S[X]$,　$X \subseteq Y \implies R[X] \subseteq R[Y]$.

(4)　$R[\bigcup_{k \in K} X_k] = \bigcup_{k \in K} R[X_k]$,　$(\bigcup_{k \in K} R_k)[X] = \bigcup_{k \in K} R_k[X]$.

(5) $R[\bigcap_{k\in K} X_k] \subseteqq \bigcap_{k\in K} R[X_k], \quad (\bigcap_{k\in K} R_k)[X] \subseteqq \bigcap_{k\in K} R_k[X].$

(6) $(R-S)[X] \supseteqq R[X] - S[X].$

(7) $(\bigcap_{k\in K} R_k)[\{x\}] = \bigcap_{k\in K} R_k[\{x\}], \quad (R-S)[\{x\}] = R[\{x\}] - S[\{x\}].$

(8) If there is a correspondence $X \to \varphi(X)$ between subclasses of V such that $\varphi(\bigcup_{k\in K} X_k) = \bigcup_{k\in K} \varphi(X_k)$ always holds, then there exists a relation R for which $R[X] = \varphi(X)$ always holds.

(9) $(x)(R[\{x\}] \subseteqq S[\{x\}]) \Longrightarrow R \subseteqq S.$

(10) Def. $R^{-1} = \{\langle x, y \rangle | \langle y, x \rangle \in R\}.$

(11) $(\bigcup_{k\in K} R_k)^{-1} = \bigcup_{k\in K} R_k^{-1}, \quad (\bigcap_{k\in K} R_k)^{-1} = \bigcap_{k\in K} R_k^{-1}, \quad (R-S)^{-1} = R^{-1} - S^{-1}.$

(12) Def. $RS = \{\langle z, x \rangle | (Ey)[zRy \text{ and } ySx]\}.$

(13) $(RS)[X] = R[S[X]].$

(14) $(R^{-1})^{-1} = R, \quad (RS)T = R(ST), \quad (RS)^{-1} = S^{-1}R^{-1}.$

(15) Def. $R^1 = R, \quad R^2 = RR, \quad R^3 = RRR, \ldots.$

(16) $R^m R^n = R^{m+n}, \quad (R^n)^{-1} = (R^{-1})^n. \qquad (m, n = 1, 2, \ldots)$

(17) $R \subseteqq S \Longrightarrow R^{-1} \subseteqq S^{-1}, \quad R \subseteqq S \Longrightarrow R^n \subseteqq S^n \ (n = 1, 2, \ldots),$

$R \subseteqq S \Longrightarrow (RT \subseteqq ST \text{ and } TR \subseteqq TS).$

(18) $R(\bigcup_{k\in K} S_k) = \bigcup_{k\in K} RS_k, \quad (\bigcup_{k\in K} R_k) S = \bigcup_{k\in K} R_k S.$

(19) $R(\bigcap_{k\in K} S_k) \subseteqq \bigcap_{k\in K} RS_k, \quad (\bigcap_{k\in K} R_k) S \subseteqq \bigcap_{k\in K} R_k S.$

(20) $(R-S)T \supseteqq RT - ST, \quad R(S-T) \supseteqq RS - RT.$

(21) Definition of the identical relation I: $I = \{\langle x, x \rangle | x \in V\}.$

(22) $IX = XI = X.$

(23) $(X)[JX = X] \Longrightarrow J = I, \quad (X)[XJ = X] \Longrightarrow J = I.$

(24) $J \subseteqq I \Longrightarrow J^{-1} = J^2 = J.$

(25) $(x)[R \cdot \{\langle x, x \rangle\} = S \cdot \{\langle x, x \rangle\}] \Longrightarrow R = S,$

$(x)[\{\langle x, x \rangle\} \cdot R = \{\langle x, x \rangle\} \cdot S] \Longrightarrow R = S.$

(26) $RR^{-1}R \supseteqq R, \quad R(R^{-1}R \cap I) = (RR^{-1} \cap I)R = R.$

(27) $R(V \times V)R = S(V \times V)S$

$\Longleftrightarrow [R^{-1}R \cap I = S^{-1}S \cap I \text{ and } RR^{-1} \cap I = SS^{-1} \cap I].$

(28) Def. S is called symmetric if and only if $S^{-1} = S.$

(29) If S_k is symmetric for every $k \in K$, then $\bigcup_{k\in K} S_k$ as well as $\bigcap_{k\in K} S_k$ are symmetric.

(30) $R \cup R^{-1}, R \cap R^{-1}, R^{-1}R, RR^{-1}$ are symmetric.

(31) Def. T is called transitive if and only if. $T^2 \subseteqq T.$

(32) If T is transitive, so is also $T^{-1}.$

(33) If T is transitive and $J \subseteq I$, then $T \cup J$ is also transitive.

(34) If T_k is transitive for every $k \in K$, so is also $\bigcap_{k \in K} T_k$.

(35) Def. F is called a function if and only if $FF^{-1} \subseteq I$.

(36) Def. E is called an equivalence relation if and only if $E^{-1}E = E$.

(37) E is an equivalence relation if and only if E is symmetric as well as transitive.

Remark. By the definitions (35) and (36), "F is a function" means "F is a function on its natural domain," and "E is an equivalence relation" means "E is an equivalence relation of its natural domain, which is equal to its converse domain." For an equivalence relation, reflexivity holds only in the sense $E \subseteq I \cap (E^{-1}E \cup EE^{-1})$.

(38) If $J \subseteq I$, then J is an equivalence relation.

(39) If E_k is an equivalence relation for every $k \in K$, then $\bigcap_{k \in K} E_k$ is also an equivalence relation.

(40) If F is a function and $F \subseteq G$, then G is also a function. In this case holds $FG^{-1} = GF^{-1} = GG^{-1}$.

(41) If F and G are functions, then FG is also a function.

(42) If $F_k \cup F_l$ is a function for any pair $k, l \in K$, or if $F_k F_l^{-1} \subseteq I$ for any pair $k, l \in K$, then $\bigcup_{k \in K} F_k$ is a function.

(43) For any function F hold $(R \cap S)F = RF \cap SF$, $F^{-1}(R \cap S) = F^{-1}R \cap F^{-1}S$, $(R - S)F = RF - SF$, and $F^{-1}(R - S) = F^{-1}R - F^{-1}S$.

(44) For any equivalence relation E and for any function F hold $EE^{-1}E = E$ and $FF^{-1}F = F$.

(45) If F is a function, then $F^{-1}F$ is an equivalence relation. Conversely, any equivalence relation E can be expressed in the form $E = F^{-1}F$, where F is a function.

(46) Def. F is called an *one-to-one correspondence* if and only if F and F^{-1} are both functions.

Remark. Evidently hold theorems corresponding to (40)-(44) for one-to-one correspondences.

(47) Def. P is called a *partial order* if and only if P is transitive and $P \cap P^{-1} \subseteq I$.

(48) If P is a partial order and $J \subseteq I$, then $(P - I) \cup J$ is also a partial order.

Remark. By (47), any relation which lies between "$<$" and "$\leqq$" is called here a partial order. $P - I$ corresponds to "$<$". The linear order L (corresponding to "$\leqq$") of the class C can be defined by "L is a linear order of C if and only if L is a partial order and $L \cup L^{-1} = C \times C$."

(49) Def. R is called closed if and only if $RR^{-1}R = R$.

(50) Any function F and any equivalence relation E are closed.

(51) If R is closed, so is also R^{-1}. If R and S are closed, so is also $R \cap S$.

(52) For any pair of functions F and G, $F^{-1}G$ is closed. Conversely, any closed relation R can be expressed in the form $R = F^{-1}G$, where F and G are functions.

(53) Def. $\overline{R} = \bigcup\limits_{n=1}^{\infty} R(R^{-1}R)^n$.

(54) $\overline{R} \supseteq R$, $\overline{\overline{R}} = \overline{R}$.

(55) $\overline{R}$ is closed, i.e. $\overline{R}\,\overline{R}^{-1}\overline{R} = \overline{R}$. R is closed if and only if $R = \overline{R}$.

(56) $R \subseteq S \Rightarrow \overline{R} \subseteq \overline{S}$, $R \subseteq S \subseteq \overline{R} \Rightarrow \overline{S} = \overline{R}$, $\overline{R}^{-1} = \overline{R^{-1}}$.

(57) $\overline{\bigcup\limits_{k \in K} R_k} \supseteq \bigcup\limits_{k \in K} \overline{R}_k$, $\overline{\bigcap\limits_{k \in K} R_k} \subseteq \bigcap\limits_{k \in K} \overline{R}_k$.

(58) $\overline{\bigcup\limits_{k \in K} \overline{R}_k} = \overline{\bigcup\limits_{k \in K} R_k}$.

(59) If R is symmetric, then $\overline{R} = \bigcup\limits_{n=1}^{\infty} R^n$ and $\overline{R}$ is an equivalence relation.

(60) $\overline{R}$ is the minimum closed relation including R.

(61) $\overline{R^{-1}R}$ and $\overline{RR^{-1}}$ are both equivalence relations.

(62) $\overline{R^{-1}R} = \overline{R}^{-1}\overline{R} = R^{-1}\overline{R} = \overline{R}^{-1}R$, $\overline{RR^{-1}} = \overline{R}\,\overline{R}^{-1} = R\overline{R}^{-1} = \overline{R}R^{-1}$.

(63) Def. If H is a relation such that $H[R]$ is a relation for every relation R, then $A \cup \bigcup\limits_{n=1}^{\infty} H^n[A]$ for any relation A is called the relation recursively generated from A by H.

(64) The recursively generated relation from A by H is the minimum relation F satisfying $A \cup H[F] \subseteq F$.

Remark. If we take $A = \{\langle a(\alpha), \langle 0, \alpha \rangle \rangle\}$ and
$H = \{\langle \langle b(z, x, \alpha), \langle x+1, \alpha \rangle \rangle, \langle z, \langle x, \alpha \rangle \rangle \rangle \,|\, x, z \in N\}$, then the condition for F becomes

$$f(0, \alpha) = a(\alpha),$$
$$f(x+1, \alpha) = b(f(x, a), x, \alpha),$$

where we take N as the set of natural numbers, $F = \{\langle f(x, \alpha), \langle x, \alpha \rangle \rangle \,|\, x \in N\}$, and α as a finite sequence of parameters.

2. Associable system of relations

The domain and the converse domain of R are classified by the equivalence relations $R^{-1}R$ and RR^{-1} respectively. The relation R is essentially a blockwise one-to-one correspondence between the classified domain of R and the classified converse domain of R. I will now define the notion of the associable system of relations, which enables us to study mutual properties between relations of different classified domains and converse domains.

Definition

(1) A relation S is called an extension of R, if and only if $R \subseteqq S$ and $\bar{S}R^{-1}\bar{S} \subseteqq \bar{R}$.

(2) A system of relations is called associable, if and only if they have their common extension.

THEOREM 1. *A necessary and sufficient condition for two relations R and S to be associable is that the conditions*

$$\overline{(R \cup S)}R^{-1}\overline{(R \cup S)} \subseteqq \bar{R},$$
$$\overline{(R \cup S)}S^{-1}\overline{(R \cup S)} \subseteqq \bar{S}$$

are both satisfied.

Proof. If T is a common extension of R and S, then $\overline{(R \cup S)}R^{-1}\overline{(R \cup S)}$ $\subseteqq \bar{T}R^{-1}\bar{T} \subseteqq \bar{R}$. Similarly for the other formula. If the two conditions are satisfied, then $R \cup S$ is a common extension of R and S.

THEOREM 2.

(1) *Any relation is associable with itself.*

(2) *Any relation R is associable with $\bar{R}$.*

(3) *Any relation is associable with its extension.*

(4) *If $R^{-1}S = RS^{-1} = \emptyset$, i.e. if the domains of R and S as well as their converse domains are disjoint, then R and S are associable.*

Proof. (1), (2), (3) Evident.

(4) $\overline{(R \cup S)}R^{-1}\overline{(R \cup S)} = \cup V_0 V_1^{-1} V_2 \ldots V_{2m-1}^{-1} V_{2m} R^{-1} W_0 W_1^{-1} W_2 \ldots W_{2n-1}^{-1} W_2 = \bar{R}$, where V's and W's stand for R or S. Similarly, $\overline{(R \cup S)}S^{-1}(R \cup S) = \bar{S}$. q.e.d.

THEOREM 3. *If R and S are associable, so are also R^{-1} and S^{-1},*

Proof. Evident.

THEOREM 4. *If a system of R_k's $(k \in K)$ is associable, so is also the system of all S_k's satisfying $R_k \subseteq S_k \subseteq \overline{R}_k$ for every k in K.*

Proof. Let T be a common extension of R_k's, i.e. $R_k \subseteq T$ and $\overline{T} R_k^{-1} \overline{T} \subseteq \overline{R}_k$. By (55), (56), and (62),

$$\overline{T} S_k^{-1} \overline{T} \subseteq \overline{T} \overline{S}_k^{-1} \overline{T} = \overline{T} \overline{R}_k^{-1} \overline{T} = \overline{T} \overline{R}_k^{-1} R_k \overline{R}_k^{-1} \overline{T}$$
$$= \overline{T} R_k^{-1} \overline{R}_k \overline{R}_k^{-1} \overline{T} \subseteq \overline{T} R_k^{-1} \overline{T} \overline{T}^{-1} \overline{T} = \overline{T} R_k^{-1} \overline{T} \subseteq \overline{R}_k = \overline{S}_k.$$

Namely, T is also a common extension of all S_k's.   q.e.d.

THEOREM 5. *If R and S are associable, then*

$$\overline{R} \overline{S}^{-1} = (\overline{R} \cap \overline{S})(\overline{R} \cap \overline{S})^{-1} \quad and \quad \overline{R}^{-1} \overline{S} = (\overline{R} \cap \overline{S})^{-1}(\overline{R} \cap \overline{S}).$$

Proof. Let $\langle x, y \rangle \in \overline{R} \overline{S}^{-1}$, i.e. there be such z that $\langle x, z \rangle \in \overline{R}$ and $\langle y, z \rangle \in \overline{S}$, then

$$\{\langle x, z \rangle\} \subseteq \{\langle x, z \rangle\}\{\langle y, z \rangle\}^{-1}\{\langle y, z \rangle\} \subseteq \overline{(R \cup S)} \overline{S}^{-1} \overline{(R \cup S)} \subseteq \overline{S},$$

similarly $\langle y, z \rangle \in \overline{R}$, therefore

$$\langle x, y \rangle \in (\overline{R} \cap \overline{S})(\overline{R} \cap \overline{S})^{-1}.$$

We obtain namely $\overline{R} \overline{S}^{-1} \subseteq (\overline{R} \cap \overline{S})(\overline{R} \cap \overline{S})^{-1}$. On the other hand, evidently $\overline{R} \overline{S}^{-1} \supseteq (\overline{R} \cap \overline{S})(\overline{R} \cap \overline{S})^{-1}$. Similarly for the other formula.   q.e.d.

THEOREM 6. *If P and Q resp. Q and R are associable, then*

$$\overline{P} \overline{Q}^{-1} \overline{R} \subseteq \overline{Q}.$$

Proof. By the preceding theorem,

$$\overline{P} \overline{Q}^{-1} \overline{R} = (\overline{P} \cap \overline{Q})(\overline{P} \cap \overline{Q})^{-1} \overline{R} \subseteq \overline{(Q \cup R)} \overline{Q}^{-1} \overline{(Q \cup R)} \subseteq \overline{Q}.$$ q.e.d.

THEOREM 7. *If all pairs of R_k's are associable, then $\overline{\bigcup_{k \in K} R_k} = \bigcup_{k \in K} \overline{R}_k$.*

Proof. Evidently $\overline{\bigcup_{k \in K} R_k} \supseteq \bigcup_{k \in K} \overline{R}_k$. On the other hand, by the preceding theorem

$$\overline{R}_{k_0} \overline{R}_{l_1}^{-1} \overline{R}_{k_1} \overline{R}_{l_2}^{-1} \overline{R}_{k_2} \ldots \overline{R}_{l_{n-1}}^{-1} \overline{R}_{k_{n-1}} \overline{R}_{l_n}^{-1} \overline{R}_{k_n}$$
$$\subseteq \overline{R}_{k_0} \overline{R}_{l_1}^{-1} \overline{R}_{k_1} \overline{R}_{l_2}^{-1} \overline{R}_{k_2} \ldots \overline{R}_{l_{n-1}}^{-1} \overline{R}_{l_n}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\subseteq \overline{R}_{k_0} \overline{R}_{l_1}^{-1} \overline{R}_{k_1} \overline{R}_{l_2}^{-1} \overline{R}_{l_3}$$
$$\subseteq \overline{R}_{k_0} \overline{R}_{l_1}^{-1} \overline{R}_{l_2}$$
$$\subseteq \overline{R}_{l_1},$$

so we have $\overline{\bigcup_{k \in K} R_k} \subseteqq \bigcup_{k \in K} \overline{R_k}$.   q.e.d.

THEOREM 8. *A necessary and sufficient condition for a system of relations* R_k ($k \in K$) *to be associable is that the relations are associable by pairs.*

Proof. Necessity is evident. It is also sufficient, because by the preceding theorem

$$(\bigcup_{j \in K} \overline{R_j}) R_k^{-1} (\overline{\bigcup_{l \in K} R_l}) \subseteqq (\bigcup_{j \in K} \overline{R_j}) \overline{R_k}^{-1} (\bigcup_{l \in K} \overline{R_l}) = \bigcup_{\substack{j \in K \\ l \in K}} \overline{R_j} \overline{R_k}^{-1} \overline{R_l} \subseteqq \bigcup_{\substack{j \in K \\ l \in K}} \overline{R_k} = \overline{R_k}. \quad \text{q.e.d.}$$

THEOREM 9. *A necessary and sufficient condition for two equivalence relations* C *and* D *to be associable is that* $CD = C \cap D$.

Proof. Necessity is evident by (39), (50), and the theorem 5. Conversely: if $CD = C \cap D$, then $(\overline{C \cup D}) C^{-1} (\overline{C \cup D}) = \cup P_1 \ldots P_m C^{-1} Q_1 \ldots Q_n = C \cup (C \cap D)$ $= C$, where P's and Q's stand for C or D. Similarly $(C \cup D) D^{-1} (\overline{C \cup D}) = D$. Therefore the condition is also sufficient.

3. Decomposition of the closed relation and of the transitive relation

Any closed relation can be decomposed into the product of the converse of a function and a function as shown in (52) (Riguet [5], [6]). In this paragraph, this theorem is given in a more refined form in some respect, and some applications of the theorem are given thereafter.

THEOREM 10.

(1) *Any closed relation* R *can be expressed in the form* $R = F^{-1}G$, *where* F *and* G *are functions. (Proposition (52)).*

(2) *If* $R = F^{-1}G$ *and* F, G *are both functions, then generally* $RR^{-1} \subseteqq F^{-1}F$ *and* $R^{-1}R \subseteqq G^{-1}G$. *We can find out, however, functions* F_0 *and* G_0 *out of sub-relations of* F *and* G *respectively, for which* $R = F_0^{-1}G_0$, RR^{-1} $= F_0^{-1}F_0$, *and* $R^{-1}R = G_0^{-1}G_0$.

(3) *If* $R = F^{-1}G$ *and* F, G *are functions, then* $FF^{-1} = GG^{-1}$ *if and only if* $RR^{-1} = F^{-1}F$ *and* $R^{-1}R = G^{-1}G$.

Proof. (2) $RR^{-1} = (F^{-1}G)(F^{-1}G)^{-1} = F^{-1}GG^{-1}F \subseteqq F^{-1}IF = F^{-1}F$, similarly $R^{-1}R \subseteqq G^{-1}G$. If we put $F_0 = GG^{-1}F$ and $G_0 = FF^{-1}G$, then $F_0 \subseteqq IF = F$, $G_0 \subseteqq IG$ $= G$, $F_0^{-1}G_0 = F^{-1}GG^{-1}FF^{-1}G = (F^{-1}G)(F^{-1}G)^{-1}(F^{-1}G) = RR^{-1}R = R$, $F_0^{-1}F_0$ $= F^{-1}GG^{-1}GG^{-1}F = F^{-1}(GG^{-1}G)(F^{-1}G)^{-1} = (F^{-1}G)(F^{-1}G)^{-1} = RR^{-1}$, and $G_0^{-1}G_0$

$= G^{-1}FF^{-1}FF^{-1}G = G^{-1}(FF^{-1}F)(F^{-1}G) = (G^{-1}F)(F^{-1}G) = R^{-1}R$.   By (40), F_0 and G_0 are both functions.   (3) If $R = F^{-1}G$, $RR^{-1} = F^{-1}F$, $R^{-1}R = G^{-1}G$, and F and G are functions,

$$FF^{-1} = IFF^{-1}I \cong GG^{-1}FF^{-1}GG^{-1} = G(F^{-1}G)^{-1}(F^{-1}G)G^{-1}$$
$$= GR^{-1}RG = GG^{-1}GG^{-1} = GG^{-1}.$$

Similarly $GG^{-1} \cong FF^{-1}$, so $FF^{-1} = GG^{-1}$.   Conversely, if $R = F^{-1}G$, $FF^{-1} = GG^{-1} \cong I$, then

$$RR^{-1} = (F^{-1}G)(F^{-1}G)^{-1} = F^{-1}GG^{-1}F = F^{-1}(FF^{-1}F) = F^{-1}F,$$
$$R^{-1}R = (F^{-1}G)^{-1}(F^{-1}G) = G^{-1}FF^{-1}G = G^{-1}(GG^{-1}G) = G^{-1}G.   \text{q.e.d.}$$

Now, let us take up an equivalence relation $C = F^{-1}F$, where F is a function and $J = FF^{-1}$. The class K of all closed relations R, satisfying $RR^{-1} = R^{-1}R = C$, corresponds one-to-one to the class of transformations T on J. The one-to-one correspondence is given by $T = FRF^{-1}$ and $R = F^{-1}TF$. If we denote the correspondence by $R \longleftrightarrow T$, we have clearly

(1) $C \longleftrightarrow J$,

(2)   If $R \longleftrightarrow T$, then $R^{-1} \longleftrightarrow T^{-1}$,

(3)   If $R \longleftrightarrow T$ and $S \longleftrightarrow V$, then $RS \longleftrightarrow TV$.

This means that the elements of the class K can be taken as transformations on C. In our theory, transformations and one-to-one correspondences are most naturally understood as those on class relations. The theorem 10 shows that any closed relation can be taken as one-to-one correspondence of classes between two systems of classes.

THEOREM 11. *If R and S are closed relations and $R \cong S$, then*

(1) *$RS^{-1}R$ and $SR^{-1}S$ are closed relations satisfying $R \cong RS^{-1}R \cong SR^{-1}S \cong S$,*

(2) *R, $RS^{-1}R$, $SR^{-1}S$, S can be expressed by functions F, G, F_0, G_0, F_1, G_1, H in such a way that $R = F_0^{-1}G_0$, $RS^{-1}R = F_0^{-1}H^{-1}HG_0$, $SR^{-1}S = F_1^{-1}G_1$, and $S = F^{-1}G$, where functions F, G, F_0, G_0, F_1, G_1, H satisfy the conditions $F_0F_0^{-1} = G_0G_0^{-1}$, $F_1F_1^{-1} = G_1G_1^{-1} = HF_0F_0^{-1}H^{-1} = HG_0G_0^{-1}H^{-1}$, $FF^{-1} = GG^{-1}$, $F \cong F_1 \cong HF_0$, $G \cong G_1 \cong HG_0$, $FF_1^{-1}F = F_1$, $GG_1^{-1}G = G_1$.*

Proof. (1) $R = RR^{-1}R \cong RS^{-1}R = RS^{-1}RR^{-1}R \cong SS^{-1}SR^{-1}S = SR^{-1}S \cong SS^{-1}S = S$. $RS^{-1}R$ is closed, because $(RS^{-1}R)(RS^{-1}R)^{-1}(RS^{-1}R) = R(S^{-1}RR^{-1}SR^{-1}RS^{-1})R \cong R(S^{-1}SS^{-1}SS^{-1}SS^{-1})R = RS^{-1}R$;   and $SR^{-1}S$ is   closed   too,   because

$(SR^{-1}S)(SR^{-1}S)^{-1}(SR^{-1}S) = (SR^{-1}SS^{-1}RS^{-1}S)R^{-1}S \subseteqq (SS^{-1}SS^{-1}SS^{-1}S)R^{-1}S$
$= SR^{-1}S.$ (2) By the theorem 10, decompose S into two functions F, G satisfying $S = F^{-1}G$, $FF^{-1} = GG^{-1}$, which imply $SS^{-1} = F^{-1}F$, $S^{-1}S = G^{-1}G$. $G = FS$ and $F = GS^{-1}$, because $FS = FF^{-1}G = GG^{-1}G = G$, $GS^{-1} = GG^{-1}F = FF^{-1}F = F$. If we put $\hat{G} = FR$, $\hat{F} = GR^{-1}$, $G_1 = \hat{F}S$, $F_1 = \hat{G}S^{-1}$, then $F_1 \supseteqq \hat{F}$ and $G_1 \supseteqq \hat{G}$, because $F_1 = \hat{G}S^{-1}$
$= FRS^{-1} = FF^{-1}FRS^{-1} = FSS^{-1}RS^{-1} \supseteqq FS(R^{-1}RR^{-1}) = FSR^{-1} = GR^{-1} = \hat{F}$, and similarly $G_1 \supseteqq \hat{G}$. Also $F \supseteqq F_1$ and $G \supseteqq G_1$, because $F = FF^{-1}F = FSS^{-1} \supseteqq FRS^{-1}$
$= F_1$, and similarly $G \supseteqq G_1$. Nextly, $\hat{F}^{-1}\hat{G} = RG^{-1}FR = RS^{-1}R$, and $F_1^{-1}G_1 = S\hat{G}^{-1}\hat{F}S$
$= S(\hat{F}^{-1}\hat{G})^{-1}S = SR^{-1}SR^{-1}S = SR^{-1}S$; because $SR^{-1}SR^{-1}S = SR^{-1}S$ can be proved as follows: $SR^{-1}SR^{-1}S \supseteqq SR^{-1}RR^{-1}S = SR^{-1}S$, and on the other hand $SR^{-1}SR^{-1}S \subseteqq SS^{-1}SR^{-1}S = SR^{-1}S$. $\hat{F}\hat{F}^{-1} = \hat{G}\hat{G}^{-1}$, because $\hat{F}\hat{F}^{-1} = GR^{-1}RG^{-1}$
$= GG^{-1}GR^{-1}RG^{-1}GG^{-1} = F(F^{-1}G)R^{-1}R(G^{-1}F)F^{-1} = FSR^{-1}RS^{-1}F^{-1} \supseteqq FRR^{-1}RR^{-1}F^{-1}$
$= FRR^{-1}F^{-1} = \hat{G}\hat{G}^{-1}$, and similarly $\hat{G}\hat{G}^{-1} \supseteqq \hat{F}\hat{F}^{-1}$. By (40), we obtain $F_1F_1^{-1}$
$= \hat{G}\hat{G}^{-1}$ and $G_1G_1^{-1} = \hat{F}\hat{F}^{-1}$ as follows: $F_1F_1^{-1} = \hat{G}S^{-1}S\hat{G}^{-1} = \hat{G}G^{-1}FF^{-1}G\hat{G}^{-1}$
$= \hat{G}G^{-1}GG^{-1}G\hat{G}^{-1} = \hat{G}G^{-1}G\hat{G}^{-1} = \hat{G}\hat{G}^{-1}\hat{G}\hat{G}^{-1} = \hat{G}\hat{G}^{-1}$, and similarly for the other formula. Accordingly, we have $F_1F_1^{-1} = G_1G_1^{-1} = \hat{F}\hat{F}^{-1} = \hat{G}\hat{G}^{-1}$, which implies $\hat{F}^{-1}\hat{F} = (RS^{-1}R)(RS^{-1}R)^{-1}$ and $\hat{G}^{-1}\hat{G} = (RS^{-1}R^1)^{-1}(RS^{-1}R)$. Moreover, we can prove $F_1^{-1}F = F_1^{-1}F_1$ and $G_1^{-1}G = G_1^{-1}G_1$ as follows: $F_1^{-1}F_1 \subseteqq F_1^{-1}F = SR^{-1}F^{-1}F$
$= SR^{-1}SS^{-1} = SR^{-1}(RR^{-1})R(R^{-1}SS^{-1}) \subseteqq SR^{-1}SS^{-1}RS^{-1} = SR^{-1}F^{-1}FRS^{-1} = F_1^{-1}F_1$,
and similarly for the other formula. Accordingly, by making use of (40), we get $FF_1^{-1}F = F_1F_1^{-1}F = F_1F_1^{-1}F_1 = F_1$ and similarly $GG_1^{-1}G = G_1$.

Now, by the theorem 10, R can be decomposed into two functions F_0, G_0 satisfying $R = F_0^{-1}G_0$, $F_0^{-1}F_0 = G_0^{-1}G_0$, which imply $RR^{-1} = F_0^{-1}F_0$, $R^{-1}R = G_0^{-1}G_0$. Let us define the relation H by $H = \hat{F}F_0^{-1}$. H is a function, because $HH^{-1} = \hat{F}F_0^{-1}F_0\hat{F}^{-1} = \hat{F}RR^{-1}\hat{F}^{-1} \subseteqq \hat{F}(RS^{-1}R)(RS^{-1}R)^{-1}\hat{F}^{-1} = \hat{F}\hat{F}^{-1}\hat{G}\hat{G}^{-1}\hat{F}\hat{F}^{-1}$
$= \hat{F}\hat{F}^{-1}\hat{F}\hat{F}^{-1}\hat{F}\hat{F}^{-1} = \hat{F}\hat{F}^{-1} \subseteqq I$; and $H = \hat{G}G_0^{-1}$, because $\hat{G}G_0^{-1} = \hat{G}\hat{G}^{-1}\hat{G}G_0^{-1}G_0G_0^{-1}$
$= \hat{F}\hat{F}^{-1}\hat{G}G_0^{-1}F_0F_0^{-1} = \hat{F}RS^{-1}RR^{-1}F_0^{-1} = \hat{F}(RS^{-1}R)(RS^{-1}R)^{-1}F_0^{-1} = \hat{F}\hat{F}^{-1}\hat{F}F_0^{-1} = \hat{F}F_0^{-1}$
$= H$, as $RS^{-1}RR^{-1} = (RS^{-1}R)(RS^{-1}R)^{-1}$ is easily provable. Lastly, we can obtain $\hat{F} = HF_0$, $\hat{G} = HG_0$ as follows: $HF_0 = \hat{F}F_0^{-1}F_0 = \hat{F}RR^{-1} = (GR^{-1})RR^{-1} = GR^{-1} = \hat{F}$, and similarly for the other formula. q.e.d.

THEOREM 12. *If a transitive relation T satisfies the condition $T \cap T^{-1}$*
$\supseteqq (TT^{-1} \cup T^{-1}T) \cap I$,[1] *then T can be expressed in the form $T = F^{-1}SF$, where*

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[1] By (33), we can find out for any transitive relation S, a transitive relation $T = S \cup J$, which satisfies the condition.

F is a function and S is a partial order, i.e. $S^2 \subseteqq S$ and $S \cap S^{-1} \subseteqq I$.

Proof. If we take $H = T \cap T^{-1}$, H is an equivalence relation including $(TT^{-1} \cup T^{-1}T) \cap I$, because evidently $H^{-1} = H$ and $H^2 = (T \cap T^{-1})^2 \subseteqq T^2 \cap (T^{-1})^2 \subseteqq T \cap T^{-1}$. By (45), there is a function F such that $H = F^{-1}F$. If we put $S = FTF^{-1}$, then $T = F^{-1}SF$; because $T \supseteqq T^2 \supseteqq TH \supseteqq T(T^{-1}T \cap I) = T$ by (26), and similarly $T = HT$, so $T = HTH = F^{-1}FTF^{-1}F = F^{-1}SF$.

Now I prove that S is a partial order. Firstly, $S^2 = FTF^{-1}FTF^{-1} = FTHTF^{-1} \subseteqq FTTTF^{-1} \subseteqq FTF^{-1} = S$. Secondly, $FF^{-1}S = FF^{-1}FTF^{-1} = FTF^{-1} = S$ and similarly $SFF^{-1} = S$, $FF^{-1}S^{-1} = S^{-1}FF^{-1} = S^{-1}$. As FF^{-1} and F are functions, we can compute by (43) as follows: $S \cap S^{-1} = FF^{-1}S \cap FF^{-1}S^{-1} = FF^{-1}(S \cap S^{-1}) = FF^{-1}(SFF^{-1} \cap S^{-1}FF^{-1}) = FF^{-1}(S \cap S^{-1})FF^{-1} = F(F^{-1}S \cap F^{-1}S)FF^{-1} = F(F^{-1}SF \cap F^{-1}S^{-1}F)F^{-1} = F(T \cap T^{-1})F^{-1} = FHF^{-1} = FF^{-1}FF^{-1} = FF^{-1} \subseteqq I$.

<div align="right">q.e.d.</div>

By this theorem, we can see that any transitive relation T is essentially a partial order on the classes classified by a suitable equivalence relation.

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