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ALMOST FIXED POINT AND BEST APPROXIMATIONS THEOREMS IN H-SPACES

O. HADŽIĆ

Using the methods of KKM theory, almost fixed point and best approximations theorems in H-spaces are proved.

1. INTRODUCTION

The notion of an H-space was introduced by Bardaro and Ceppitelli in [**1**] and since 1988, many results from Nonlinear Functional Analysis in such spaces have been obtained [**1, 2, 3, 4, 5, 6, 7, 13, 14, 15, 16, 17, 18**].

In this paper we shall introduce some generalisations of the Zima condition [**8, 9, 10**]) on a subset of an H-space. Some fixed point and almost fixed point theorems for single-valued and multi-valued mappings $f\colon K \to 2^X$, where X is a not necessarily locally convex topological vector space and K satisfies the Zima condition, have been proved in [**8, 9, 10**].

2. PRELIMINARIES

Let A be a subset of a topological space X. By 2^A we denote the family of all nonempty subsets of A and by $\mathcal{F}(A)$ the family of all nonempty finite subsets of A.

In [1] the following two definitions are given.

DEFINITION 1. *A pair $(X, \{\Gamma_A\})$ is said to be an H-space if X is a topological space and $\{\Gamma_A\}_{A\in\mathcal{F}(X)}$ is a given family of contractible subsets Γ_A of X, such that $A \subset B \subset X$ implies $\Gamma_A \subset \Gamma_B$.*

DEFINITION 2. *A nonempty subset D of an H-space $(X, \{\Gamma_A\})$ is called H-convex if for each $A \in \mathcal{F}(D)$, $\Gamma_A \subset D$.*

We shall introduce a condition of generalised Zima type in the following way.

DEFINITION 3. *Let $(X, \{\Gamma_A\})$ be an H-space with a uniformity $\mathcal{V}$ and let K be a nonempty subset of X. We say that K is of generalised Zima type if for every*

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$V \in \mathcal{V}$ there exists $U \in \mathcal{V}$ such that for every $D \in \mathcal{F}(K)$ and every H-convex subset M of K the following implication holds:

$$(1) \qquad M \cap U(z) \neq \emptyset, \text{ for every } z \in D \quad \Rightarrow \quad M \cap V(u) \neq \emptyset, \text{ for every } u \in \Gamma_D.$$

REMARK. If $K = X$ and $U = V$, for every $V \in \mathcal{V}$, then X is of generalised Zima type if X is a l. c. H-space in the sense of [4].

EXAMPLE. Let X be a topological vector space with a fundamental system of neighbourhoods of zero $\mathcal{V}$ and let $K \subset X$ be of Zima type, that is, for every $V \in \mathcal{V}$ there exists $U \in \mathcal{V}$ such that

$$(2) \qquad co(U \cap (K - K)) \subseteq V,$$

where co is the convex hull operation.

We shall prove that (2) implies (1).

Let (2) hold and $D \in \mathcal{F}(K)$ and $M \subseteq K$, where M is convex. Suppose that

$$(3) \qquad M \cap (z + U) \neq \emptyset, \text{ for every } z \in D.$$

If $D = \{z_1, z_2, \ldots, z_n\}$, it follows from (3) that there exists $\{v_1, v_2, \ldots, v_n\} \subseteq M$ such that

$$v_i \in M \cap (z_i + U),\ i \in \{1, 2, \ldots, n\}.$$

Hence, there exists $\{w_1, w_2, \ldots, w_n\} \subseteq U$ such that $v_i = z_i + w_i$, $i \in \{1, 2, \ldots, n\}$. If $u \in \Gamma_D = co\, D$ then

$$u = \sum_{i=1}^{n} \lambda_i z_i,\ \lambda_i \geqslant 0,\ i \in \{1, 2, \ldots, n\},\ \sum_{i=1}^{n} \lambda_i = 1,$$

and so:

$$\sum_{i=1}^{n} \lambda_i v_i = \sum_{i=1}^{n} \lambda_i z_i + \sum_{i=1}^{n} \lambda_i w_i \in u + co(U \cap (K - K)) \subseteq (u + V) \cap M.$$

From this it follows that $M \cap (u + V) \neq \emptyset$, for every $u \in co\, D$.

Now, we shall give an example of a subset of Zima type in a non locally convex topological vector space.

Let $S(0, 1)$ be the space of all the equivalence classes of real measurable functions on $[0, 1]$, and for every $\widetilde{x} \in S(0, 1)$ let

$$\|\widetilde{x}\| = \int_0^1 \frac{|x(t)|}{1 + |x(t)|} \mu(dt),\ \{x(t)\} \in \widetilde{x}.$$

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Then $\|\cdot\|$ is a paranorm ($\|\cdot\|$ is not homogeneous) and $S(0,1)$ is a non locally convex topological vector space in which the fundamental system of neighbourhoods of zero is given by the family $\mathcal{V} = \{V_\varepsilon\}_{\varepsilon>0}$ where

$$V_\varepsilon = \{x;\ \|x\| < \varepsilon\}.$$

Convergence in this topology coincides with convergence in measure.

Let $\alpha > 0$ and $K_\alpha \subset S(0,1)$ be defined by

$$K_\alpha = \{\widetilde{x};\ \widetilde{x} \in S(0,1);\ |x(t)| \leqslant \alpha,\ t \in [0,1]\}.$$

It can be shown that for every $s \in [0,1]$ and every $\widetilde{x}, \widetilde{y} \in K_\alpha$:

$$\|s(\widetilde{x} - \widetilde{y})\| \leqslant (1 + 2\alpha)s\,\|\widetilde{x} - \widetilde{y}\| = C(K_\alpha)s\,\|\widetilde{x} - \widetilde{y}\|, \tag{4}$$

where $C(K_\alpha) = 1 + 2\alpha > 1$. From (4) it follows that

$$co\big(V_{\varepsilon/(C(K_\alpha))} \cap (K_\alpha - K_\alpha)\big) \subset V_\varepsilon, \text{ for every } \varepsilon > 0$$

which means that K_α is of Zima type, for every $\alpha > 0$.

3. An almost fixed point theorem

If $(X;\ \{\Gamma_A\})$ is an H-space with uniformity $\mathcal{V}$, $K \subset X$ and $F\colon K \to 2^X$ then F has a V-almost point ($V \in \mathcal{V}$) if there exists $y \in K$ such that

$$F(y) \cap V(y) \neq \emptyset.$$

Theorem 1. *Let $(X;\ \{\Gamma_A\})$ be an H-space with uniformity $\mathcal{V}$ and let K be a nonempty, convex and precompact subset of X. Let $F\colon K \to 2^X$ be a lower semicontinuous mapping such that $F(x) \cap K \neq \emptyset$ for every $x \in K$, and $F(x)$ is H-convex for every $x \in K$. If $K \cup F(K)$ is of generalised Zima type then F has a V-almost fixed point, for every $V \in \mathcal{V}$.*

Proof: Let $V \in \mathcal{V}$ and $U \in \mathcal{V}$ be such that (1) holds for every $D \in \mathcal{F}(K \cup F(K))$ and every H-convex subset $M \subset K \cup F(K)$. We shall suppose that every $V \in \mathcal{V}$ is open. For every $x \in K$ let

$$G(x) = \{y;\ y \in K,\ F(y) \cap U(x) = \emptyset\}.$$

Since F is lower semicontinuous and $U(x)$ is open it follows that $G(x)$ is closed for every $x \in K$. From the precompactness of K it follows that there exists $D_1 \in \mathcal{F}(K)$ such that

$$K \subseteq \bigcup_{z \in D_1} U(z). \tag{5}$$

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Since for every $x \in K$, $F(x) \cap K \neq \emptyset$, it follows from (5) that $F(x) \cap \left(\bigcup_{z \in D_1} U(z) \right) \neq \emptyset$. Hence $\bigcap_{z \in D_1} G(z) = \emptyset$. From this we conclude that there exists $D_2 \subset D_1$ such that

$$\Gamma_{D_2} \not\subseteq \bigcup_{z \in D_2} G(z).$$

This means that there exists $y \in \Gamma_{D_2}$ such that $y \notin G(z)$, for every $z \in D_2$, this means that

$$F(y) \cap U(z) \neq \emptyset, \text{ for every } z \in D_2.$$

Since $F(y) \subseteq K \cup F(K)$ and $K \cup F(K)$ is of generalised Zima type, it follows that $F(y) \cap V(y) \neq \emptyset$, since $y \in \Gamma_{D_2}$. Hence y is a V-almost fixed point for F. □

4. A BEST APPROXIMATION THEOREM

In this part of the paper we suppose that $(X; \{\Gamma_A\})$ is a metrisable H-space with metric d.

If $\emptyset \neq K \subset X$ and $F: K \to 2^X$ we say that F satisfies the *Z_δ-condition* on K if for every $y \in K$ and every $\varepsilon > 0$:

$$\text{(6)} \qquad co(L(F(y), \delta(\varepsilon)) \cap K) \subset L(F(y), \varepsilon)$$

where δ: $[0, \infty) \to [0, \infty)$ and for $M \subset X$, $r > 0$

$$L(M, r) = \{x; x \in X, d(x, M) < r\}.$$

EXAMPLE 2. Let $(X, \| \ \|)$ be a paranormed space, $\emptyset \neq K \subset X$, $F: K \to 2^X_{co}$ and suppose there exists $C > 0$ such that for every $s \in [0, 1]$

$$\text{(7)} \qquad \|s(u - v)\| \leqslant Cs\|u - v\|, \text{ for every } u \in K, v \in F(K).$$

We shall prove that (7) implies (6) for $\delta(\varepsilon) = \varepsilon / C$. Let $z \in co(L(F(y), \delta(\varepsilon)) \cap K)$. This means that there exists $\{x_1, x_2, \ldots, x_n\} \in L(F(y), \delta(\varepsilon)) \cap K$ such that $z = \sum_{i=1}^{n} \lambda_i x_i$, $\lambda_i \geqslant 0$, $i \in \{1, 2, \ldots, n\}$, $\sum_{i=1}^{n} \lambda_i = 1$.

Hence $x_i \in K$ and $d(x_i, F(y)) < \delta(\varepsilon)$, for every $i \in \{1, 2, \ldots, n\}$. Hence, there exists $v_i \in F(y)$, $i \in \{1, 2, \ldots, n\}$ such that $\|x_i - v_i\| < \delta(\varepsilon)$, for every $i \in \{1, 2, \ldots, n\}$.

Then (7) implies

$$\left\| \sum_{i=1}^{n} \lambda_i x_i - \sum_{i=1}^{n} \lambda_i v_i \right\| \leqslant C \sum_{i=1}^{n} \lambda_i \|x_i - v_i\| < C \cdot \frac{\varepsilon}{C} = \varepsilon.$$

Since $F(y)$ is convex, $\sum_{i=1}^{n} \lambda_i v_i \in F(y)$ and so $d(z, F(y)) \leqslant \left\| y - \sum_{i=1}^{n} \lambda_i v_i \right\| < \varepsilon$, which means that $z \in L(F(y), \varepsilon)$.

EXAMPLE 3. If (X, d, W) is a Takahashi convex metric space (as defined by Takahashi in [15]) with continuous W and

(a) $$d(W(x_1, x_2, \lambda), W(z_1, z_2, \lambda)) \leqslant \lambda d(x_1, z_1) + (1-\lambda) d(x_2, z_2)$$

then for any $\{x_1, x_2, \ldots, x_n\} \subset X$, any $y \in co_W\{x_1, x_2, \ldots, x_n\}$ and any W-convex set A:

(8) $$d(y, A) \leqslant \min_{1 \leqslant i \leqslant n} d(x_i, v_i)$$

for arbitrary $v_i \in A$ $(i \in \{1, 2, \ldots, n\})$, see [11].

In this case (6) holds for $\delta(\varepsilon) = \varepsilon$, for every $\varepsilon > 0$. Indeed if M is an arbitrary convex set in X then $L(M, \varepsilon)$ is a convex set as well. Suppose that $x_i \in L(M, \varepsilon)$, $i \in \{1, 2, \ldots, n\}$. Then $d(x_i, M) < \varepsilon$, for every $i \in \{1, 2, \ldots, n\}$ and so there exists $v_i \in M$, $i \in \{1, 2, \ldots, n\}$ such that $d(x_i, v_i) < \varepsilon$, $i \in \{1, 2, \ldots, n\}$. From (8) it follows that for every $y \in co_W\{x_1, x_2, \ldots, x_n\}$:

$$d(y, M) < \varepsilon$$

and so $y \in L(M, \varepsilon)$, which means that $L(M, \varepsilon)$ is a W-convex set.

In the proof of the next theorem we shall use the following result of Horvath [12]:

Let (M, d) be a complete metric space and let $\{F_i\}_{i \in I}$ be a family of closed subsets in M. If the family $\{F_i\}_{i \in I}$ has the finite intersection property and $\inf_{i \in I} \alpha(F_i) = 0$, where α is the Kuratowski measure of noncompactness, then $\bigcap_{i \in I} F_i$ is compact and nonempty.

THEOREM 2. *Let $(X; \{\Gamma_A\})$ be a metrisable H-space with metric d, $\emptyset \neq M$ an H-convex and complete subset of X, and let $F\colon M \to \mathcal{K}(X)$ (the family of all nonempty H-convex and compact subsets of X) be a continuous mapping which satisfies the Z_δ-condition on M, where δ is continuous and*

(9) $$\inf_{x \in M} \alpha[\{y;\, y \in M,\, \delta(d(y, F(y))) \leqslant d(x, F(y))\}] = 0.$$

Then there exists $y_0 \in M$ such that

$$\delta(d(y_0, F(y_0))) \leqslant \inf_{x \in M} d(x, F(y_0)).$$

PROOF: Let $G(x)$, $x \in M$, be defined in the following way:

$$G(x) = \{y;\, y \in M,\, \delta(d(y, F(y))) \leqslant d(x, F(y))\}.$$

We shall prove that G is an H-KKM mapping, that is, that for every $D = \{x_1, x_2, \dots, x_n\} \subseteq M$

$$\Gamma_D \subseteq \bigcup_{z \in D} G(z). \tag{10}$$

Suppose that $\Gamma_D \not\subseteq \bigcup_{z \in D} G(z)$. Then there exists $y \in \Gamma_D$ such that $y \notin G(x_i)$ for every $i \in \{1, 2, \dots, n\}$. This means that

$$\delta(d(y, F(y))) > d(x_i, F(y)), \text{ for every } i \in \{1, 2, \dots, n\},$$

and so $x_i \in L(F(y), \delta(d(y, F(y)))) \cap M$, for every $i \in \{1, 2, \dots, n\}$. This implies that

$$\begin{aligned} y &\in co(L(F(y), \delta(d(y, F(y)))) \cap M) \\ &\subset L(F(y), d(y, F(y))) \end{aligned}$$

which means that $d(y, F(y)) < d(y, F(y))$. This is a contradiction. Hence (10) holds and G is an H-KKM mapping.

In order to prove that $G(x)$ is closed for every $x \in M$ we shall prove that the mapping $y \mapsto \delta(d(y), F(y))$ $(y \in M)$ is lower semicontinuous and for every $x \in M$, the mapping $y \mapsto d(x, F(y))$ is upper semicontinuous. Let $\gamma > 0$ and

$$\begin{aligned} P_\gamma &= \{y;\ y \in M,\ \delta(d(y, F(y))) > \gamma\} \\ Q_\gamma &= \{y;\ y \in M,\ d(x, F(y)) < \gamma\}. \end{aligned}$$

We prove that P_γ and Q_γ are open.

Since $y \mapsto (y, F(y))$ is upper semicontinuous and

$$P_\gamma = \{y;\ y \in M,\ (y, F(y)) \subseteq \{(z, v);\ (z, v) \in M \times X;\ \delta(d(z, v)) > \gamma\}\}$$

it follows that P_γ is open.

Analogously, since F is lower semicontinuous and

$$Q_\gamma = \{y;\ y \in M,\ F(y) \cap \{v;\ v \in X,\ d(x, v) < \gamma\} \neq \emptyset\}$$

therefore Q_γ is open.

Hence $G(x)$ is closed, for every $x \in M$, and since G is an H-KKM mapping it follows that $\{G(x)\}_{x \in M}$ has the finite intersection property. From (9) it follows that $\bigcap_{x \in M} G(x) \neq \emptyset$. If $y_0 \in \bigcap_{x \in M} G(x)$ then

$$\delta(d(y_0, F(y_0))) \leqslant \inf_{x \in M} d(x, F(y_0)).$$

∎

Corollary 1. *Let (X, d, W) be a Takahashi convex metric space with continuous W such that (a) holds. Let $\emptyset \neq M$ be a convex and complete subset of X and let $F\colon M \to \mathcal{K}(X)$ be a continuous mapping such that*

$$\inf_{x \in M} \alpha[\{y;\, y \in M,\, d(y, F(y)) \leqslant d(x, F(y))\}] = 0.$$

Then there exists $y_0 \in M$ such that

$$d(y_0, F(y_0)) = \inf_{x \in M} d(x, F(y_0)).$$

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University of Novi Sad
Department of Mathematics
21000 Novi Sad
Trg Dositeja Obradovica 4
Yugoslavia