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ON INTEGRAL ESTIMATES OF NONNEGATIVE POSITIVE DEFINITE FUNCTIONS

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Abstract

Let $\ell > 0$ be arbitrary. We introduce the extremal quantities

$$G(\ell) := \sup_f \int_{-\ell}^{\ell} f\,dx \Big/ \int_{-1}^{1} f\,dx, \quad C(\ell) := \sup_f \sup_{a\in\mathbb{R}} \int_{a-\ell}^{a+\ell} f\,dx \Big/ \int_{-1}^{1} f\,dx,$$

where the supremum is taken over all not identically zero nonnegative positive definite functions. We investigate how large these extremal quantities can be. This problem was originally posed by Yu. Shteinikov and S. Konyagin (for the case $\ell = 2$) and is an extension of the classical problem of Wiener. In this note we obtain exact values for the right limits $\overline{\lim}_{\varepsilon\to 0+} G(k+\varepsilon)$ and $\overline{\lim}_{\varepsilon\to 0+} C(k+\varepsilon)$ $(k \in \mathbb{N})$ taken over doubly positive functions, and sufficiently close bounds for other values of ℓ.

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1. Introduction

We fix some notation and basic concepts which will be used throughout the paper. The symbols $[\cdot]$ and $\lceil\cdot\rceil$ stand for the lower (or in other words, the usual) and the upper integer part, respectively. A function $f : \mathbb{R} \to \mathbb{C}$ satisfying

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i\overline{c_j} f(x_i - x_j) \geq 0$$

for any n-tuples $(x_i)_{i=1}^n \in \mathbb{R}^n$ and complex numbers $(c_i)_{i=1}^n$ is called *positive definite*, denoted by $f \gg 0$. If the function f is positive definite and nonnegative in the ordinary

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sense, then we say that f is *doubly positive*, which we will also write as $f \ggg 0$. The symbol $\star$ stands for the convolution.

Let $\ell > 0$ be arbitrary. We introduce the extremal quantities

$$G(\ell) := \sup_f \frac{\int_{-\ell}^{\ell} f\,dx}{\int_{-1}^{1} f\,dx}, \quad C(\ell) := \sup_f \sup_{a\in\mathbb{R}} \frac{\int_{a-\ell}^{a+\ell} f\,dx}{\int_{-1}^{1} f\,dx},$$

where the supremum is taken over all not identically zero nonnegative positive definite functions. We investigate how large these extremal quantities can be.

The problem formulated above was originally posed by S. Konyagin and Yu. Shteinikov, who wanted to use the estimate (in the case $\ell = 2$) for a problem in number theory in [14]. In all our investigations, we take the liberty to discuss only the continuous case. We believe that the transfer between the discrete and continuous settings should not cause any difficulty.

The problem in question is closely related to Wiener's celebrated problem [13, 17], that is, the question whether, in any L^p norm and for any fixed $\delta > 0$, the ratio

$$\frac{\int_{-\pi}^{\pi} f^p(x)\,dx}{\int_{-\delta}^{\delta} f^p(x)\,dx}$$

is bounded for all 2π-periodic positive definite functions f. Clearly, in an appropriate sense this holds for $p = \infty$, as we have $\|f\|_\infty = f(0)$ for any positive definite function. Also, it can be proved by means of the Parseval identity that, for any given $\delta > 0$,

$$\int_{-\pi}^{\pi} f^2\,dx \le \frac{2\pi}{\delta} \int_{-\delta}^{\delta} f^2\,dx$$

whenever $f \in L^\infty$ (say) [13, 17]. This, of course, extends to any even $p = 2m$ powers, since if f is positive definite then so is f^m. However, it is known that for no other exponents $p \notin 2\mathbb{N}$ does such a finite bound hold. The first counterexamples were constructed by Wainger [16], and the strongest ones (with arbitrarily large gaps and only idempotent polynomials in place of $f \in L^p$) can be found in [1].

Furthermore, in the noncompact case of $\mathbb{R}$, any bound between integrals on $[-1, 1]$ and $[-k, k]$ must grow to infinity with the length k as δ is fixed normalised to 1. This is explained in [9] as 'Wiener's property fails with $k \to \infty$'. However, the case is similar for $\delta \to 0$ in the torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ and on the real line $\mathbb{R}$ with $\delta := 1$ and $k \to \infty$. The ratio in the estimate must depend on the ratio of the corresponding intervals. In this sense, both $\mathbb{T}$ and $\mathbb{R}$ behave the same: there is a finite upper bound exactly for $p \in 2\mathbb{N}$ and the bound happens to be linear in the ratio of the compared intervals.

At this point let us note that the brave question under study boldly extends the classical Wiener problem to the case of L^1 where it is known to fail in general. The price we pay is that we restrict to doubly positive functions instead of general positive definite functions. However, this is in fact not a restriction but a *generalisation*. Indeed, for any power $p = 2m$, where Wiener's problem has a positive answer, an estimate can be easily deduced from the current setting if we observe the following: for any $f \gg 0$, trivially $f^{2m} \geq 0$ and by Schur's theorem $f^{2m} \gg 0$, whence $f^{2m} \ggg 0$. Thus the L^1-problem of Konyagin and Shteinikov can be applied to deduce an answer to Wiener's problem even if there is no Parseval identity to help in this approach. In other words, the positive answer in the question of Konyagin and Shteinikov suggests that the positive cases of Wiener's problem are not so intimately connected to Parseval's formula, but the key seems to relate more to double positivity.

The original question of Konyagin and Shteinikov was answered positively in [7] where Gorbachev found the following bound. (In fact, this result was originally formulated for the discrete case.)

THEOREM 1.1 (Gorbachev). *For any $f \ggg 0$ and $L > 0$, we have*

$$\int_{-2L}^{2L} F dx \leq \pi^2 \int_{-L}^{L} F\, dx.$$

In terms of $G(k)$, this result can be reformulated as $G(2) \leq \pi^2$. By iterating the above estimate one can obtain a bound for any interval length ratio ℓ. It happens to be nonlinear, but linear growth might be expected by virtue of the known results in Wiener's problem.

In what follows, we obtain bounds for the whole range of ℓ which are of linear growth. This allows us a direct derivation of the positive answers in Wiener's problem when the exponent is $p = 2m$. For the case $k = 2$, our upper bound is $G(2) \leq 5$ which is somewhat better than π^2.

2. The result

Let $G(k + 0)$ and $C(k + 0)$ be the right limits $\overline{\lim}_{\varepsilon\to 0+} G(k + \varepsilon)$ and $\overline{\lim}_{\varepsilon\to 0+} C(k + \varepsilon)$, respectively. We note that both functions $G(\ell)$ and $C(\ell)$ are nondecreasing in $(0, \infty)$, and that $G(\ell) \leq C(\ell)$; also, $C(\ell) = G(\ell) = 1$ on $(0, 1]$. Our result reads as follows.

THEOREM 2.1. *For the extremal constant functions $G(\ell)$ and $C(\ell)$ taken over doubly positive functions, the following estimates hold.*

(1) Lower bound. *For any $\ell \in \mathbb{R}\backslash\mathbb{N}$, $G(\ell), C(\ell) \geq 2[\ell] + 1$. Moreover, for all $k \in \mathbb{N}$, $G(k) \geq 2k - 1$ and $C(k) \geq 2k$.*

(2) Upper bound. *For any $\ell \geq 1$,*

$$G(\ell) \leq C(\ell) \leq \frac{([2\ell] + 1)([2\ell] + 2)}{2([2\ell] + 1 - \ell)} \leq \lceil 2\ell \rceil + 1. \tag{2.1}$$

(3) Sharpness. *As a consequence of the above, our bounds are exact for $G(k + 0)$ and $C(k + 0)$, that is, $\lim_{\ell\to k+0} G(\ell) = \lim_{\ell\to k+0} C(\ell) = 2k + 1$ for all $k \in \mathbb{N}$.*

We remark that instead of the space of doubly positive functions, we could consider the space of smooth doubly positive functions or only measurable doubly positive functions. However, the constants would not essentially differ.

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The proof of Theorem 2.1 is composed of two lemmas. Before presenting them, let us explain the idea implemented in Lemma 2.2 since the actual formulas may hide it a little.

Our strategy is the following. We consider the so-called periodically extended Dirac delta, that is, $\Phi := \sum_{k=-\infty}^{\infty} \delta_{kp}$. This 'function' is obviously nonnegative and positive definite since it can be regarded as the characteristic function of a group, namely, the discrete group $p\mathbb{Z}$. Here the period p is chosen to be $1+\varepsilon$ in order to minimise the presence of values pz ($z \in \mathbb{Z}$) in the segment $[-1, 1]$, but at the same time make these values as densely occurring in other intervals as possible. It is easy to show that $\int_{-1}^{1} \Phi\,dx = 1$ and that $\int_{-\ell}^{\ell} \Phi\,dx = 2k+1$ and $\int_{-0}^{\ell-0} \Phi\,dx = k+1$, for any given length ℓ with $kp < \ell < (k+1)p$. The only technical matter is to make this construction fit the class of doubly positive functions. We will do this below.

Lemma 2.2. *For all $k \in \mathbb{N}$, we have $G(k+0) \geq 2k+1$. Moreover, $C(k) \geq 2k$.*

Proof. Let us fix $k \in \mathbb{N}$ and $\varepsilon > 0$. We are to estimate $C(k)$ and $G(k+\varepsilon)$ from below.

Let $f_n(x) := \cos^{2n}(\pi x/p)$, where $1 \leq p < 1.1$ and n is assumed to be large enough. Clearly, for all values of the parameters n and p the function f_n is doubly positive and p-periodic. It is easy to see that with any given fixed value of $\delta \in (0, 0.1)$,

$$\frac{\int_{-\delta}^{\delta} f_n(x)\,dx}{\int_{-p/2}^{p/2} f_n(x)\,dx} \to 1 \quad (\text{as } n \to +\infty),$$

that is, the function f_n is concentrated in the segment $(-\delta, \delta)$ with respect to the period (in the limit, when $n \to \infty$). Thus, we see that f_n is concentrated in $\bigcup_{m\in\mathbb{N}} \Omega_m$ where $\Omega_m := (-\delta + mp, \delta + mp)$.

To make estimates for $\int_{-k-\varepsilon}^{k+\varepsilon} f_n(x)\,dx$ for some $0 < \varepsilon < 1$, we have to find how many segments Ω_m are contained in the intervals $[-k-\varepsilon, k+\varepsilon]$. If we choose $\delta < p-1$, then the interval $[-1, 1]$ contains only one Ω_m, namely Ω_0, and is disjoint from the rest.

Now if we take δ small enough and p sufficiently close to 1 (more exactly, if $k(p-1)+\delta < \varepsilon$), then the interval $[-k-\varepsilon, k+\varepsilon]$ already contains all Ω_m with $-k \leq m \leq k$. Thus $G(k+\varepsilon) \geq (2k+1)$, whence also $G(k+0) \geq 2k+1$.

Furthermore, the inequality $C(k) \geq 2k$ can be easily seen from considering, for example, the interval $[1, 2k+1]$ which contains Ω_m for all $m = 1, \ldots, 2k$ whenever $2k(p-1)+\delta < 1$. □

As the functions $G(\ell)$ and $C(\ell)$ are both nondecreasing, and $C(\ell) \geq G(\ell)$, we obtain the first lower bound of Theorem 2.1. As for $G(k)$, clearly we have $G(1) = 1$, while for $k > 1$ we can easily use the monotonicity of G to derive $G(k) \geq G(k-1+0) \geq 2k-1$. The other estimate $C(k) \geq 2k$ is contained in the above lemma, whence part (1) of Theorem 2.1 is proved.

Lemma 2.3. *For any $\ell > 1$,*

$$C(\ell) \leq \frac{([2\ell]+1)([2\ell]+2)}{2([2\ell]+1-\ell)}. \tag{2.2}$$

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Proof. Fix the interval $I := [-1/2, 1/2]$. For temporary use let us denote by $\chi := \chi_I$ the characteristic function of I, and write $\chi_a(x) := \chi(x-a)$ for indices $a \in \mathbb{R}$. We shall also use the triangle function $T := \chi \star \chi = (1-|x|)_+$ (where $\xi_+ := \max(\xi, 0)$) which is an important example of a nonnegative positive definite function.

Consider the functions $g_a := \chi - \chi_a$ and $h_a := g_a \star \widetilde{g_a}$, where $\widetilde{g_a}(x) := \overline{g_a(-x)}$. Since g_a is a real-valued function, $\widetilde{g_a}(x) = g_a(-x) = \chi(x) - \chi_{-a}(x)$. Then, obviously, $h_a(x) = T(x) - T(x+a) - T(x-a) + T(x) = 2T(x) - (T(x+a) + T(x-a))$. Because h is defined as a convolution square, it is obvious that $h_a \gg 0$ for any $a \in \mathbb{R}$.

Let us introduce here an additional parameter p with $0 < p \le 1$. Take the sum $H_{a,k,p} := \sum_{j=0}^{k} h_{a+j(2-p)}$ with some $k \in \mathbb{N}$. Then we can estimate $H_{a,k,p}$ as follows:

$$\begin{aligned} H_{a,k,p}(x) &:= \sum_{j=0}^{k} h_{a+j(2-p)}(x) \\ &= 2(k+1)T(x) - \bigg(\sum_{j=0}^{k} T(x+a+j(2-p)) \ + \ \sum_{j=0}^{k} T(x-a-j(2-p)) \bigg) \\ &\le 2(k+1)\chi_{[-1,1]}(x) - p(\chi_{[a-1+p,a+k(2-p)+1-p]}(x) + \chi_{[-a-k(2-p)-1+p,1-a-p]}(x)). \end{aligned}$$

Note that $H \gg 0$ together with its summands $h_{a+j(2-p)}$. Multiplying by any (say, continuous) doubly positive function f and applying Schur's theorem, we get $Hf \gg 0$, whence

$$0 \le \widehat{Hf}(0) = \int_{-\infty}^{\infty} Hf \le 2(k+1) \int_{-1}^{1} f(x)\,dx - 2p \int_{a+p-1}^{a+k(2-p)+1-p} f(x)\,dx,$$

because f, as a positive definite real-valued function, is necessarily even.

Now let $b := a + p - 1$. It follows that

$$p \int_{b}^{b+k(2-p)+2-2p} f(x)\,dx \le (k+1) \int_{-1}^{1} f(x)\,dx.$$

That is, in terms of $C(\ell)$,

$$C\bigg(\frac{(k+1)(2-p)-p}{2}\bigg) \le \frac{k+1}{p}.$$

Take an arbitrary $\ell > 1$, and write it in the form $\ell = \frac{1}{2}((k+1)(2-p)-p)$ with suitable values of k and p. This is equivalent to $p = (2(k+1) - 2\ell)/(k+2)$. Further, the double inequality $0 < p \le 1$ is equivalent to $\ell - 1 < k \le 2\ell$. Consequently,

$$C(\ell) \le \frac{(k+1)(k+2)}{2(k+1-\ell)} \tag{2.3}$$

for any integer k between ℓ and 2ℓ (and with the appropriate choice of p). So it remains to minimise the estimate (2.3) in the range $k \in (\ell - 1, 2\ell]$ for any fixed $\ell > 1$.

In order to do so, consider the auxiliary functions

$$\varphi : (\ell - 1, 2\ell] \to \mathbb{R}, \quad \varphi(x) := \frac{(x+1)(x+2)}{x+1-\ell} = x + \ell + 2 + \frac{\ell(\ell+1)}{x+1-\ell}$$

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and

$$\psi : (\ell, 2\ell] \to \mathbb{R}, \quad \psi(x) := \varphi(x) - \varphi(x-1).$$

Straightforward computation shows that $\psi(x) = 0$ is equivalent to

$$\frac{(x+1)(x+2)}{x+1-\ell} = \frac{x(x+1)}{x-\ell},$$

the unique solution of which on the interval $(\ell, 2\ell]$ is $x = 2\ell$. Since ψ is continuous we deduce that the sign of the function ψ does not vary on the interior of its domain. Further, as $\ell > 1$,

$$\psi(\ell+1) = \varphi(\ell+1) - \varphi(\ell) = 1 - \frac{\ell(\ell+1)}{2} < 0.$$

Thus the function ψ is negative in $(\ell, 2\ell)$, whence φ is nonincreasing on the set of integers in $(\ell-1, 2\ell]$. Therefore, the minimum of φ is certainly achieved at the unique integer in $(2\ell-1, 2\ell]$, in other words at $[2\ell]$. We substitute $k = [2\ell]$ in (2.3), which indeed yields the desired inequality (2.2). □

The last inequality in (2.1) can be obtained easily by considering separately the cases where $2\ell = m \in \mathbb{N}$ (providing equality), and where $2\ell \notin \mathbb{N}$ (leading to strict inequality).

As in the above argument concerning the lower bound, the proof of the upper estimate follows because $G(\ell) \le C(\ell)$. This completes part (2) of Theorem 2.1.

Finally, we are in a position to prove part (3), that is, the sharpness statement. The inequality $G(k+0) \ge 2k+1$ is clear, because the lower estimate in part (1) provides $G(k+\varepsilon) \ge [2(k+\varepsilon)] + 1 = 2k+1$ for arbitrary $\varepsilon > 0$. Moreover, from Lemma 2.3,

$$\begin{aligned} C(k+0) = \lim_{\varepsilon \to +0} C(k+\varepsilon) &\le \frac{([2k+2\varepsilon]+1)([2k+2\varepsilon]+2)}{2([2k+2\varepsilon]+1-(k+\varepsilon))} \\ &= \lim_{\varepsilon \to +0} \frac{(2k+1)(2k+2)}{2(2k+1-(k+\varepsilon))} \le 2k+1. \end{aligned}$$

Altogether, we have $2k+1 \le G(k+0) \le C(k+0) \le 2k+1$ and equality holds everywhere, as needed.

Therefore, the proof of Theorem 2.1 is complete.

3. Concluding remarks

Let us see what general framework for the construction of the above estimates and proofs can be set up. Basically, what we do is to look for an auxiliary function H, positive definite itself, and satisfying

$$H \le A\chi_{[-1,1]} - B\chi_{[a,a+\ell]} - B\chi_{[-a-\ell,-a]}$$

or

$$H \le A\chi_{[-1,1]} - B\chi_{[-\ell,\ell]}.$$

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By Schur's theorem, we also have $fH \gg 0$ for any $f \gg 0$, whence

$$0 \le \widehat{fH}(0) = \int_{-\infty}^{\infty} fH \le A \int_{-1}^{1} f - 2B \int_{a}^{a+\ell} f$$

and $C(\ell/2) \le A/2B$, or

$$0 \le \widehat{fH}(0) = \int_{-\infty}^{\infty} fH \le A \int_{-1}^{1} f - B \int_{-\ell}^{\ell} f$$

and $G(\ell) \le A/B$.

Let $\ell > 0$ be arbitrary. Let us now consider the extremal quantities

$$\begin{aligned}
\sigma(a,\ell) &:= \inf\{A/2B : \exists H \gg 0, H \le A\chi_{[-1,1]} - B\chi_{[a,a+\ell]} - B\chi_{[-a-\ell,-a]}\},\\
\sigma(\ell) &:= \sup_{a\in\mathbb{R}} \sigma(a,\ell),\\
\gamma(\ell) &:= 2\sigma(0,\ell) = \inf\{A/B : \exists H \gg 0, H \le A\chi_{[-1,1]} - B\chi_{[-\ell,\ell]}\}.
\end{aligned}$$

Clearly, from the above remarks, $C(\ell/2) \le \sigma(\ell)$ and $G(\ell) \le \gamma(\ell)$, always. Let us make a few additional remarks here. First, the setting here is quite general, but at least in $\mathbb{R}$ any 'reasonable' positive definite function H can be represented as a 'convolution square', say, $H = G \star \widetilde{G}$ of some function $G \in L^2$ (see, for example, [5]). The construction of $H := H_{a,k,p}$ worked along somewhat different lines, for we instead represented H as the sum of other convolution squares with well-controlled supports, but in principle the direct convolution square representation is also possible.

The above-defined extremal problems σ and γ are very much like the so-called Turán or Delsarte extremal problems. The main difference is that here we want to compare integrals over given intervals to integrals over given central pieces, while in the Turán and Delsarte problems we normalise with respect to $f(0)$ and compare to this normalisation either *the full integral*, or (in the case of the so-called 'pointwise Turán problem') *a particular one-point value*. In the recent work [9], more concrete applications of the Turán and Delsarte problems are worked out for the case of Wiener's problem in several dimensions. This type of approach seems to be reasonable here, too.

Since for $H \gg 0$ we necessarily have $0 < \widehat{H}(0) = \int H$, it follows immediately that $A \ge B\ell$, whence $\gamma(\ell) \ge \ell$ and $\sigma(\ell) \ge \ell/2$. But in view of our lower estimations of $C(\ell)$ and $G(\ell)$, it is apparent that these are far from being sharp. From the other side, it could well be that we have $\sigma(\ell) = C(\ell/2)$ and $\gamma(\ell) = G(\ell)$. The essential part of the above constructions (that is, the ones for the upper estimation) targeted the computation (or estimation) of $\gamma(\ell)$ and $\sigma(\ell)$. We conjecture that in principle this approach is best possible.

For further details about the Turán and Delsarte problems and their applications in, for example, packing problems, see [2–4, 6, 10–12, 15].

Very recently, Gorbachev and Tikhonov generalised the original extremal questions to comparison of integrals of doubly positive definite functions on general open sets of $\mathbb{R}^n$; see [8].

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