

EXISTENCE OF SEAMOUNT STEADY VORTEX FLOWS

B. EMAMIZADEH¹ and F. BAHRAMI^{2 3}

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Abstract

In this paper we will study a feature of a localised topographic flow. We will prove existence of an ideal fluid containing a bounded vortex, approaching a uniform flow at infinity and passing over a localised seamount. The domain of the fluid is the upper half-plane and the data prescribed is the rearrangement class of the vorticity field.

1. Introduction

In this paper we will study a localised topographic feature on the half-plane $\Pi = \{x_2 > 0\}$. More precisely, we prove the existence of an ideal fluid flow containing a bounded vortex and approaching a uniform flow $-\lambda x_2$ at infinity, passing over a localised seamount which is represented by a compactly supported nonnegative function h . The data prescribed is the rearrangement class of the vorticity field.

It is well known that steady flows can be thought of as critical points of energy. Therefore, to reach our goal we set up an appropriate energy functional Ψ_λ and look for elements in a given rearrangement class, \mathcal{F} , which maximise Ψ_λ . To do this, we use a standard rescaling to convert the energy Ψ_λ into a new parametrised energy functional $\hat{\Psi}_c$. We then look for the critical points for $\hat{\Psi}_c$ with respect to the rescaled class of rearrangements \mathcal{F}_c . The variational problems to be considered suffer two technical difficulties; firstly the awkward nature of the set of rearrangements (as a set in L^p) and secondly a loss of compactness arising from the unbounded domain Π . To overcome these difficulties, we first solve the problem on a bounded domain using Burton's results [3] on rearrangements. Passage to the unbounded domain is

¹Department of Mathematics, The Petroleum Institute, P.O. Box 2533, Abu Dhabi, UAE;
e-mail: bemamizadeh@pi.ac.ae.

²Department of Mathematics, Tarbiat Modarres University, P.O. Box 14155-4838, Tehran, Iran.

³Current address: Department of Mathematics, Tabriz University, Tabriz, Iran;
e-mail: fbahrami@tabrizu.ac.ir.

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accomplished by deriving estimates to show (as anticipated in [2]) that a solution valid in a sufficiently large bounded domain is in fact valid in Π .

Similar rearrangement variational approaches have been extensively used recently. The reader could refer to [5–9].

2. Notation, definitions and statement of the main result

We denote by p an arbitrary fixed number in $(2, +\infty)$. For any number $r \geq 1$, r^* denotes the conjugate of r ; that is, $1/r + 1/r^* = 1$. Let Π denote the upper half-plane

$$\Pi = \{x = (x_1, x_2) \mid x_2 > 0\},$$

and for $\xi > 0$ we set

$$\Pi(\xi) = \{x \in \Pi \mid |x_1| < \xi, x_2 < \xi\}.$$

The open disc centred at x with radius R is denoted by $B_R(x)$. In this paper we denote the Green’s function for Δ with homogeneous Dirichlet boundary conditions on Π by G . It is well known that for $x, y \in \mathbb{R}^2$,

$$G(x, y) = \frac{1}{2\pi} \log \frac{|x - \tilde{y}|}{|x - y|}, \tag{2.1}$$

where $\tilde{\cdot}$ denotes reflection in the x_1 -axis. Note that G is positive, $G(x, y) = G(y, x)$ and

$$G(x, y) = \frac{1}{4\pi} \log \left(1 + \frac{4x_2y_2}{|x - y|^2} \right). \tag{2.2}$$

For a measurable function ζ on Π and $x \in \mathbb{R}^2$, we define

$$T\zeta(x) = \int_{\Pi} G(x, y)\zeta(y) dy, \tag{2.3}$$

whenever the integral exists. For a measurable set $A \subseteq \mathbb{R}^2$, we denote the two-dimensional Lebesgue measure by $|A|$ and the *essential diameter* of the set A by

$$\text{essdiam}(A) = \inf\{l \mid A = A' \cup N \text{ for some } A' \subseteq \mathbb{R}^2 \text{ with } \text{diam}(A') = l, |N| = 0\},$$

where $\text{diam}(A') = \sup\{|x - y| \mid x, y \in A'\}$.

For a measurable function ζ on Π , the *strong support* of ζ , denoted $\text{supp}(\zeta)$, is the set $\{x \in \Pi \mid \zeta(x) > 0\}$. Let us fix $\zeta_0 \in L^p(\Pi)$ which is a nonnegative, non-trivial function with compact support and assume $|\text{supp}(\zeta_0)| = \pi a^2$, for some $a > 0$.

Moreover we suppose that $\|\zeta_0\|_1 = 1$. By \mathcal{F} we denote the set of rearrangements of ζ_0 on Π which have compact support. By $\mathcal{F}(\xi)$ we denote the subset of \mathcal{F} comprising functions vanishing outside $\Pi(\xi)$. Let us recall that ζ is a rearrangement of ζ_0 whenever $|\{x \in \Pi \mid \zeta(x) \geq \alpha\}| = |\{x \in \Pi \mid \zeta_0(x) \geq \alpha\}|$, for every $\alpha \in \mathbb{R}$.

For a measurable function ζ on Π , we define the *energy* functional

$$\Psi_\lambda(\zeta) = \frac{1}{2} \int_\Pi \zeta T\zeta + \int_\Pi \eta\zeta - \lambda \int_\Pi x_2\zeta, \tag{2.4}$$

whenever the integrals exist, where λ is a positive fixed number. In (2.4), $\eta = Th$, where $h \in L^p(\Pi)$ is a nonnegative function with compact support. To simplify future calculations we assume that $\|h\|_1 = 1$.

We now define the variational problem

$$P_\lambda : \sup_{\zeta \in \mathcal{F}} \Psi_\lambda(\zeta) \tag{2.5}$$

and the corresponding solution set is denoted Σ_λ . In order to introduce the second variational problem which is a *rescaled* version of P_λ we need first some preparation. For this purpose fix $c > 0$ and let ζ be a measurable function on Π . We define

$$\mathcal{C}\zeta(x) = c^2\zeta(cx), \quad x \in \Pi. \tag{2.6}$$

By \mathcal{F}_c we denote the set of all rearrangements of $\mathcal{C}\zeta_0$ on Π with compact support. For a measurable function ζ on Π , $c > 0$, we define

$$\hat{\Psi}_c(\zeta) = \frac{1}{2} \int_\Pi \zeta T\zeta + \int_\Pi \eta^c\zeta - \int_\Pi x_2\zeta, \tag{2.7}$$

whenever the integrals exist; here $\eta^c = T(\mathcal{C}h)$. Now we define the rescaled variational problem. For $c > 0$, we set

$$\hat{P}_c : \sup_{\zeta \in \mathcal{F}_c} \hat{\Psi}_c(\zeta), \tag{2.8}$$

and the corresponding solution set is denoted by $\hat{\Sigma}_c$. We also need the following truncated variational problem:

$$\hat{P}_{c,\xi} : \sup_{\zeta \in \mathcal{F}_{c,\xi}} \hat{\Psi}_c(\zeta), \tag{2.9}$$

where $\mathcal{F}_{c,\xi}$ is the subset of \mathcal{F}_c comprising functions vanishing outside $\Pi(\xi)$. In order to guarantee $\mathcal{F}_{c,\xi}$ is not empty, henceforth we assume $\xi > 2a$, $c \geq 1$. The solution set for $\hat{P}_{c,\xi}$ is denoted $\hat{\Sigma}_{c,\xi}$. The main result of this paper is the following theorem.

THEOREM 2.1. *There exists $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0)$, P_λ has a solution. Moreover if $\zeta_\lambda \in \Sigma_\lambda$ and $\psi_\lambda = T\zeta_\lambda + \eta - \lambda x_2$, then ψ_λ satisfies the semilinear elliptic partial differential equation*

$$-\Delta \psi_\lambda = \varphi_\lambda \circ \psi_\lambda + h \tag{2.10}$$

almost everywhere in Π , where φ_λ is an increasing function unknown a priori.

The steady barotropic vorticity equation is given by

$$[\psi, \omega + h] = 0, \tag{2.11}$$

where ψ is the stream function, ω is the vorticity and h is the height of the bottom topography; the symbol $[\cdot, \cdot]$ denotes the Jacobian. A weak formulation of (2.11) is given by

$$\int_{\Pi} (\omega + h) [\psi, u] = 0, \tag{2.12}$$

for all $u \in C_0^\infty(\Pi)$, see for example [13, 14]. Writing (2.12) in the context of Theorem 2.1 we obtain

$$\int_{\Pi} (\zeta_\lambda + h) [T\zeta_\lambda - \lambda x_2, u] = 0, \tag{2.13}$$

for all $u \in C_0^\infty(\Pi)$. At the end of this paper we will prove the validity of (2.13).

3. Preliminaries

In this section we present some lemmas which are crucial in our analysis.

LEMMA 3.1. *Let $0 < a < \infty$ and $1 < r < \infty$. Then there are positive constants M_1, M_2, M_3 and β , with $\beta < 1$, such that if $\zeta \in L^r(\Pi)$ and ζ vanishes outside a set of area πa^2 , then for $x \in \mathbb{R}^2$ we have*

$$|T\zeta(x)| \leq \begin{cases} (M_1 + M_2 \log |x_2|) \|\zeta\|_r, & |x_2| \geq a; \\ M_3 |x_2|^\beta \|\zeta\|_r, & |x_2| \leq a. \end{cases} \tag{3.1}$$

PROOF. See [4]. □

LEMMA 3.2. (i) *Suppose $0 < a < \infty$, then for any $\zeta \in L^p(\Pi)$ vanishing outside a set of area πa^2 we have $T\zeta \in C^1(\mathbb{R}^2)$ and $|T\zeta(x)| \leq N |x_2| \|\zeta\|_p$, for all $x \in \mathbb{R}^2$, where N is a constant depending only on p and a .*

(ii) Let $1 \leq q < \infty$ and U be a bounded open subset of Π . Then $T : L^p(U) \rightarrow L^q(U)$ is compact, in the sense that if ζ_n is a sequence of functions, bounded in $L^p(\Pi)$ and vanishing outside U , then the sequence $T\zeta_n|_U$ has a subsequence converging in the q -norm. Moreover, if $\zeta \in L^p(\Pi)$ vanishes outside U , then $T\zeta \in W_{loc}^{2,p}(\Pi)$ and verifies the Poisson equation

$$-\Delta u = \zeta \tag{3.2}$$

almost everywhere in Π .

(iii) Suppose $\zeta \in L^p(\Pi)$ has bounded support, then $\nabla T\zeta(x) = O(|x|^{-2})$, $T\zeta(x) = O(|x|^{-1})$ as $|x| \rightarrow +\infty$ and $\int_{\Pi} |\nabla T\zeta|^2 = \int_{\Pi} \zeta T\zeta < \infty$.

PROOF. See [4]. □

LEMMA 3.3. Suppose $\Phi : L^p(\Pi(\xi)) \rightarrow \mathbb{R}$ is a weakly sequentially continuous, strictly convex functional, then Φ attains a maximum relative to $\zeta \in \mathcal{F}(\xi)$, for $\xi > 2a$. Moreover, if ζ is a maximiser and $\psi \in \partial\Phi(\zeta)$, the subdifferential of Φ at ζ , then $\zeta = \varphi \circ \psi$ almost everywhere in $\Pi(\xi)$, for some increasing function φ .

PROOF. See [3]. □

LEMMA 3.4. The problem $\hat{P}_{c,\xi}$ is solvable; moreover, if $\hat{\zeta}_{c,\xi} \in \hat{\Sigma}_{c,\xi}$, then

$$\hat{\zeta}_{c,\xi} = \hat{\varphi}_{c,\xi} \circ (T\hat{\zeta}_{c,\xi} + \eta^c - x_2)$$

almost everywhere in $\Pi(\xi)$ for some increasing function $\hat{\varphi}_{c,\xi}$.

PROOF. Let us begin by noting that $T : L^p(\Pi(\xi)) \rightarrow L^{p'}(\Pi(\xi))$ satisfies the conditions

$$\int_{\Pi} vTw = \int_{\Pi} wTv, \quad \int_{\Pi} vTv > 0,$$

for every non-trivial $v, w \in L^p(\Pi(\xi))$, which readily follow from the symmetry of G and Lemma 3.2 (iii). Therefore T is strictly positive, symmetric and by Lemma 3.2 compact. It then follows that $\hat{\Psi}_c$ on $L^p(\Pi(\xi))$ is strictly convex and weakly sequentially continuous. Now by applying Lemma 3.3 we conclude that $\hat{P}_{c,\xi}$ is solvable. Note that $\hat{\Psi}_c$ is differentiable and its Gâteaux derivative at any ζ can be identified with $T\zeta + \eta^c - x_2$. Therefore if $\hat{\zeta}_{c,\xi} \in \hat{\Sigma}_{c,\xi}$, then by Lemma 3.3 there exists an increasing function $\hat{\varphi}_{c,\xi}$ such that

$$\hat{\zeta}_{c,\xi} = \hat{\varphi}_{c,\xi} \circ (T\hat{\zeta}_{c,\xi} + \eta^c - x_2)$$

almost everywhere in $\Pi(\xi)$. □

The following lemma is straightforward.

LEMMA 3.5. For $c > 0$, let \mathcal{C} be defined as in (2.6). Then $\mathcal{C} : \mathcal{F} \rightarrow \mathcal{F}_c$ is a bijection; moreover, if $\zeta \in \mathcal{F}$, then

- (i) $\|\mathcal{C}\zeta\|_q = c^{2/q} \|\zeta\|_q, 1 \leq q \leq \infty,$
- (ii) $|\text{supp}(\mathcal{C}\zeta)| = c^{-2} |\text{supp}(\zeta)|,$
- (iii) $\mathcal{C}^{-1}\zeta(x) = (1/c^2)\zeta(x/c).$

LEMMA 3.6. There exists $c_1 > 0$ such that if $c > c_1, \xi > 3a$, then for $\hat{\zeta}_{c,\xi} \in \hat{\Sigma}_{c,\xi}$, we have

$$\hat{\Psi}_c(\hat{\zeta}_{c,\xi}) \geq \frac{1}{4\pi} \log \frac{c}{2a} + C_1, \tag{3.3}$$

where C_1 is a constant independent of c, ξ .

PROOF. Let ζ^* denote the Schwartz-symmetrisation of $\hat{\zeta}_{c,\xi}$ with respect to $x_0 = (0, 2a)$. Then $\text{supp}(\zeta^*) = B_{a/c}(x_0) \subseteq \Pi(\xi)$; thus $\hat{\Psi}_c(\hat{\zeta}_{c,\xi}) \geq \hat{\Psi}_c(\zeta^*)$. We now proceed to find a lower bound for $\hat{\Psi}_c(\zeta^*)$. From the definition of $\hat{\Psi}_c$ we have

$$\begin{aligned} \hat{\Psi}_c(\zeta_{c,\xi}^*) &\geq \frac{1}{4\pi} \int_{B_{a/c}(x_0)} \int_{B_{a/c}(x_0)} \log \frac{|x - \bar{y}|}{|x - y|} \zeta^*(x)\zeta^*(y) dx dy - \int_{B_{a/c}(x_0)} x_2 \zeta^* \\ &= I_1 - I_2. \end{aligned}$$

We now estimate I_1 as follows:

$$\begin{aligned} I_1 &\geq \frac{1}{4\pi} \log \frac{c}{2a} + \frac{1}{4\pi} \log(4a) \\ &\quad + \frac{1}{4\pi} \int_{B_{a/c}(x_0)} \int_{B_{a/c}(x_0)} \log \frac{|x - \bar{y}|}{4a} \zeta^*(x)\zeta^*(y) dx dy, \end{aligned} \tag{3.4}$$

where we have used $\|\zeta^*\|_1 = 1$. Now we show the limit of the integral in (3.4) is zero as c tends to infinity:

$$\begin{aligned} &\left| \frac{1}{4\pi} \int_{B_{a/c}(x_0)} \int_{B_{a/c}(x_0)} \log \frac{|x - \bar{y}|}{4a} \zeta^*(x)\zeta^*(y) dx dy \right| \\ &\leq \sup_{x,y \in B_{a/c}(x_0)} \frac{1}{4\pi} \left| \log \frac{|x - \bar{y}|}{4a} \right| \xrightarrow{c \rightarrow +\infty} 0. \end{aligned}$$

Next we estimate I_2 . Note that $I_2 = 2a + \int_{B_{a/c}(x_0)} (x_2 - 2a)\zeta^*$. Hence

$$\left| \int_{B_{a/c}(x_0)} (x_2 - 2a)\zeta^* \right| \leq \sup_{x \in B_{a/c}(x_0)} |x_2 - 2a| \xrightarrow{c \rightarrow +\infty} 0.$$

Therefore

$$\hat{\Psi}_c(\zeta^*) \geq \frac{1}{4\pi} \log \frac{c}{2a} + \frac{1}{4\pi} \log(4a) - 2a + o(1), \quad c \rightarrow +\infty.$$

This clearly verifies (3.3). □

Let us assume that $c \geq c_1$ and $\xi \geq 3a$. From Lemma 3.4 it follows that $\hat{P}_{c,\xi}$ is solvable and if $\hat{\zeta}_{c,\xi} \in \hat{\Sigma}_{c,\xi}$, then there exists an increasing function $\hat{\varphi}_{c,\xi}$ such that $\hat{\zeta}_{c,\xi} = \hat{\varphi}_{c,\xi} \circ (T\hat{\zeta}_{c,\xi} + \eta^c - x_2)$ almost everywhere in $\Pi(\xi)$. From this it follows that

$$\text{supp}(\hat{\zeta}_{c,\xi}) = \{x \in \Pi(\xi) \mid T\hat{\zeta}_{c,\xi} + \eta^c - x_2 \geq \gamma_{c,\xi}\} \tag{3.5}$$

for some constant $\gamma_{c,\xi}$, modulo a set of measure zero. Note that the inequality in (3.5) can be changed to a strict inequality, since the level sets of $T\hat{\zeta}_{c,\xi} + \eta^c - x_2$ (sets on which $T\hat{\zeta}_{c,\xi} + \eta^c - x_2$ is constant) on $\text{supp}(\hat{\zeta}_{c,\xi})$ have measure zero, see [10, Lemma 7.7].

In Lemma 3.7 we derive a lower bound for $\gamma_{c,\xi}$ when c and ξ are sufficiently large.

LEMMA 3.7. *There exist $c_2 > 0$ and $\xi_2 > 0$ such that if $c \geq c_2$ and $\xi \geq \xi_2$, then*

$$\gamma_{c,\xi} \geq \frac{1}{2\pi} \log \frac{c}{2a} + C_2, \tag{3.6}$$

where C_2 is a constant independent of c and ξ .

PROOF. If $\gamma_{c,\xi} < 0$ then for every $x \in \Pi(\xi)$, with $0 < x_2 < |\gamma_{c,\xi}|$, we have

$$T\hat{\zeta}_{c,\xi} + \eta^c - x_2 > T\hat{\zeta}_{c,\xi} + \eta^c - |\gamma_{c,\xi}| \geq \gamma_{c,\xi}.$$

By considering the area of $\text{supp}(\hat{\zeta}_{c,\xi})$ it follows that $(-2\xi\gamma_{c,\xi}) \leq \pi a^2/c^2$, then there exist c'_2 and ξ'_2 such that $\gamma_{c,\xi} \geq -1/2$, for all $c \geq c'_2$ and $\xi \geq \xi'_2$. Now we consider

$$\phi(\hat{\zeta}_{c,\xi}) = \frac{1}{2} \int_{\Pi} (T\hat{\zeta}_{c,\xi} - x_2 - \gamma_{c,\xi}) \hat{\zeta}_{c,\xi}.$$

We then have

$$\phi(\hat{\zeta}_{c,\xi}) \leq \frac{1}{2} \int_{\Pi(\xi)} u^+ \hat{\zeta}_{c,\xi} + \frac{1}{2} \int_{\Pi(\xi)} \hat{\zeta}_{c,\xi}, \tag{3.7}$$

where $u = T\hat{\zeta}_{c,\xi} - x_2 - \gamma_{c,\xi} - 1$ and $u^+ = \max\{u, 0\}$.

By Lemma 3.2 (iii), $T\hat{\zeta}_{c,\xi}(x) \rightarrow 0$ as $|x| \rightarrow +\infty$, since $\gamma_{c,\xi} \geq -1/2$. Then $u^+ \in H_0^1(\Pi(M))$ for some $M \equiv M_{c,\xi} \geq \xi$. From [10, Lemma 7.6] and the divergence theorem, see for example [11, Theorem 1.5.1], we obtain

$$\begin{aligned} \int_{\Pi(\xi)} |\nabla u^+|^2 &\leq \int_{\Pi(M)} |\nabla u^+|^2 = \int_{\Pi(M)} \nabla u^+ \cdot \nabla u = \int_{\Pi(M)} u^+ \hat{\zeta}_{c,\xi} = \int_{\Pi(\xi)} u^+ \hat{\zeta}_{c,\xi} \\ &\leq \|\hat{\zeta}_{c,\xi}\|_2 \left(\int_{\Pi(\xi)} |u^+|^2 \right)^{1/2} \leq c \|\hat{\zeta}_0\|_2 \left(\int_{\Pi(\xi)} |u^+|^2 \right)^{1/2}. \end{aligned} \tag{3.8}$$

From the continuous embedding $W^{1,1}(\Pi(\xi)) \rightarrow L^2(\Pi(\xi))$, see [1, Lemma 5.14], we deduce

$$\left(\int_{\Pi(\xi)} |u^+|^2\right)^{1/2} \leq K \left(\int_{\Pi(\xi)} |u^+| + |u_{x_1}^+| + |u_{x_2}^+|\right), \tag{3.9}$$

where K is the constant depending on the cone determining the cone property of $\Pi(\xi)$. Let us point out that K is independent of c and ξ . Note that the sets $\{x \in \Pi(\xi), u^+(x) > 0\}$ and $\{x \in \Pi(\xi), |\nabla u^+| > 0\}$ are both contained in $\text{supp}(\hat{\zeta}_{c,\xi})$ except for a set of measure zero. This implies

$$\left(\int_{\Pi(\xi)} |u^+|^2\right)^{1/2} \leq K \left(\frac{\pi a^2}{c^2}\right)^{1/2} \left(\int_{\Pi(\xi)} |u^+|^2\right)^{1/2} + K \left(\int_{\Pi(\xi)} |u_{x_1}^+| + |u_{x_2}^+|\right).$$

If $c > 2Ka\sqrt{\pi}$ then

$$\left(\int_{\Pi(\xi)} |u^+|^2\right)^{1/2} \leq 4K \left(\frac{\pi a^2}{c^2}\right)^{1/2} \left(\int_{\Pi(\xi)} |u_{x_1}^+|^2 + |u_{x_2}^+|^2\right)^{1/2}. \tag{3.10}$$

Combining (3.8) and (3.10), we infer

$$\left(\int_{\Pi(\xi)} |\nabla u^+|^2\right)^{1/2} \leq 4K\sqrt{\pi}a\|\zeta_0\|_2. \tag{3.11}$$

From (3.8), (3.10) and (3.11) we obtain

$$\int_{\Pi(\xi)} u^+ \hat{\zeta}_{c,\xi} \leq \beta,$$

where $\beta = 16K^2\pi a^2\|\zeta_0\|_2^2$ for $c \geq c'_2 = \max\{2Ka\sqrt{\pi}, c'_2\}$, $\xi \geq \xi'_2$. Hence

$$\phi(\hat{\zeta}_{c,\xi}) \leq (\beta + 1)/2. \tag{3.12}$$

From Lemma 3.2 (i) and (iii) it follows that η is bounded from above so if we set $\eta_\infty = \sup_\Pi \eta$, then

$$\eta^c(x) = \eta(cx) \leq \eta_\infty, \quad x \in \Pi. \tag{3.13}$$

Observe that

$$\gamma_{c,\xi} = 2\hat{\Psi}_c(\hat{\zeta}_{c,\xi}) - 2\phi(\hat{\zeta}_{c,\xi}) + \int_\Pi x_2 \hat{\zeta}_{c,\xi} - 2 \int_\Pi \eta^c \hat{\zeta}_{c,\xi}.$$

By Lemma 3.6, (3.12) and (3.13) there exists $c_2 \geq \max\{c'_2, c_1\}$ such that

$$\gamma_{c,\xi} \geq \frac{1}{2\pi} \log \frac{c}{2a} + C_2,$$

for all $c \geq c_2$ and $\xi \geq \xi_2$, where C_2 is a constant independent of c and ξ . □

LEMMA 3.8. *There exist $c_3 > 0$ and $\xi_3 > 0$ such that*

$$\text{supp}(\hat{\zeta}_{c,\xi}) \subseteq \{x \in \Pi \mid x_2 \leq H(c)\}, \quad \xi \geq \xi_3, \quad c \geq c_3, \tag{3.14}$$

except for a set of measure zero, where $H(c)$ is a positive constant which depends only on c .

PROOF. Let c_2 and ξ_2 be as in Lemma 3.7. Let c'_3 be a positive constant such that for $c \geq c'_3$,

$$\frac{1}{2\pi} \log \frac{c}{2a} + C_2 > 1 + \eta_\infty.$$

Fix $c \geq c_3 = \max\{c_2, c'_3\}$ and $\xi \geq \xi_2$. From (3.5)

$$\text{supp}(\hat{\zeta}_{c,\xi}) \subseteq \{x \in \Pi(\xi) \mid T\hat{\zeta}_{c,\xi} + \eta^c - x_2 \geq 1 + \eta_\infty\},$$

except for a set of measure zero. From Lemmas 3.1, 3.5 and (3.13) it follows that for $x \in \text{supp}(\hat{\zeta}_{c,\xi})$ we have $(M_1 + M_2 \log(x_2))(c^{2/p^*} \|\zeta_0\|_p) - x_2 \geq 1$, provided $x_2 \geq a/c$, so there exists a number $H(c)$, independent of ξ such that $0 < x_2 < H(c)$ for all $x \in \text{supp}(\hat{\zeta}_{c,\xi})$ except for a set of measure zero. \square

In the next lemma we derive an upper bound for the essential diameter of $\text{supp}(\hat{\zeta}_{c,\xi})$, when c and ξ are sufficiently large.

LEMMA 3.9. *There exist $\xi_4 > 0$ and $c_4 > 0$ such that*

$$\text{essdiam}(\text{supp}(\hat{\zeta}_{c,\xi})) \leq \frac{\sigma \xi a}{c}, \quad \xi \geq \xi_4, \quad c \geq c_4, \tag{3.15}$$

where σ is a positive constant independent of c and ξ .

PROOF. Let c_2 and ξ_2 be as in Lemma 3.7. Then for $c > c_2$ and $\xi > \xi_2$ we have

$$\gamma_{c,\xi} \geq \frac{1}{2\pi} \log \frac{c}{2a} + C_2. \tag{3.16}$$

Let $R \geq 1$. For $x \in \text{supp}(\hat{\zeta}_{c,\xi})$ let $B(x, \xi) = \{y \in \Pi(\xi) \mid |y - x| < Ra/c\}$. Then by (3.5), (3.13) and (3.16)

$$T\hat{\zeta}_{c,\xi}(x) - \frac{1}{2\pi} \log \frac{c}{2a} \geq x_2 + C_2 - \eta_\infty,$$

for almost all $x \in \text{supp}(\hat{\zeta}_{c,\xi})$. Hence

$$\int_{B(x,\xi)} \log \left(\frac{2a|x - \bar{y}|}{c|x - y|} \right) \hat{\zeta}_{c,\xi}(y) dy + \int_{\Pi(\xi) \setminus B(x,\xi)} \log \left(\frac{2a|x - \bar{y}|}{c|x - y|} \right) \hat{\zeta}_{c,\xi}(y) dy \geq 2\pi(x_2 + C_2 - \eta_\infty), \tag{3.17}$$

for almost all $x \in \text{supp}(\hat{\zeta}_{c,\xi})$. Similarly to the proof of [4, Lemma 1], it can be shown that there exist constants M'_1, M'_2 and $M'_3 > 0$ independent of c and ξ such that

$$\int_{B(x,\xi)} \log \left(\frac{2a|x - \bar{y}|}{c|x - y|} \right) \hat{\zeta}_{c,\xi}(y) dy \leq \begin{cases} (M'_1 + M'_2 |\log x_2|) \|\zeta_0\|_p, & x_2 \geq a; \\ M'_3 \|\zeta_0\|_p, & 0 < x_2 < a. \end{cases} \tag{3.18}$$

Note that $|x - y| \geq Ra/c$ and $|x - \bar{y}| \leq 5\xi$ for $y \in \Pi(\xi) \setminus B(x, \xi)$, hence

$$\int_{\Pi(\xi) \setminus B(x,\xi)} \log \left(\frac{2a|x - \bar{y}|}{c|x - y|} \right) \hat{\zeta}_{c,\xi}(y) dy \leq \left(\log \frac{10\xi}{R} \right) \int_{\Pi(\xi) \setminus B(x,\xi)} \hat{\zeta}_{c,\xi}(y) dy. \tag{3.19}$$

Then by rearranging the terms in (3.17) and applying the estimates (3.18) and (3.19) we obtain

$$\left(\log \frac{R}{10\xi} \right) \int_{\Pi(\xi) \setminus B(x,\xi)} \hat{\zeta}_{c,\xi}(y) dy \leq \begin{cases} (M'_1 + M'_2 |\log x_2|) \|\zeta_0\|_p + 2\pi(\eta_\infty - C_2 - x_2), & x_2 \geq a; \\ M'_3 \|\zeta_0\|_p + 2\pi(\eta_\infty - C_2 - x_2), & 0 < x_2 < a. \end{cases}$$

Therefore

$$\left(\log \frac{R}{10\xi} \right) \int_{\Pi(\xi) \setminus B(x,\xi)} \hat{\zeta}_{c,\xi}(y) dy < \kappa,$$

where κ is a constant independent of c and ξ . Now we set $R = 10\xi e^{2x}$, thus

$$\int_{\Pi(\xi) \setminus B(x,\xi)} \hat{\zeta}_{c,\xi}(y) dy < \frac{1}{2}. \tag{3.20}$$

We then claim that $\text{essdiam}(\text{supp}(\hat{\zeta}_{c,\xi})) \leq 2Ra/c$. To seek a contradiction suppose the claim is false. Then $\text{supp}(\hat{\zeta}_{c,\xi}) = S \cup N$, where $\text{diam}(S) > 2Ra/c, |N| = 0$ and (3.20) holds for all $x \in S$. Choose x' and $x'' \in S$ such that $B(x', \xi) \cap B(x'', \xi)$ is empty, then

$$1 \leq \int_{\Pi(\xi) \setminus B(x',\xi)} \hat{\zeta}_{c,\xi}(y) dy + \int_{\Pi(\xi) \setminus B(x'',\xi)} \hat{\zeta}_{c,\xi}(y) dy < 1,$$

which is a contradiction. □

To prove Theorem 2.1 we need the following crucial result.

LEMMA 3.10. *There exists $c_0 > 0$ such that for all $c \geq c_0$, we can find $\xi(c)$ such that $\text{supp}(\hat{\zeta}_{c,\xi}) \subseteq \Pi(\xi(c))$, $\xi \geq \xi(c)$, except for a set of measure zero.*

PROOF. Let σ , c_3 and c_4 be as in Lemmas 3.8–3.9. Fix $c > c_0 = \max\{c_3, c_4, 4\sigma a\}$. For $\xi \geq 3a$, set $d_\xi = \min\{\xi' \mid \text{supp}(\hat{\zeta}_{c,\xi}) \subseteq \Pi(\xi')\}$, except for a set of measure zero. To seek a contradiction suppose the assertion in the lemma is false. Then there exist sequences $\{\xi_n\}$ and $\{d_{\xi_n}\}$ which tend to infinity. Choose n_0 such that $d_{\xi_n} \geq \max\{\xi_3, \xi_4, 2H(c)\}$, for $n \geq n_0$. Since $\hat{\zeta}_{c,\xi_n}$ maximises $\hat{\Psi}_c$ relative to $\mathcal{F}_{c,d_{\xi_n}}$ and by the choice of c and d_{ξ_n} , $n \geq n_0$ and from Lemma 3.9, we have

$$\text{essdiam}(\text{supp}(\hat{\zeta}_{c,\xi_n})) \leq d_{\xi_n}/4.$$

Therefore $\text{supp}(\hat{\zeta}_{c,\xi_n}) = S_{c,\xi_n} \cup N$ where $\text{diam}(S_{c,\xi_n}) \leq d_{\xi_n}/4$ and $|N| = 0$. If there exists $x \in S_{c,\xi_n}$ with $|x_1| \leq d_{\xi_n}/2$, since $d_{\xi_n} > 2H(c)$, this contradicts the minimality of d_{ξ_n} . Thus

$$\text{supp}(\hat{\zeta}_{c,\xi_n}) \subseteq \{x \in \Pi(d_{\xi_n}) \mid |x_1| > d_{\xi_n}/2\}, \quad n \geq 0,$$

except for a set of measure zero. We choose a subsequence, again denoted $\{\xi_n\}$, such that

$$\text{supp}(\hat{\zeta}_{c,\xi_n}) \subseteq \{x \in \Pi(d_{\xi_n}) \mid x_1 > d_{\xi_n}/2\}, \quad n \geq n_0,$$

except for a set of measure zero. Let $d(c)$ be the smallest number such that $\text{supp}(\mathcal{C}h) \subseteq \Pi(d(c))$ except for a set of measure zero. Note that there exists $n_1 \geq n_0$ such that $d_{\xi_{n_1}}/2 \geq 2d(c)$. If we set $\hat{\zeta}(x_1, x_2) = \hat{\zeta}_{c,\xi_{n_1}}(x_1 + d(c), x_2)$, then $\text{supp}(\hat{\zeta}) \subseteq \Pi(d_{\xi_{n_1}})$ and by (2.2)

$$\begin{aligned} \hat{\Psi}_c(\hat{\zeta}) - \hat{\Psi}_c(\hat{\zeta}_{c,\xi_{n_1}}) &= \frac{1}{4\pi} \int_{\Pi} \int_{\Pi} \log \left(1 + \frac{4x_2y_2}{|x-y|^2} \right) \hat{\zeta}(x)\mathcal{C}h(y) \, dx \, dy \\ &\quad - \frac{1}{4\pi} \int_{\Pi} \int_{\Pi} \log \left(1 + \frac{4x_2y_2}{|x-y|^2} \right) \hat{\zeta}_{c,\xi_{n_1}}(x)\mathcal{C}h(y) \, dx \, dy > 0. \end{aligned}$$

This is a contradiction to the maximality of $\hat{\zeta}_{c,\xi_{n_1}}$. □

REMARK. Note that from Lemma 3.10 it follows that \hat{P}_c is solvable for $c > c_0$.

4. Proof of Theorem 2.1

Let c_0 be as in Lemma 3.10 and fix $c \geq c_0$. It follows that $\hat{\Sigma}_c$ is nonempty. Now consider $\hat{\zeta}_c \in \hat{\Sigma}_c$. From Lemma 3.5 (iii), applying (2.3) and (2.6), we deduce that

$$\hat{\Psi}_c(\hat{\zeta}_c) = \Psi_\lambda(\zeta_\lambda),$$

where $\mathcal{C}^{-1}\hat{\xi}_c = \zeta_\lambda$, $\lambda = 1/c$. It is then clear that ζ_λ maximises Ψ_λ relative to \mathcal{F} and $\text{supp}(\zeta_\lambda) \subseteq \Pi(c\xi(c))$, except for a set of measure zero. It is clear that ζ_λ also maximises Ψ_λ relative to $\mathcal{F}(c\xi(c))$ for $c = 1/\lambda$. Now by applying Lemma 3.3, we can find an increasing function φ_λ such that

$$\zeta_\lambda = \varphi_\lambda \circ (T\zeta_\lambda + \eta - \lambda x_2),$$

for almost every $x \in \Pi(c\xi(c))$, $c = 1/\lambda$. Notice that we can assume $\varphi_\lambda(s) \geq 0$, for all $s \in \text{dom } \varphi_\lambda$ (domain of φ_λ). Since ζ_λ is an increasing function of $T\zeta_\lambda + \eta - \lambda x_2$ on $\Pi(R)$, $R > c\xi(c)$, there exists a constant γ_λ such that

$$T\zeta_\lambda + \eta - \lambda x_2 \geq \gamma_\lambda$$

almost everywhere on $\text{supp}(\zeta_\lambda)$ and

$$T\zeta_\lambda + \eta - \lambda x_2 < \gamma_\lambda$$

almost everywhere in $\Pi \setminus \Pi(c\xi(c))$, $c = 1/\lambda$. Now define

$$\varphi(s) = \begin{cases} \varphi_\lambda(s), & s \in \text{dom } \varphi_\lambda, s \geq \gamma_\lambda; \\ 0, & s < \gamma_\lambda. \end{cases}$$

Clearly φ is increasing and $\zeta_\lambda = \varphi \circ (T\zeta_\lambda + \eta - \lambda x_2)$ almost everywhere in Π . Now, applying Lemma 3.2, we obtain (2.10). Note that $\lambda_0 = 1/c$. This completes the proof of Theorem 2.1.

Now we show that if λ is sufficiently large, then P_λ has no solution.

PROPOSITION 4.1. *There exists λ_1 such that for $\lambda > \lambda_1$ we have $\sup_{\zeta \in \mathcal{F}} \Psi_\lambda(\zeta) = 0$; moreover, this supremum is not attained.*

PROOF. By Lemma 3.2 there exists a constant N depending only on p , $|\text{supp}(\zeta_0)|$ and $|\text{supp}(h)|$ such that for all $\zeta \in \mathcal{F}$ we have

$$\begin{aligned} \Psi_\lambda(\zeta) &= \frac{1}{2} \int_\Pi \zeta T\zeta + \int_\Pi \eta \zeta - \lambda \int_\Pi x_2 \zeta \\ &\leq \int_\Pi \left(\frac{1}{2} N (\|\zeta_0\|_p + \|h\|_p) - \lambda \right) x_2 \zeta < 0, \end{aligned} \tag{4.1}$$

for $\lambda > N(\|\zeta_0\|_p + \|h\|_p)/2$. Let $\lambda > N(\|\zeta_0\|_p + \|h\|_p)/2$ and let ζ_n denote a rearrangement of ζ_0 with bounded support in $\{x \in \Pi \mid x_2 < 1/n\}$. Therefore

$$\Psi_\lambda(\zeta_n) = \frac{1}{2} \int_\Pi \zeta_n T\zeta_n + \int_\Pi \eta \zeta - \lambda \int_\Pi x_2 \zeta > -\frac{\lambda}{n} \|\zeta_0\|_1.$$

Hence $\sup_{\zeta \in \mathcal{F}} \Psi_\lambda(\zeta) = 0$ but by (4.1) this supremum is not attained. □

In the remainder of this section we sketch a proof that the solutions of P_λ represent weak solutions of the barotropic equation (2.11).

PROPOSITION 4.2. *Let ζ_λ be a solution of P_λ , then ζ_λ is a solution of (2.13).*

PROOF. Since $\zeta_\lambda + h$ has compact support we infer existence of an open set $\Omega \subseteq \Pi$ such that $\text{supp}(\zeta_\lambda + h) \subset \Omega$. Therefore it suffices to prove (2.13) only for test functions $u \in C_0^\infty(\Omega)$. To this end we fix $u \in C_0^\infty(\Omega)$ and denote by $f_t(x)$ the unique solution of the Hamiltonian system

$$\frac{dz}{dt} = \nabla^\perp u(z),$$

satisfying the initial condition $z(0) = x \in \Omega$; where $\nabla^\perp = (\partial/\partial x_2, -\partial/\partial x_1)$. It is well known that the mapping $x \rightarrow f_t(x)$, $t \in [-T, T]$, T small, defines a one-parameter family of measure-preserving diffeomorphisms of Ω , see for example [12]. Now following [13, 14] we obtain

$$\Psi_\lambda(\zeta_\lambda \circ f_t^{-1}) = \Psi_\lambda(\zeta_\lambda) + t \int_\Omega (\zeta_\lambda + h)[T\zeta_\lambda - \lambda x_2, u] + o(t), \quad (4.2)$$

as $t \rightarrow 0^+$. Hence if we set $\alpha(t) = \Psi_\lambda(\zeta_\lambda \circ f_t^{-1})$, for $t \in [-T, T]$, we infer from (4.2) that

$$\alpha'(0) = \int_\Omega (\zeta_\lambda + h)[T\zeta_\lambda - \lambda x_2, u].$$

Moreover, since $\zeta_\lambda \in \Sigma_\lambda$ and $\zeta_\lambda \circ f_t^{-1} \in \mathcal{F}$, it follows that α has a global minimum at zero, whence $\alpha'(0) = 0$. Thus (2.13) follows. \square

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