ON THE HAUSDORFF AND TRIGONOMETRIC MOMENT PROBLEMS

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Let K be a subset of $BV(0, 1)$—the space of functions of bounded variation on the closed interval $[0, 1]$. By the Hausdorff moment problem for K we shall mean the determination of necessary and sufficient conditions that corresponding to a given sequence $\mu = \{\mu_n | n = 0, 1, 2, \ldots\}$[1] there should be a function $\alpha \in K$ so that

$$\mu_n = \int_0^1 t^n d\alpha(t), \; n = 0, 1, 2, \ldots \; . \tag{1}$$

For various collections K this problem has been solved—see **(3**, Chapter III**)**.

By the trigonometric moment problem for K we shall mean the determination of necessary and sufficient conditions that corresponding to a sequence $c = \{c_n | n = 0, \pm 1, \pm 2, \ldots\}$[2] there should be a function $\alpha \in K$ so that

$$c_n = \int_0^1 e^{-2n\pi i t} d\alpha(t), \; n = 0, \pm 1, \pm 2, \ldots \; . \tag{2}$$

For various collections K this problem has also been solved—see, for example **(4**, Chapter IV, § **4)**. It is noteworthy that these two problems have been solved for essentially the same collections K.

Recently **(2)**, we gave new solutions of the trigonometric moment problem for certain classes K, namely those K determined by $K' = L_p(0, 1)$, $1 < p \leqslant 2$, where K' is defined now and henceforth, if the functions of K are absolutely continuous, to consist of all functions equal almost everywhere to the derivative of a function in K. These solutions were determined by use of the known solutions of the Hausdorff moment problem for these particular classes K.

Here we propose to generalize this procedure. Specifically, we propose to show that if the Hausdorff moment problem can be solved for a particular class K, then so can the trigonometric moment problem. This forms the content of the theorem below, and we shall illustrate our theory in a number of cases.

Received June 6, 1960.

[1]We shall use μ as a generic symbol for sequences whose indices run from zero to infinity.

[2]We shall use c as a generic symbol for sequences whose indices run from minus infinity to infinity.

https://doi.org/10.4153/CJM-1961-038-8 Published online by Cambridge University Press

To this end, we must first establish a number of results concerning certain numbers $a_{r,m}$ defined by

$$(3) \qquad a_{r,m} = \int_0^1 t^m e^{2\pi i r t} dt, \; r = 0, +1, \pm 2, \ldots, \quad m = 0, 1, 2, \ldots .$$

Since these numbers are essentially both the Hausdorff moments of the trigonometric powers of t, and the trigonometric moments of the algebraic powers of t, it is perhaps not surprising that they have an important role to play. Their properties are given in the following lemmas.

LEMMA 1.

$$(4) \qquad a_{r,m} = (1 - ma_{r,m-1})/2\pi ir, \qquad rm \neq 0,$$

$$(5) \qquad |a_{r,m}| \leqslant (m+1)^{-1},$$

$$(6) \qquad |a_{r,m}| \leqslant (\pi|r|)^{-1}, \qquad r \neq 0,$$

$$(7) \qquad a_{r,m} = \begin{cases} (m+1)^{-1}, r = 0, \\ 0, \qquad m = 0, r \neq 0, \\ \sum_{n=0}^{m-1} \binom{m}{n} (-1)^n n!/(2\pi ir)^{n+1}, rm \neq 0. \end{cases}$$

Proof. On integration by parts, (4) follows from (3). If $m \neq 0$, (6) comes from applying (5), which is trivial, to the right-hand side of (4). The first two parts of (7) are immediate, and the third part follows from the second on repeated application of (4). From (7), (6) is obvious if $m = 0$.

LEMMA 2. *If* $|c_r| \leqslant M$, $r = 0, \pm 1, \pm 2, \ldots,$ *and*

$$\lim_{N\to\infty} \sum_{-N}^{N}{}' \frac{c_r}{r}$$

exists, (*where the prime denotes the omission of the term corresponding to* $r = 0$)*, then for* $m = 0, 1, 2, \ldots$

$$\lim_{N\to\infty} \sum_{r=-N}^{N} c_r a_{r,m}$$

exists.

Proof. Since, from (7), $a_{r,0} = 0$, $r \neq 0$, and $a_{0,0} = 1$, it follows that

$$(8) \qquad \sum_{-N}^{N} c_r a_{r,0} = c_0,$$

and the limit exists for $m = 0$. Now if $m > 0$, then from (7) and (4),

$$(9) \qquad \sum_{r=-N}^{N} c_r a_{r,m} = \frac{1}{m+1} c_0 + \frac{1}{2\pi i} \sum_{-N}^{N}{}' \frac{c_r}{r} - \frac{m}{2\pi i} \sum_{r=-N}^{N}{}' \frac{c_r}{r} a_{r,m-1}.$$

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But the first two terms on the right of this equation have limits as $N \to \infty$, so that it suffices to show that the third term has such a limit. But from (6)

$$\sum_{r=-\infty}^{\infty}{}' \left| \frac{c_r}{r} a_{r,m-1} \right| \leqslant \frac{M}{\pi} \sum_{-\infty}^{\infty}{}' \frac{1}{r^2} = \frac{\pi M}{3}, \tag{10}$$

so that the series

$$\sum_{-\infty}^{\infty}{}' \frac{c_r}{r} a_{r,m-1}$$

converges absolutely. Thus the limit of the last term in (9) also exists, and the lemma is proved.

With each sequence c, satisfying the hypotheses of Lemma 2 we can now associate a sequence $\mu(c)$ defined by

$$\mu_m(c) = \lim_{N\to\infty} \sum_{r=-N}^{N} c_r a_{r,m}. \tag{11}$$

The sequence $\mu(c)$ has certain properties that we summarize as a lemma.

Lemma 3. *If c satisfies the hypotheses of Lemma 2, then*

$$\mu_0(c) = c_0, \tag{12}$$

$$\mu_m(c) = \frac{1-m}{1+m} \frac{c_0}{2} + \mu_1 - \frac{m}{2\pi i} \sum_{-\infty}^{\infty}{}' \frac{c_r}{r} a_{r,m-1}\, m > 0. \tag{13}$$

Proof. Equation (12) follows immediately from (8) and (11). Now from (7), $a_{r,1} = (2\pi i r)^{-1}$, $r \neq 0$, $a_{0,1} = \frac{1}{2}$. Hence from (11) and (9),

$$\begin{aligned}
\mu_m(c) &= \lim_{N\to\infty} \sum_{r=-N}^{N} c_r a_{r,m} \\
&= \lim_{N\to\infty} \left(\frac{1}{m+1} c_0 + \frac{1}{2\pi i} \sum_{-N}^{N}{}' \frac{c_r}{r} - \frac{m}{2\pi i} \sum_{-N}^{N}{}' \frac{c_r}{r} a_{r,m-1} \right) \\
&= \lim_{N\to\infty} \left(\frac{1}{m+1} c_0 - \frac{1}{2} c_0 + \sum_{-N}^{N} c_r a_{r,1} - \frac{m}{2\pi i} \sum_{-N}^{N}{}' \frac{c_r}{r} a_{r,m-1} \right) \\
&= \frac{1-m}{1+m} \frac{c_0}{2} + \mu_1 - \frac{m}{2\pi i} \sum_{-\infty}^{\infty}{}' \frac{c_r}{r} a_{r,m-1},
\end{aligned}$$

since by (10), this last series converges absolutely.

We are now ready to state and prove our theorem.

Theorem. *Necessary and sufficient conditions that a sequence c be represented in the form (2) for some $\alpha \in K$ are that*

(i) $|c_r| \leqslant M, r = 0, \pm 1, \pm 2, \ldots,$

(ii) $\displaystyle\lim_{N\to\infty} \sum_{-N}^{N}{}' \frac{c_r}{r}$ *exists,*

(iii) *$\mu(c)$ is represented in the form (1) with $\alpha \in K$.*

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Proof of necessity. Suppose

$$c_n = \int_0^1 e^{-2n\pi i t}\, d\alpha(t),\ n = 0, \pm 1, \pm 2, \dots\ ,$$

where $\alpha \in K$. Clearly,

$$|c_n| \leqslant \int_0^1 dV(t) = M,$$

where $V(t)$ is the total variation of α, so that (i) is necessary.

Now let

$$\beta(t) = 2\pi\alpha(t/2\pi) - c_0 t, \qquad 0 \leqslant t \leqslant 2\pi,$$

and define $\beta(t)$ outside this interval by

$$\beta(t + 2\pi) = \beta(t).$$

Then $\beta(t)$ is periodic and of bounded variation, so that if

$$d_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}\beta(t)dt,\ n = 0, \pm 1, \pm 2, \dots\ ,$$

it follows from the Dini–Dirichlet test (**4**, Chapter II, Theorem 8.1)

$$\lim_{N\to\infty} \sum_{-N}^{N} d_n = \frac{1}{2}(\beta(0+) + \beta(0-)). \tag{14}$$

But if $n \neq 0$,

$$d_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}\beta(t)dt = \int_0^{2\pi} e^{-int}\alpha(t/2\pi)dt - \frac{c_0}{2\pi}\int_0^{2\pi} te^{-int}dt$$

$$= 2\pi\int_0^1 e^{-2n\pi i t}\alpha(t)dt + \frac{c_0}{in},$$

and integrating by parts, we obtain, if $n \neq 0$,

$$d_n = c_n/in.$$

Thus, (14) becomes

$$\lim_{N\to\infty} {\sum_{-N}^{N}}' \frac{c_r}{r} = \frac{i}{2}(\beta(0+) + \beta(0-)) - i\, d_0,$$

and (ii) is necessary.

Now let

$$d_n' = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}d\beta(t).$$

Then, since

$$a_{r,n} = \int_0^1 t^n e^{2\pi i r t}dt = (2\pi)^{-(n+1)}\int_0^{2\pi} t^n e^{irt}dt,$$

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it follows from Parseval's theorem for Fourier series (**4**, Chapter IV, Theorem 8.7 (iv)) that

$$\int_0^{2\pi} t^n\, d\beta(t) = \sum_{r=-\infty}^{\infty} d'_r (2\pi)^{n+1} a_{r,n}\ (C, 1). \tag{15}$$

But the left-hand side of (15) is equal to

$$\int_0^{2\pi} t^n d\beta(t) = 2\pi \int_0^{2\pi} t^n d\alpha(t/2\pi) - c_0 \int_0^{2\pi} t^n dt$$

$$= (2\pi)^{n+1} \left\{ \int_0^1 t^n d\alpha(t) - c_0/(n+1) \right\}.$$

Also

$$d'_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} d\beta(t) = \int_0^{2\pi} e^{-int} d\alpha(t/2\pi) - \frac{c_0}{2\pi} \int_0^{2\pi} e^{-int} dt$$

$$= \int_0^1 e^{-2n\pi i t} d\alpha(t) - \frac{c_0}{2\pi} \int_0^{2\pi} e^{-int} dt$$

$$= \begin{cases} c_n & n \neq 0 \\ 0 & n = 0, \end{cases}$$

so that (15) becomes

$$(2\pi)^{n+1} \left\{ \int_0^1 t^n d\alpha(t) - \frac{c_0}{n+1} \right\} = (2\pi)^{n+1} \sum_{-\infty}^{\infty}{}' c_r\, a_{r,n} \qquad (C, 1).$$

Thus, since $a_{0,n} = (n+1)^{-1}$

$$\int_0^1 t^n d\alpha(t) = \sum_{-\infty}^{\infty} c_r\, a_{r,n} \qquad (C, 1).$$

But, since by (i), (ii), and Lemma 2

$$\lim_{N\to\infty} \sum_{r=-N}^{N} c_r\, a_{r,n}$$

exists and equals $\mu_n(c)$, and since the $(C, 1)$ method is consistent, we must have

$$\mu_n(c) = \int_0^1 t^n d\alpha(t),$$

and (iii) is necessary.

Proof of sufficiency. From (iii), $\alpha \in K$ exists so that

$$\mu_n(c) = \int_0^1 t^n d\alpha(t).$$

Let

$$c'_n = \int_0^1 e^{-2n\pi i t}\, d\alpha(t).$$

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We shall show that $c_n' = c_n$. Firstly, $c_0' = c_0$, for from (12),

$$c_0 = \mu_0(c) = \int_0^1 d\alpha(t) = c_0'.$$

Then, if $n \neq 0$,

$$c_n' = \int_0^1 e^{-2n\pi i t} d\alpha(t)$$
$$= \sum_{m=0}^{\infty} \frac{(-2n\pi i)^m}{m!} \int_0^1 t^m d\alpha(t) = \sum_{m=0}^{\infty} \frac{(-2n\pi i)^m}{m!} \mu_m(c),$$

the interchange of integration and summation being justified by the uniform convergence of the exponential series.

But then using (13) and (12), if $n \neq 0$,

$$(16) \quad c_n' = \sum_{m=0}^{\infty} \frac{(-2n\pi i)^m}{m!} \mu_m(c)$$
$$= \mu_0(c) + \sum_{m=1}^{\infty} \frac{(-2n\pi i)^m}{m!} \left(\frac{1-m}{1+m} \cdot \frac{c_0}{2} + \mu_1 - \frac{m}{2\pi i} \sum_{-\infty}^{\infty}{}' \frac{c_r}{r} a_{r,m-1} \right)$$
$$= c_0 \left(1 + \frac{1}{2} \sum_{m=1}^{\infty} \frac{(1-m)(-2n\pi i)^m}{(m+1)!} \right) + \mu_1 \sum_{m=1}^{\infty} \frac{(-2n\pi i)^m}{m!}$$
$$+ n \sum_{m=1}^{\infty} \frac{(-2n\pi i)^{m-1}}{(m-1)!} \sum_{-\infty}^{\infty}{}' \frac{c_r}{r} a_{r,m-1}.$$

Now

$$\sum_{m=1}^{\infty} \frac{x^m}{(m+1)!} = \frac{e^x - 1}{x} - 1,$$

and

$$\sum_{m=1}^{\infty} \frac{mx^m}{(m+1)!} = e^x - \frac{e^x - 1}{x},$$

so that the coefficient of c_0 in (16) is equal to

$$1 + \frac{1}{2} \left(\frac{e^{-2n\pi i} - 1}{-2n\pi i} - 1 - e^{-2n\pi i} + \frac{e^{-2n\pi i} - 1}{-2n\pi i} \right) = 0.$$

Also,

$$\sum_{m=1}^{\infty} \frac{x^m}{m!} = e^x - 1,$$

so that the coefficient of μ_1 in (16) is also zero. Thus if $n \neq 0$

$$(17) \qquad c_n' = n \sum_{m=1}^{\infty} \frac{(-2n\pi i)^{m-1}}{(m-1)!} \sum_{-\infty}^{\infty}{}' \frac{c_r}{r} a_{r,m-1}.$$

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But the series

$$\sum_{m=1}^{\infty} \frac{|(-2n\pi i)|^{m-1}}{(m-1)!} \sum_{-\infty}^{\infty}{}' \left| \frac{c_r}{r} a_{r,m-1} \right| < \infty .$$

For from (10) it is smaller than

$$\frac{\pi M}{3} \sum_{m=1}^{\infty} \frac{(2|n|\pi)^{m-1}}{(m-1)!} = \frac{\pi M}{3} e^{2|n|\pi} < \infty .$$

Hence we can interchange the orders of summation in (17) and obtain

$$c_n' = n \sum_{-\infty}^{\infty}{}' \frac{c_r}{r} \sum_{m=1}^{\infty} \frac{(-2n\pi i)^{m-1}}{(m-1)!} a_{r,m-1} \tag{18}$$

$$= n \sum_{-\infty}^{\infty}{}' \frac{c_r}{r} \sum_{m=0}^{\infty} \frac{(-2n\pi i)^{m}}{m!} a_{r,m}.$$

But

$$\sum_{m=0}^{\infty} \frac{(-2n\pi i)^m}{m!} a_{r,m} = \sum_{m=0}^{\infty} \frac{(-2n\pi i)^m}{m!} \int_0^1 t^m e^{2r\pi i t} dt$$

$$= \int_0^1 \left(\sum_{m=0}^{\infty} \frac{(-2n\pi i t)^m}{m!} \right) e^{2r\pi i t} dt = \int_0^1 e^{2(r-n)\pi i t} dt$$

$$= \begin{cases} 0, & r \neq n \\ 1, & r = n, \end{cases}$$

and using this in (18) we obtain $c_n' = c_n$, that is

$$c_n = \int_0^1 e^{-2n\pi i t} d\alpha(t), \; n = 0, \pm 1, \pm 2, \ldots ,$$

with $\alpha \in K$.

As an example of the use of the theorem to obtain solutions of the trigonometric moment problem, let us take $K = BV(0, 1)$. Then from **(3**, Chapter III, Theorem 2b**)**, a necessary and sufficient condition that sequence μ be the Hausdorff moment sequence of a function in $BV(0, 1)$ is that for some constant L

$$\sum_{m=0}^{k} |\lambda_{k,m}| < L, \; k = 0, 1, 2, \ldots ,$$

where

$$\partial_{j,m} = \binom{k}{m} (-1)^{k-m} \Delta^{k-m} \mu_m,$$

and Δ is the advancing difference operator.

Thus, given a sequence c, we find as necessary and sufficient conditions that c be the trigonometric moment sequence of a function in $BV(0, 1)$, are that (i) and (ii) of the theorem be satisfied, and that for some constant L,

$$\sum_{m=0}^{k} |\lambda_{k,m}(c)| < L, k = 0, 1, 2, \dots ,$$

where

(19) $$\lambda_{k,m}(c) = \binom{k}{m} (-1)^{k-m} \Delta^{k-m} \mu_m$$

$$= \lim_{N \to \infty} \sum_{r=-N}^{N} c_r \, a_{r,k,m},$$

where

$$a_{r,k,m} = \binom{k}{m} (-1)^{k-m} \Delta^{k-m} a_{r,m} = \binom{k}{m} \int_0^1 t^m (1-t)^{k-m} e^{2\pi i r t} dt.$$

We list in Table I the conditions for representation as a trigonometric moment sequence for some of the more common classes K. In all cases (i) and (ii) of the theorem must hold and the column marked (iii) gives the third condition that must hold. The last column gives the place from which the conditions for the Hausdorff representation are taken.

TABLE I

	K	(iii)	Reference (**3**, Chapter III)
1	$BV(0,1)$	$\sum_{m=0}^{k} \lvert\lambda_{k,m}(c)\rvert < L,\ k = 0, 1, 2, \dots.$	Theorem 2b
2	Increasing functions on [0,1]	$\lambda_{k,m}(c) \geqslant 0,\ k = 0, 1, 2, \dots, 0 \leqslant m \leqslant k,$	Theorem 4a
3	$K' = L_p(0, 1), 1 < p < \infty$	$(k+1)^{p-1} \sum_{m=0}^{k} \lvert\lambda_{k,m}(c)\rvert^p < L,\ k = 0, 1, 2, \dots .$	Theorem 5
4	$K' = L_\infty(0, 1)$	$(k+1) \lvert\lambda_{r,m}\rvert < L,\ k = 0, 1, 2, \dots, 0 \leqslant m \leqslant k.$	Theorem 6

Case 2 is of particular note, since the trigonometric moment problem for this K was given a particularly elegant solution by Bochner (**1**, § 20).

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https://doi.org/10.4153/CJM-1961-038-8 Published online by Cambridge University Press