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ASYMPTOTICALLY STABLE ATTRACTING SETS IN THE NAVIER-STOKES EQUATIONS

P.E. KLOEDEN

The planar Navier-Stokes equations with periodic boundary conditions are shown to have a nearby asymptotically stable attracting set whenever a Galerkin approximation involving the eigenfunctions of the Stokes operator has such an attracting set, provided the approximation has sufficiently many terms and its attracting set is sufficiently strongly stable. Lyapunov functions are used to characterize the stability of these attracting sets, which are compact sets of arbitrary geometric shape. This generalizes earlier results of Constantin, Foias and Temam and of the author for asymptotically stable steady solutions in the Navier-Stokes equations and such Galerkin approximations.

1. Introduction

In many numerical and theoretical studies in fluid dynamics, particularly in meteorology and oceanography, simpler truncated systems, called Galerkin approximations or spectral systems, are studied instead of the full system of partial differential equations. These are finite dimensional systems of ordinary differential equations, usually only with linear and quadratic terms, which are obtained by truncating infinite

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dimensional systems involving the time-dependent coefficients of Fourier-like series expansions of the solutions of the partial differential equations. An implicit assumption here is that the qualitative behaviour of the solutions of the truncated system closely resembles that of the solutions of the full system of partial differential equations. This is known not to be true, in the Lorenz equations for example, when the truncation is too severe or the type of behaviour under consideration is too complicated.

Recently Constantin, Foias and Temam [1] have shown for the Navier-Stokes equations that the presence of an asymptotically stable steady solution in a Galerkin approximation, defined in terms of the eigenfunctions of the Stokes operator, of sufficiently high order implies the existence of a nearby asymptotically stable solution in the Navier-Stokes equations. Their proof makes considerable use of the spectral properties of the linear operators in the Galerkin approximations and the Navier-Stokes equations linearized about steady solutions. Such a simple spectral theory is not available for more complicated attracting sets such as periodic or almost periodic solutions, let alone for strange attractors. There is however an extensive theory, presented in, for example, Yoshizawa [6], in which the stability of an attracting set is characterized in terms of Lyapunov functions. This was used by Kloeden [3] to obtain a result similar to that of Constantin, Foias and Teman [1] for steady solutions of the Navier-Stokes equations and their Galerkin approximations. More recently Wells and Dutton [5] have used Lyapunov functions for quite generally shaped attractors that bifurcate from a steady solution. In this paper we consider compact attracting sets of arbitrary geometric shape and origin. We show that if a Galerkin approximation to the planar Navier-Stokes equations with periodic boundary conditions has an asymptotically stable attracting set, then the Navier-Stokes equations has a nearby asymptotically stable attracting set provided the Galerkin approximation has sufficiently many terms and its attracting set is sufficiently stable.

In section 2 we present necessary background material on the Navier-Stokes equations and their Galerkin approximations and in section 3 we outline the Lyapunov theory for compact attracting sets. In particular we state a theorem from Yoshizawa [6] guaranteeing the existence of a Lyapunove function characterizing the stability of an asymptotically stable set in an autonomous system of ordinary differential equations. Our main result is stated and proved in section 4. Our proof makes full use of the properties of a Lyapunov function characterizing the assumed asymptotic stability of a Galerkin approximation. Finally in the appendix we establish a bound on certain terms of appearing in the proof of our main result.

2. Preliminaries

We consider the Navier-Stokes equations on the unit square spatial domain $Q = [0,1]^2$ in $\mathbb{R}^2$

$$\frac{\partial \underset{\sim}{u}}{\partial t} - \nu\Delta\underset{\sim}{u} + \underset{\sim}{u} \cdot \nabla\underset{\sim}{u} + \nabla p = \underset{\sim}{f} ,$$

$$\operatorname{div} \underset{\sim}{u} = 0 , \tag{2.1}$$

$$\underset{\sim}{u}(\underset{\sim}{x},0) = \underset{\sim}{u}_0(\underset{\sim}{x}) ,$$

with periodic boundary conditions

$$\underset{\sim}{u}(0,x_2,t) = \underset{\sim}{u}(1,x_2,t) ,$$

$$\underset{\sim}{u}(x_1,0,t) = \underset{\sim}{u}(x_1,1,t) \tag{2.2}$$

for all $\underset{\sim}{x} = (x_1,x_2) \in Q$ and $t \geq 0$. Further, we restrict attention to flows with zero spatial average.

$$\int_Q \underset{\sim}{u} = \underset{\sim}{0} ,$$

which requires $\int_Q \underset{\sim}{f} = \underset{\sim}{0}$.

Following the terminology of Temam [4] , we write

$$H_p^m(Q) = \{g \in H^m(Q) \ ; \ g \text{ spatially periodic on } Q\} ,$$

$$\dot{H}_p^m(Q) = \{g \in H_p^m(Q) \ ; \ \int_Q g = 0\} ,$$

and $\mathbb{H}_p^m(Q) = \dot{H}_p^m(Q)^2$. Here $H_p^m(Q)$ is the usual Sobolev space of real-valued functions on Q with mth-order generalized derivatives in $L_2(Q)$ and norm $|\underset{\sim}{u}|_m$. Writing $\mathbb{L}_p$ for $\mathbb{H}_p^0$, we also write

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$$H = \{\underset{\sim}{u} \in \mathbb{L}_p(Q)\ ;\quad \operatorname{div}\ \underset{\sim}{u} = 0\}\ ,$$

$$V = \{\underset{\sim}{u} \in \mathbb{H}^1_p(Q)\ ;\quad \operatorname{div}\ \underset{\sim}{u} = 0\}$$

and use the following inner products and norms

$$(\underset{\sim}{u},\underset{\sim}{v}) = \int_Q \underset{\sim}{u}\cdot\underset{\sim}{v}\ ,\quad |\underset{\sim}{u}| = (\underset{\sim}{u},\underset{\sim}{v})^{\frac{1}{2}} \quad \text{on}\quad H$$

and

$$((\underset{\sim}{u},\underset{\sim}{v})) = \int_Q \nabla\underset{\sim}{u} : \nabla\underset{\sim}{v}\ ,\quad \|\underset{\sim}{u}\| = ((\underset{\sim}{u},\underset{\sim}{v}))^{\frac{1}{2}} \quad \text{on}\quad V\ .$$

We write P for the orthogonal projector from $\mathbb{L}\ (Q)$ onto H and note that it is also the orthogonal projector from $\mathbb{H}^1_p(Q)$ onto V. In the present setting, we then have $P\Delta = \Delta$ on the domain

$$dom(\Delta) = \{u \in \dot{H},\ \Delta u \in H = \mathbb{H}^2_p(Q) \cap \dot{H}\}\}$$

where, as above, the dot indicates zero spatial averages. The Navier-Stokes equations then take the form

$$\frac{\partial \underset{\sim}{u}}{\partial t} - \nu\Delta\underset{\sim}{u} + P(\underset{\sim}{u}\ .\ \nabla\underset{\sim}{u}) = P\underset{\sim}{f}\ . \tag{2.3}$$

A strong solution $\underset{\sim}{u}$ of (2.3) on a time interval $0 \le t < T$ satisfies

$$\underset{\sim}{u} \in L_2\big(0,T;\mathrm{dom}(\Delta)\big) \cap L_\infty(0,T;v)\quad .$$

The global existence of a unique strong solution for each $u_0 \in \dot{V}$, $\underset{\sim}{f} \in L_2(0,T;\dot{H})$ and any $0 < T < \infty$ is established in Temam [4] using a

using a priori estimates based on the relationships

$$\frac{1}{2}\frac{d}{dt}|\underset{\sim}{u}(t)|^2 + \nu\|\underset{\sim}{u}(t)\|^2 = \big(\underset{\sim}{f}(t),\ \underset{\sim}{u}(t)\big)\ ,$$

$$\frac{1}{2}\frac{d}{dt}\|\underset{\sim}{u}(t)\|^2 + \nu|\Delta\underset{\sim}{u}(t)|^2 = \big(\underset{\sim}{f}(t),\Delta\underset{\sim}{u}(t)\big) = -\Big(\big(\underset{\sim}{f}(t),\underset{\sim}{u}(t)\big)\Big)\ , \tag{2.4}$$

and on the Poincaré inequalities

$$\lambda_1|\underset{\sim}{u}|^2 \le \|\underset{\sim}{u}\|^2\ ,\quad \lambda_1\|\underset{\sim}{u}\|^2 \le |\Delta\underset{\sim}{u}|^2 \tag{2.5}$$

where $\lambda_1 > 0$ is defined below. These relationships are obtained by multiplying (2.3) by $\underset{\sim}{u}$ and $\Delta\underset{\sim}{u}$, respectively, integrating over Q and using the identities

$$\int_Q (\underset{\sim}{u}\ .\ \nabla\underset{\sim}{u})\ .\ \underset{\sim}{u} = 0 = \int_Q (\underset{\sim}{u}\ .\ \nabla\underset{\sim}{u})\ .\ \Delta\underset{\sim}{u}\ ,$$

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the latter being peculiar to this two-dimensional spatially periodic setting. (See Temam [4], page 19).

In this paper we restrict attention to time-independent forcing terms $\underset{\sim}{f}$, which we assume belong to the space $\dot{V}$. The crucial a priori estimate is then

(2.6) $$||\underset{\sim}{u}(t)|| \leq ||\underset{\sim}{u}_0||e^{-\nu\lambda_1 t} + (\nu\lambda_1)^{-1}||\underset{\sim}{f}|| \quad (1-e^{-\nu\lambda_1 t}) ,$$

from which we see that all solutions asymptote towards the bounded subset

$$B = \{\underset{\sim}{u} \in \dot{V};\ ||\underset{\sim}{u}|| \leq K_1 = ||\underset{\sim}{f}||/\nu\lambda_1\}$$

of V . This set B is in fact invariant under the Navier-Stokes equations that is any solution of (2.3) starting in B remains in B . We thus need only consider these solutions in B . Furthermore, the initial data $\underset{\sim}{u}_0 \in \mathrm{dom}(\Delta) \cap B$ we have the additional estimate

(2.7) $$|\Delta\underset{\sim}{u}(t)|^2 \leq |\Delta\underset{\sim}{u}_0|^2 e^{-\nu\lambda_1 t} + K_2(\nu,||\underset{\sim}{f}||)^2(1-e^{-\nu\lambda_1 t}) ,$$

which is derived in Foias and Temam [2]. The closed and bounded subset

$$A = \{\underset{\sim}{u} \in \mathrm{dom}(\Delta) \cap B;\ |\Delta\underset{\sim}{u}| \leq K_2\}$$

of $\mathrm{dom}(\Delta)$ is also invariant under the Navier-Stokes equations. In the sequel we restrict attention to solutions in A , which is in fact a compact subset of the spaces $\dot{V}$ and $\dot{H}$.

The Stokes operator, $-P\Delta$, is equal to $-\Delta$ here. It has eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \to \infty$ and corresponding eigenfunctions $\underset{\sim}{\phi}_1, \underset{\sim}{\phi}_2, \underset{\sim}{\phi}_3, \ldots$ $\mathrm{dom}(\Delta)$, satisfying

(2.7) $$-\Delta\underset{\sim}{\phi}_k = \lambda_k\underset{\sim}{\phi}_k \quad \textit{and } \mathrm{div}\ \underset{\sim}{\phi}_k = 0.$$

These eigenfunctions are complete and orthonormal in $\dot{H}$, and orthogonal in $\dot{V}$ with

(2.8) $$||\underset{\sim}{\phi}_k||^2 = \lambda_k .$$

We denote by P_n the orthogonal projector of $\mathbb{L}_p$ onto the finite-dimensional subspace of $\dot{H}$ spanned by $\{\underset{\sim}{\phi}_1, \underset{\sim}{\phi}_2, \ldots, \underset{\sim}{\phi}_n\}$ and write $Q_n = I - P_n$. From (2.8) we obtain the following generalizations of the Poincaré inequalities

(2.9) $$\lambda_{n+1}|Q_n\underset{\sim}{u}|^2 \leq ||Q_n\underset{\sim}{u}||^2 , \quad \lambda_{n+1}||Q_n\underset{\sim}{u}||^2 \leq |\Delta Q_n\underset{\sim}{u}|^2$$

Defining $\underset{\sim}{u}_n = P_n\underset{\sim}{u}$ and $\underset{\sim}{q}_n = Q_n\underset{\sim}{u}$, we can consider the Navier-Stokes equations (2.3) as a system on the finite-dimensional space P_nV coupled with a system on the infinite-dimensional space Q_nV :

$$\frac{\partial \underset{\sim}{u}_n}{\partial t} - \nu\Delta\underset{\sim}{u}_n + P_n(\underset{\sim}{u}_n.\nabla\underset{\sim}{u}_n) = P_n\underset{\sim}{f} - P_n(\underset{\sim}{u}_n.\nabla\underset{\sim}{q}_n+\underset{\sim}{q}_n.\nabla\underset{\sim}{u}_n+\underset{\sim}{q}_n.\nabla\underset{\sim}{q}_n)$$

(2.10)

$$\frac{\partial \underset{\sim}{q}_u}{\partial t} - \nu\Delta\underset{\sim}{q}_n + Q_n(\underset{\sim}{q}_n.\nabla\underset{\sim}{q}_n) = Q_n\underset{\sim}{f} - Q_n(\underset{\sim}{q}_n.\Delta\underset{\sim}{u}_n+\underset{\sim}{u}_n.\Delta\underset{\sim}{q}_n+\underset{\sim}{u}_n.\Delta\underset{\sim}{u}_n) ,$$

since $P_n\Delta = \Delta P_n$ and $Q_n\Delta = \Delta Q_n$ here. The finite-dimensional system

$$\text{(2.11)} \qquad \frac{\partial \bar{\underset{\sim}{u}}_n}{\partial t} - \nu\Delta\bar{\underset{\sim}{u}}_n + P_n(\bar{\underset{\sim}{u}}_n.\nabla\bar{\underset{\sim}{u}}_n) = P_n\underset{\sim}{f}$$

on P_nV obtained by discarding the coupling term is called the *nth-order Galerkin approximation* of the Navier-Stokes equations relative to the given basis of eigenfunctions. We can write it as a system of linear-quadratic ordinary differential equations on $\mathbb{R}^n$ in terms of the time-dependent coefficients of the linear span of $\bar{\underset{\sim}{u}}_n(x,t) = \sum_{j=1}^{n} c_{nj}(t)\underset{\sim}{\phi}_j(\underset{\sim}{x})$. We note that the various priori estimates such as (2.6) and (2.7), for the Navier-Stokes equations are also valid for any nth-order Galerkin approximation, so all solutions of (2.11) will be attracted to the subset P_nA and remain there. The solutions of the corresponding system of differential equations in $\mathbb{R}^n$ will thus be attracted to, and remain in, the compact subset of $\mathbb{R}^n$ corresponding to P_nA , namely

$$\{\underset{\sim}{y} \in \mathbb{R}^n ; \sum_{j=1}^{n} \lambda_j y_j^2 \le K_1^2 \ \& \ \sum_{j=1}^{n} \lambda_j^2 y_j^2 \le K_2^2\} .$$

3. Asymptotically stable attracting sets

We consider an autonomous system of ordinary differential equations

$$\text{(3.1)} \qquad \frac{d\underset{\sim}{y}}{dt} = \underset{\sim}{F}(\underset{\sim}{y})$$

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on $\mathbb{R}^n$. For the particular needs of this paper, this system will correspond to an nth-order Galerkin approximation (2.11) of the Navier-Stokes equations, although our discussion in this section holds more generally. We suppose that all solutions of (3.1) are attracted to some bounded subset of $\mathbb{R}^n$, hence exist for all future time, and that $\underset{\sim}{F}$ is uniformly Lipschitz on such a bounded subset.

Let Y be a nonempty, compact subset of $\mathbb{R}^n$, which is invariant under system (3.1). Following Yoshizawa [*6*], we say that Y is *uniformly stable* for (3.1) if for each $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$\mathrm{dist}(\underset{\sim}{y}(t;\underset{\sim}{y}_0),Y) < \varepsilon \quad \text{for all} \quad t \geq 0$$

whenever $\mathrm{dist}(\underset{\sim}{y}_0,Y) < \delta$, where $\underset{\sim}{y}(t;\underset{\sim}{y}_0)$ is the solution of (3.1) with initial value $\underset{\sim}{y}(0;\underset{\sim}{y}_0) = \underset{\sim}{y}_0$. If in addition there exists a $\delta_0 > 0$ and a $T(\varepsilon) > 0$ for each $\varepsilon > 0$ such that

$$\mathrm{dist}(\underset{\sim}{y}(t;\underset{\sim}{y}_0),Y) < \varepsilon \quad \text{for all} \quad t \geq T(\varepsilon)$$

whenever $\mathrm{dist}(\underset{\sim}{y}_0,Y) < \delta_0$, we say that Y is *uniformly asymptotically stable* for (3.1).

Yoshizawa [*6*] has given various necessary conditions and sufficient conditions for a compact subset Y to be uniformly stable or uniformly asymptotically stable. These involve the existence of an energy-like Lyapunov function that is a real-valued function defined on a neighbourhood of Y and possessing special properties. For such a function V we define the *upper-Dini derivative* relative to (3.1) as

$$D^+_{(3.1)}V(\underset{\sim}{y}) = \limsup_{h\to 0+}\{V(\underset{\sim}{y}+h\underset{\sim}{F}(\underset{\sim}{y})) - F(\underset{\sim}{y})\}/h \quad .$$

The following theorem of necessary conditions for uniform asymptotic stability is a restatement of Theorem 22.5 of Yoshizawa [*6*]. These conditions are also sufficient, as can be deduced from sections 14 and 16 of [*4*], but we do not require that here. We define

$$S(Y;R_0) = \{\underset{\sim}{y} \in \mathbb{R}^n;\ \mathrm{dist}(\underset{\sim}{y},Y) < R_0\} \quad .$$

THEOREM 3.1 *Suppose that the nonempty, compact subset* Y *of* $\mathbb{R}^n$ *is uniformly asymptotically stable for* (3.1) *and that* $\underset{\sim}{F}$ *is uniformly Lipschitzian on some sufficiently large neighbourhood of* $\underset{\sim}{Y}$. *Then there exists a function*

$$V \;:\; S(Y;R_0) \to \mathbb{R}^+$$

for some $R_0 > 0$ *for which:*

(I) V *is uniformly Lipschitzian on* $S(Y;R_0)$, *that is there exists a constant* $0 < L < \infty$ *such that*

$$|V(\underset{\sim}{y}) - V(\underset{\sim}{y}')| \le L|\underset{\sim}{y}' - \underset{\sim}{y}''|$$

for all $\underset{\sim}{y}, \underset{\sim}{y}' \in S(Y;R_0)$;

(II) *there exist continuous monotonically increasing functions* $\alpha, \beta : (0,R_0) \to \mathbb{R}^+$ *with* $\alpha(0) = \beta(0) = 0$ *and* $\alpha(r) < \beta(r)$ *for* $r > 0$ *such that*

$$\alpha(\mathrm{dist}(\underset{\sim}{y},Y)) \le V(\underset{\sim}{y}) \le \beta(\mathrm{dist}(\underset{\sim}{y},Y))$$

for all $\underset{\sim}{y} \in S(Y;R_0)$; *and*

(III) *there exists a constant* $c > 0$ *such that*

$$D^+_{(3.1)} V(\underset{\sim}{y}) \le -cV(\underset{\sim}{y})$$

for all $\underset{\sim}{y} \in S(Y;R_0)$.

We note here that

$$\mathrm{dist}(\underset{\sim}{y},Y) = \inf\{|\underset{\sim}{y} - \underset{\sim}{y}'| \;;\; y' \in Y\} ,$$

where the infimum is actually attained for a compact set Y .

4. Main result

The main result of this paper is to show that when a sufficiently high order Galerkin approximation of the Navier-Stokes equations has a uniformly asymptotically stable compact attracting set, then the Navier-

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Stokes equations have a nearby, compact in H , attracting set which is also uniformly asymptotically stable. We prove it here, for simplicity, for the two-dimensional Navier-Stokes equations with periodic boundary conditions. Our proof could be extended to the usual case of vanishing boundary conditions, or to the three-dimensional Navier-Stokes equations provided the global existence of sufficiently smooth solutions is assumed.

This generalizes a recent result of Constantin, Foias and Temam [1], who considered the simplest type of attractor, a steady solution. They made considerable use of the spectral properties of the linearization about this steady solution. Independently, Kloeden [2] obtained the same result using Lyapunov functions rather than spectral properties to characterize the stability. The advantage of this approach, to be used here, is that it extends naturally to more complicated attracting sets for which the spectral properties may not be so apparent. There is however a coupling between the size of the Galerkin approximation and the stability characteristics of its attracting set, which will be revealed in the proof of the theorem.

THEOREM 4.1 *Suppose that* $\underset{\sim}{f} \in \dot{V}$ *and that* Λ_n *is a nonempty compact subset of* $P_n\dot{H}$ *which is uniformly asymptotically stable for the nth-order Galerkin approximation of the two-dimensional Navier-Stokes equations with periodic boundary condition. In addition suppose that the eigenvalue* λ_{n+1} *of the corresponding Stokes operator is sufficiently large in comparison with the stability characteristics of* Λ_n *(as specified in the proof).*

Then there exists a nonempty compact subset Λ *of* $\dot{H}$ *which is uniformly asymptotically stable for the Navier-Stokes equations and is close to* Λ_n *in the Hausdorff metric relative to the* H*-norm.*

Proof. We identify the nth-order Galerkin approximation (2.11) with a vector differential equation (3.1) by writing $\underset{\sim}{y} \in \mathbb{R}^n$ for the coordinates of elements in P_nH relative to the basis $\{\phi_1, \phi_2, \ldots, \phi_n\}$. Since the inequalities (2.6) and (2.7) also hold for a Galerkin

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approximation, the attractor Λ_n is in fact a subset of $P_n A$. As above, we can identify Λ_n with a compact subset Y_n of $\mathbb{R}^n$, which is then uniformly asymptotically stable for the differential equation (3.1) corresponding to the Galerkin approximation. By Theorem 3.1 there thus exists a Lyapunov function V defined on some neighbourhood $S(Y_n;R_0)$ of Y_n in $\mathbb{R}^n$, with corresponding functions α,β and constants c, L and R_0 (which all depend on n, omitted for typographical convenience).

Now let $\underset{\sim}{u}(t) \in A$ be any solution of the Navier-Stokes equations (2.3) and set $\underset{\sim}{u}_n(t) = P_n \underset{\sim}{u}(t)$, $\underset{\sim}{q}_n(t) = Q_n \underset{\sim}{u}(t)$. We can then consider $\underset{\sim}{u}_n(t)$ as a solution of the first system of (2.10)

$$\text{(4.1)} \qquad \frac{\partial \underset{\sim}{u}_n}{t} - \nu\Delta \underset{\sim}{u}_n + P_n(\underset{\sim}{u}_n \cdot \nabla u_n) = P_n \underset{\sim}{f} - P_n(\underset{\sim}{u}_n \cdot \nabla \underset{\sim}{q}_n(t) + \underset{\sim}{q}_n(t) \cdot \nabla \underset{\sim}{u}_n + \underset{\sim}{q}_n(t) \cdot \nabla \underset{\sim}{q}_n(t))$$

in $P_n \dot{H}$ corresponding to a given forcing term $\underset{\sim}{q}_n(t)$. Identifying $\underset{\sim}{y} \in \mathbb{R}^n$ with the coordinates of $\underset{\sim}{u}_n$ in $P_n \dot{H}$ and $\underset{\sim}{G}(\underset{\sim}{y},t)$ with those of

$$\text{(4.2)} \qquad - P_n(\underset{\sim}{u}_n \cdot \nabla \underset{\sim}{q}_n(t) + \underset{\sim}{q}_n(t) \cdot \nabla \underset{\sim}{u}_n + \underset{\sim}{q}_n(t) \cdot \nabla \underset{\sim}{q}_n(t)) \,,$$

we may regard (4.1) as the perturbation

$$\text{(4.3)} \qquad \frac{d\underset{\sim}{y}}{dt} = \underset{\sim}{F}(\underset{\sim}{y}) + \underset{\sim}{G}(\underset{\sim}{y},t)$$

of the appropriate equation (3.1).

We use the Lyapunov function V for the unperturbed system (3.1) to investigate the stability and boundedness properties of the system (4.3) for an arbitrary, continuous perturbation $\underset{\sim}{G}$. This requires the perturbation $\underset{\sim}{G}$ to be sufficiently small so that the solutions of (4.3) under consideration remain in $S(Y_n;R_0)$, the domain of definition of V. From inequalities (2.9) we have for any $\underset{\sim}{u} \in A$

$$\text{(4.4)} \qquad |\underset{\sim}{q}_n| \le ||\underset{\sim}{q}_n||/\lambda_{n+1}^{1/2} \le ||\underset{\sim}{u}||/\lambda_{n+1}^{1/2} \le K_1/\lambda_{n+1}^{1/2}$$

and

(4.5) $$\|q_n\| \le \|\Delta q_n\|/\lambda_{n+1}^{1/2} \le |\Delta y|\ \lambda_{n+1}^{1/2} \le K_2/\lambda_{n+1}^{1/2}$$

where $q_n = Q_n u$. Using these we shall show in the appendix that

(4.6) $$|G| = |P_n(u_n \cdot \nabla q_n(t) + q_n(t)\cdot \nabla u_n + q_n(t)\cdot \nabla q_n(t))| \le K_2 K_3/\lambda_{n+1}^{1/2}$$

for any $u = u_n + q_n \in A$, where the constant K_3 depends on K_1 and K_2 (see (5.6)). We now assume that λ_{n+1} satisfies

(4.7) $$\lambda_{n+1} > (4\ L\ K_2\ K_3/c\ \alpha(R_0))^2$$

and define $r_0 = \beta^{-1}(\frac{1}{4}\alpha(R_0))$, so that $r_0 < \beta^{-1}(\alpha(R_0)) \le R_0$. Then any solution $y(t)$ of (4.3) with initial value $y(0) \in S(Y_n;r_0)$ remains in $S(Y_n;R_0)$ for all $t \ge 0$. Suppose this were not so. Then there would exist a first instant $0 < \tau < \infty$ such that $\mathrm{dist}(y(\tau),Y_n) = R_0$. As $y(t)$ remains in the domain of definition of V for $0 \le t < \tau$, we obtain from the properties of V (that is Theorem 3.1) and upper-Dini derivatives and from (4.6)

(4.8) $$\begin{aligned} D^+_{(4.3)}V(y(t)) &\le D^+_{(3.1)}V(y(t)) + L|G(y(t),t)| \\ &\le -cV(y(t)) + L\ K_2\ K_3/\lambda_{n+1}^{1/2} \end{aligned}$$

for $0 \le t < \tau$. Hence

$$\begin{aligned} V(y(t)) &\le V(y(0))e^{-ct} + (1 - e^{-ct})\ L\ K_2\ K_3/c\ \lambda_{n+1}^{1/2} , \\ &\le V(y(0)) + L\ K_2\ K_3/c\ \lambda_{n+1}^{1/2} , \\ &< \beta(\mathrm{dist}(y(0),Y_n)) + \tfrac{1}{2}\alpha(R_0) , \\ &\le \beta(r_0) + \tfrac{1}{2}\alpha(R_0) = \tfrac{3}{4}\alpha(R_0) , \end{aligned}$$

for $0 \le t < \tau$, where we have used (4.7) and the definition of r_0 . By the continuity of V and y we thus have

$$V(\underset{\sim}{y}(\tau)) \le \frac{3}{4}\,\alpha(R_0) \quad ,$$

which contracts the fact that

$$\alpha(R_0) = \alpha(\mathrm{dist}(\underset{\sim}{y}(\tau),Y_n)) \le V(\underset{\sim}{y}(\tau)) \quad .$$

Hence no such τ exists and $\underset{\sim}{y}(t) \in S(Y_n;R_0)$ for all $t \ge 0$.

We now strengthen inequality (4.7) to

$$\lambda_{n+1} > (2\ L\ K_2\ K_3/c\ \min\{r_0,\alpha(r_0)\})^2 \tag{4.9}$$

and define $\eta = 2\ L\ K_2\ K_3/c\ \lambda_{n+1}^{1/2}$, so that $\eta \le {}^1\!/_2 \min\{r_0,\alpha(r_0)\}$. We claim that the nonempty compact subset

$$\tilde{Y}_n = \{\underset{\sim}{y} \in \mathbb{R}^n ; V(y) \le \eta\} \quad ,$$

which contains $Y_n = \{\underset{\sim}{y} \in \mathbb{R}^n ; V(\underset{\sim}{y}) = 0\}$ in its interior, is uniformly asymptotically stable for the perturbed system (4.3) as long as the perturbation $\underset{\sim}{G}$ satisfies the bound (4.6) and as long as n is large enough for (4.9) to be satisfied. Our proof is adapted from that of Theorem 25.3 of Yoshizawa [*6*].

Let $\underset{\sim}{y} \in S(Y_n;R_0)\backslash\tilde{Y}_n$. Then $V(\underset{\sim}{y}) > \eta$ and from (4.8) we have

$$D^+_{(4\cdot3)}V(\underset{\sim}{y}) \le -c\ V(\underset{\sim}{y}) + \tfrac{1}{2}c\ \eta$$

$$< -c\ \eta + \frac{1}{2}\,c\ \eta = -\,\frac{1}{2}\,c\ \eta\ , \tag{4.10}$$

so any solution $\underset{\sim}{y}(t)$ of (4.3) starting in $\tilde{Y}_n$ must remain in $\tilde{Y}_n$, that is $\tilde{Y}_n$ is invariant under (4.3). Moreover for $y \in \tilde{Y}_n$, we have

$$\alpha(\mathrm{dist}(\underset{\sim}{y},Y_n)) \le V(y) \le \eta$$

so

$$\mathrm{dist}(\underset{\sim}{y},Y_n) \le \alpha^{-1}(\eta) \ . \tag{4.11}$$

Hence $Y_n \subseteq \tilde{Y}_n \subseteq \overline{S(Y_n;\alpha^{-1}(\eta))}$ and as $\alpha^{-1}(\eta) \to 0$ for $\eta \to 0$, we have $\tilde{Y}_n$ close to Y_n in the Hausdorff metric on compacta.

Let $\delta_0 = r_0 - \alpha^{-1}(\eta)$. Then $\delta_0 > 0$, as $\eta \le \frac{1}{2}\min\{r_0, \alpha(r_0)\}$, so

$$S(\tilde{Y}_n;\delta_0) \subseteq \overline{S(Y_n;\delta_0 + \alpha^{-1}(\eta))} \subseteq S(Y_n;r_0) \quad .$$

Suppose that a solution $\underset{\sim}{y}(t)$ of (4.3) with initial value $\underset{\sim}{y}_0 \in S(\tilde{Y}_n;r_0)$ does not belong to $\tilde{Y}_n$. Then $V(\underset{\sim}{y}(t)) > \eta$ and from (4.10)

$$\begin{aligned} V(\underset{\sim}{y}(t)) &\le -\frac{1}{2}c\,\eta\,t + V(\underset{\sim}{y}_0) \quad , \\ &\le -\frac{1}{2}c\,\eta\,t + \beta(\delta_0 + \alpha^{-1}(\eta)) \quad , \\ &= -\frac{1}{2}c\,\eta\,t + \beta(r_0) \quad , \end{aligned}$$

which is no greater than η for $t \ge T^* = 2(\beta(r_0) - \eta)/c\,\eta$, where $T^* > 0$ as $\eta \le \frac{1}{2}\alpha(r_0) < \beta(r_0)$. Hence for any $\varepsilon > 0$ we certainly have $\underset{\sim}{y}(t) \in S(\tilde{Y}_n;\varepsilon)$ for $t \ge T^*$ and $y_0 \in S(Y_n;r_0)$, which is the asymptotic part of the uniform asymptotic stability definition. It remains now only to show that $\tilde{Y}_n$ is uniformly stable for (4.3). The proof of this is exactly the same as on pages 135 and 136 of Yoshizawa [6] and will not be repeated here, other than to mention that it uses the invariance of $\tilde{Y}_n$ and the absorbing property of $\tilde{Y}_n$ just proven, and the uniform Lipschitz property of $\underset{\sim}{F}(\underset{\sim}{y})$ and $\underset{\sim}{G}(\underset{\sim}{y},t)$ in $\underset{\sim}{y} \in S(Y_n;R_0)$. We thus have that $\tilde{Y}_n$ is uniformly asymptotically stable for the perturbed system (4.3) and any admissible perturbation $\underset{\sim}{G}$ under consideration. Moreover $\tilde{Y}_n$ is independen of any particular admissible perturbation under consideration.

We are now ready to return to the Navier-Stokes equations (2.3), or in coupled form (2.10). To begin we restrict attention to the bounded invariant subset A of $\mathrm{dom}(\Delta)$ and define

$$\Lambda = \{\underset{\sim}{u} \in A;\ P_n\underset{\sim}{u} \in \tilde{\Lambda}_n\}$$

where

$$\tilde{\Lambda}_n = \{\underset{\sim}{u}_n = \textstyle\sum_{j=1}^{n} y_j\,\underset{\sim}{\phi}_j \in P_nA;\ \underset{\sim}{y} \in \tilde{Y}_n\} \ .$$

Clearly $\tilde{\Lambda}_n$ is closed and bounded in $P_n A$, so Λ is closed and bounded in A. Thus Λ is a compact subset of V, and also of H. Moreover, for any $\underset{\sim}{u} \in \Lambda$, with $P_n \underset{\sim}{u} = \sum_{j=1}^{n} y_j \underset{\sim}{\phi}_j (\underset{\sim}{x})$, we have from (4.4) and (4.11)

$$\begin{aligned} \operatorname{dist}_H(\underset{\sim}{u},\Lambda) &= \operatorname{dist}_H(P_n\underset{\sim}{u},\tilde{\Lambda}_n) + |Q_n\underset{\sim}{u}| \\ &= \operatorname{dist}(\underset{\sim}{y},\tilde{Y}_n) + |Q_n\underset{\sim}{u}| \\ &\le \alpha^{-1}(\eta) + K_1/\lambda_{n+1}^{1/2} \\ &\le \alpha^{-1}(2\ L\ K_2K_3/c\ \lambda_{n+1}^{1/2}) + K_1/\lambda_{n+1}^{1/2} \end{aligned}$$

which becomes arbitrarily small as n becomes arbitrarily large. The uniform asymptotic stability of Λ for the Navier-Stokes equations restricted to A then follows from that of $\tilde{Y}_n$ for the appropriate perturbed equation (4.3) in $\mathbb{R}^n$. Finally, the restriction to A can be lifted on account of inequality (2.6) and an analogous inequality to (2.7).

Apart from the proof of the auxiliary estimate (4.6), which is given in the appendix, this completes the proof of Theorem 4.1.

There are several points that need to be made about the attracting set Λ in the above theorem. More than likely it will not be minimal attracting set for the Navier-Stokes equations, but will contain such a set. (As a simple illustration, any set $[-\varepsilon,\varepsilon]$ is a uniformly asymptotically stable attractor for the differential equation $\dot{x} = -x$, yet only the subset $\{0\}$ is the minimal attractor). In the simple case where the attractor in the Galerkin approximation is a steady solution, both Constantin, Foias and Temam [1] and Kloeden [3] have shown that the Navier-Stokes equations have a nearby attracting steady solution. For more complicated types of a attractors such as periodic orbits, the attractor in the Navier-Stokes equations need not be of exactly the same form, though as we can see from the construction of Λ, it will be close (for example, in a thin tube about the original periodic orbit). Methods other than considered here will be needed to clarify the exact form of the minimal attractor contained in Λ.

Finally, we note that expressions (4.7) and (4.9) involve a coupling between the size of the eigenvalue λ_{n+1} and the stability characteristics of the n-th order Galerkin approximation, namely c_n, L_n, $\alpha_n(R_{0n})$, r_{0n} etcetera which we now label with the subscript n. These quantities may change as n changes, so there is not a priori guarantee that (4.7) and (4.9) can be made to hold simply by taking a larger value of n. The problem here is that the successively higher order Galerkin approximations may have progressively weaker attracting strength, so that in the limit there is no attraction at all. Constantin, Foias and Temam [1] show that this cannot happen in the case of an asymptotically stable steady solution. We do not believe that it can happen in the general case, but are unable to provide a proof. (This problem does not occur in the converse situation where the Navier-Stokes equations are assumed to have an asymptotically stable attracting set and it is to be shown that sufficiently large Galerkin approximations have a nearby asymptotically stable attracting set; here the stability characteristics of the original attractor are fixed and do not depend on the particular approximation being considered.)

Appendix: Uniform bound on the perturbations G

We need now to establish the uniform bound (4.6) on the perturbations G, which was used in the proof of Theorem 4.1. For this we use the inequality

$$|\underset{\sim}{u}\cdot\nabla\underset{\sim}{v}| \le c_1|\underset{\sim}{u}|^{\frac{1}{2}}\,|\Delta\underset{\sim}{u}|^{\frac{1}{2}}\,\|\underset{\sim}{v}\| \quad ,$$

which follows from various Sobolev space embedding inequalities. See Temam [3; pages 11-13]. Then

$$\begin{aligned}
|\underset{\sim}{G}| &= |P_n(\underset{\sim}{u}_n\cdot\nabla\underset{\sim}{q}_n + \underset{\sim}{q}_n\cdot\nabla\underset{\sim}{u}_n + \underset{\sim}{q}_n\cdot\nabla\underset{\sim}{q}_n)| \qquad\qquad (L_2\text{-norms})\\
&\le |\underset{\sim}{u}_n\cdot\nabla\underset{\sim}{q}_n| + |\underset{\sim}{q}_n\cdot\nabla\underset{\sim}{u}_n| + |\underset{\sim}{q}_n\cdot\nabla\underset{\sim}{q}_n|\\
&\le c_3|\underset{\sim}{u}_n|^{\frac{1}{2}}\,|\Delta\underset{\sim}{u}_n|^{\frac{1}{2}}\,\|\underset{\sim}{q}_n\| + c_3|\underset{\sim}{q}_n|^{\frac{1}{2}}\,|\Delta\underset{\sim}{q}_n|^{\frac{1}{2}}\,\|\underset{\sim}{u}_n\|\\
&\qquad + c_3|\underset{\sim}{q}_n|^{\frac{1}{2}}\,|\Delta\underset{\sim}{q}_n|^{\frac{1}{2}}\,\|\underset{\sim}{q}_n\|\\
&\le c_3|\underset{\sim}{u}_n|^{\frac{1}{2}}\,|\Delta\underset{\sim}{u}_n|^{\frac{1}{2}}\,|\Delta\underset{\sim}{q}_n|/\lambda_{n+1}^{\frac{1}{2}} + c_3|\Delta\underset{\sim}{q}_n|\,\|\underset{\sim}{u}_n\|/\lambda_{n+1}^{\frac{1}{2}}\\
&\qquad + c_3|\Delta\underset{\sim}{q}_n|\,\|\underset{\sim}{q}_n\|/\lambda_{n+1}^{\frac{1}{2}}
\end{aligned}$$

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$$\leq c_3(K_1^{\frac{1}{2}}K_2^{\frac{1}{2}} + 2K_1)\ |\Delta q_n|/\lambda_{n+1}^{\frac{1}{2}}$$

$$\leq c_3(K_1^{\frac{1}{2}}K_2^{\frac{1}{2}} + 2K_1)\ K_2/\lambda_{n+1}^{\frac{1}{2}}$$

$$\leq K_3\cdot K_2/\lambda_{n+1}^{\frac{1}{2}}$$

where we have used the generalized Poincaré inequalities (4.4) and (4.5) and the fact that $\underset{\sim}{u} \in A$.

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School of Mathematics and Physical Sciences,
Murdoch University,
Murdoch 6150
Western Australia.

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