VISIBLE ACTIONS ON FLAG VARIETIES OF TYPE B AND A GENERALISATION OF THE CARTAN DECOMPOSITION

YUICHIRO TANAKA

Abstract

We give a generalisation of the Cartan decomposition for connected compact Lie groups of type B motivated by the work on visible actions of Kobayashi ['A generalized Cartan decomposition for the double coset space $(U(n_1) \times U(n_2) \times U(n_3)) \backslash U(n)/(U(p) \times U(q))$', *J. Math. Soc. Japan* **59** (2007), 669–691] for type A groups. Suppose that G is a connected compact Lie group of type B, σ is a Chevalley–Weyl involution and L, H are Levi subgroups. First, we prove that $G = LG^\sigma H$ holds if and only if either (I) both H and L are maximal and of type A, or (II) (G, H) is symmetric and L is the Levi subgroup of an arbitrary maximal parabolic subgroup up to switching H and L. This classification gives a visible action of L on the generalised flag variety G/H, as well as that of the H-action on G/L and of the G-action on $(G \times G)/(L \times H)$. Second, we find an explicit 'slice' B with dim B = rank G in case I, and dim $B = 2$ or 3 in case II, such that a generalised Cartan decomposition $G = LBH$ holds. An application to multiplicity-free theorems of representations is also discussed.

1. Introduction and statement of main results

Let G be a connected compact simple Lie group of type B and σ a Chevalley–Weyl involution of G. The aim of this paper is to classify all the pairs (L, H) of Levi subgroups of G such that $G = LG^\sigma H$ holds. The motivation for considering this kind of decomposition comes from the theory of *visible actions* on complex manifolds introduced by Kobayashi [Ko2], and $G = LG^\sigma H$ can be interpreted as a generalisation of the Cartan decomposition to the nonsymmetric setting. (We refer the reader to [He, Ho, Ma1, Ko4] and references therein for some aspects of the Cartan decomposition from geometric and group-theoretic viewpoints.)

A generalisation of the Cartan decomposition for symmetric pairs has been used in various contexts including analysis on symmetric spaces; however, there was no analogous result for nonsymmetric cases before Kobayashi's paper [Ko4]. Motivated

FIGURE 1. Dynkin diagram of type B_n.

by visible actions on complex manifolds [Ko1, Ko2], he completely determined the pairs of Levi subgroups

$$(L, H) = (U(n_1) \times \cdots \times U(n_k), U(m_1) \times \cdots \times U(m_l))$$

of the unitary group $G = U(n)$ such that the multiplication mapping $L \times O(n) \times H \to G$ is surjective. Furthermore, he developed a method to find a suitable subset B of $O(n)$ which gives the following decomposition (a generalised Cartan decomposition, see [Ko4]):

$$G = LBH.$$

On the other hand, Sasaki recently studied visible actions in the setting where (G, H) is a pair of *complex* reductive Lie groups, and gave a generalisation of the Cartan decomposition $G = LBH$ [Sa1, Sa2].

Returning to the decomposition theory [Ko4], we considered the following problems for general compact Lie groups and examined the case where G is a connected compact simple Lie group of type D in another paper ('Visible actions on flag varieties of type D and a generalization of the Cartan decomposition', submitted for publication).

Let G be a connected compact Lie group, $\mathfrak{t}$ a Cartan subalgebra, and σ a Chevalley–Weyl involution of G with respect to $\mathfrak{t}$. (Here, we recall that an involutive automorphism μ of a connected compact Lie group K is said to be a Chevalley–Weyl involution if there is a maximal torus T of K such that $\mu(t) = t^{-1}$ for every $t \in T$ [Wo]. For instance, an involution $\sigma(g) = \bar{g}$ defines a Chevalley–Weyl involution of $G = U(n)$ with the standard maximal torus, and $G^\sigma = \{g \in U(n) : \bar{g} = g\} \simeq O(n)$.)

(1) Classify all the pairs of Levi subgroups L and H with respect to $\mathfrak{t}$ such that the multiplication mapping $\psi : L \times G^\sigma \times H \to G$ is surjective.
(2) Find a 'good' representative $B \subset G^\sigma$ such that $G = LBH$ in the case ψ is surjective.

We call such a decomposition $G = LBH$ a *generalised Cartan decomposition*. Here we note that the role of the subgroups H and L is symmetric.

In the present paper, we solve the above problems for connected compact simple Lie groups G of type B. In order to state the main results, we label the Dynkin diagram of type B_n as shown in Figure 1.

For a subset Π' of the set Π of simple roots, we denote by $L_{\Pi'}$ the Levi subgroup whose root system is generated by Π'. For example, $L_\varnothing$ is a maximal torus of G and $L_{\{\alpha_p\}^c} = U(p) \times SO(2(n - p) + 1)$ for $G = SO(2n + 1)$ $(1 \le p \le n)$. Here $(\Pi')^c$ denotes the complement $\Pi \setminus \Pi'$.

THEOREM 1.1. *Let G be a connected compact simple Lie group of type B_n, σ a Chevalley–Weyl involution, Π' and Π'' proper subsets of the simple system Π, and $L_{\Pi'}$ and $L_{\Pi''}$ the corresponding Levi subgroups. Then the following two conditions on $\{\Pi', \Pi''\}$ are equivalent.*

(i) $G = L_{\Pi'} \, G^\sigma \, L_{\Pi''}$.

(ii)   *One of the following conditions holds up to switch of the factors Π' and Π'' :*

$$\text{Case I.} \quad (\Pi')^c = \{\alpha_n\}, \quad (\Pi'')^c = \{\alpha_n\}.$$
$$\text{Case II.} \quad (\Pi')^c = \{\alpha_1\}, \quad (\Pi'')^c = \{\alpha_j\}, \quad 1 \le j \le n.$$

We note that the pair $(G, L_{\Pi'})$ forms a symmetric pair if and only if $(\Pi')^c = \{\alpha_1\}$, and that $G/L_{\Pi'} = G/L_{\Pi''}$ is a (nonsymmetric) spherical variety in case I (see [Kr]).

Theorem 1.1 implies that $G = LG^\sigma H$ holds if and only if (G, L, H) satisfies one of the following two conditions: (I) both H and L are maximal and of type A, or (II) (G, H) is symmetric and L comes from a maximal parabolic subgroup up to switching of H and L. In each case, we give a generalised Cartan decomposition $G = LBH$ explicitly with $\dim B = \operatorname{rank} G$ in case I and $\dim B = 2$ or 3 in case II. This is stated in Propositions 3.2 and 3.3.

Application to representation theory. A generalised Cartan decomposition $G = LBH$ implies that the subgroup L acts on G/H in a (strongly) visible fashion, and likewise H on G/L, and G on $(G \times G)/(L \times H)$. Then Kobayashi's theory leads us to three multiplicity-free theorems (*triunity* à la [Ko1]):

$$\text{Restriction } G \downarrow L : \operatorname{Ind}_H^G(\mathbb{C}_\lambda)|_L,$$
$$\text{Restriction } G \downarrow H : \operatorname{Ind}_L^G(\mathbb{C}_\lambda)|_H,$$
$$\text{Tensor product} : \operatorname{Ind}_H^G(\mathbb{C}_\lambda) \otimes \operatorname{Ind}_L^G(\mathbb{C}_\mu).$$

Here $\operatorname{Ind}_H^G(\mathbb{C}_\lambda)$ denotes a holomorphically induced representation of G from a character $\mathbb{C}_\lambda$ of H by the Borel–Weil theorem. See [Ko1, Ko2, Ko3, Ko5, Ko6] for the general theory on the application of visible actions (including the vector bundle setting), and Corollaries 5.4 and 5.5 for type B groups.

Special features of type B groups. We compare the main results with the previous results for type A [Ko4] and type D.

(1)   Although both type A and type D groups are rich in pairs of Levi subgroups (L, H) satisfying $G = LG^\sigma H$, our classification theorem shows that there are not many pairs for type B groups. This geometric result is reflected by the representation-theoretic fact that type B groups do not have many pairs (λ, μ) of highest weights such that the tensor product representation $a\lambda \otimes b\mu$ is multiplicity-free for arbitrary nonnegative integers a, b [Li, St].

(2)   The herringbone stitch method was used in type A, D and case II here; however, we take another approach to find a generalised Cartan decomposition in case I, that is, $L \backslash G/H = U(n) \backslash SO(2n+1)/U(n)$.

Organisation of this paper. In Section 2 we give a matrix realisation of the orthogonal group $G = \mathrm{SO}(2n + 1)$ and its subgroups which are used in Sections 3 and 4. In Section 3 we prove that (ii) implies (i). Furthermore, we find explicitly a slice B that gives a generalised Cartan decomposition $G = L_{\Pi'} B L_{\Pi''}$. The converse implication on (ii) $\Rightarrow$ (i) is proved in Section 4 by using the invariant theory for quivers. An application to multiplicity-free representations is discussed in Section 5.

2. Matrix realisation

The surjectivity of $\psi : L \times G^{\sigma} \times H \to G$ is independent of the coverings and the choice of Cartan subalgebras and Chevalley–Weyl involutions. Thus, we may and do work with the orthogonal group $\mathrm{SO}(2n + 1)$ and a fixed pair of a Cartan subalgebra and a Chevalley–Weyl involution in Sections 2–4.

Throughout this paper, we realise $G = \mathrm{SO}(2n + 1)$ as a matrix group as follows:

$$G := \{g \in \mathrm{SL}(2n + 1, \mathbb{C}) : {}^{t}g J_{2n+1} g = J_{2n+1}, \ {}^{t}\overline{g} g = I_{2n+1}\},$$

where ${}^{t}g$ denotes the transpose of g, and J_m is defined by

$$J_m := \begin{pmatrix} & & & & & 1 \\ & & & & 1 & \\ & O & & \cdot^{\cdot^{\cdot}} & & \\ & & \cdot^{\cdot} & & O & \\ 1 & & & & & \end{pmatrix} \in \mathrm{GL}(m, \mathbb{R}).$$

Then the corresponding Lie algebra $\mathfrak{g} = \mathfrak{so}(2n + 1)$ of G forms

$$\mathfrak{g} := \{X \in \mathfrak{sl}(2n + 1, \mathbb{C}) : {}^{t}X J_{2n+1} + J_{2n+1} X = O, \ {}^{t}\overline{X} + X = O\}. \tag{2.1}$$

We take a Cartan subalgebra $\mathfrak{t}$ and an involution σ of G as follows:

$$\mathfrak{t} := \bigoplus_{1 \leq i \leq n} \mathbb{R} \sqrt{-1} H_i,$$
$$\sigma : G \to G, \quad g \mapsto \bar{g},$$

where $H_i := \mathrm{E}_{i,i} - \mathrm{E}_{2n+2-i,2n+2-i}$, and $\bar{g}$ denotes the complex conjugation of $g \in G$. The differential of σ is denoted by the same letter. Then σ is a Chevalley–Weyl involution of G with respect to $\mathfrak{t}$.

We let $\{\varepsilon_i\}_{1 \leq i \leq n} \subset (\mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C})^*$ be the dual basis of $\{H_i\}_{1 \leq i \leq n}$. Then we define a simple system $\Pi := \{\alpha_1, \ldots, \alpha_n\}$ by

$$\alpha_1 := \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{n-1} := \varepsilon_{n-1} - \varepsilon_n, \alpha_n := \varepsilon_n.$$

Let $n = n_1 + \cdots + n_k$ be a partition of n with $n_1, \ldots, n_{k-1} > 0$ and $n_k \geq 0$. We put

$$s_i := \sum_{1 \leq p \leq i} n_p \quad (1 \leq i \leq k - 1),$$

$$\Pi' := \Pi \setminus \{\alpha_{s_i} \in \Pi : 1 \leq i \leq k - 1\},$$

and denote by $L_{\Pi'}$ the Levi subgroup whose root system is generated by Π'. In the matrix realisation, $L_{\Pi'}$ takes the form

$$L_{\Pi'} = \left\{ \begin{pmatrix} A_1 & & & & & & \\ & \ddots & & & & & \\ & & A_{k-1} & & & & \\ & & & B & & & \\ & & & & J_{n_{k-1}}\overline{A_{k-1}}J_{n_{k-1}} & & \\ & & & & & \ddots & \\ & & & & & & J_{n_1}\overline{A_1}J_{n_1} \end{pmatrix} : \begin{array}{l} A_i \in U(n_i)\,(1 \leq i \leq k-1), \\ \\ B \in SO(2n_k + 1) \end{array} \right\}$$

$$\simeq U(n_1) \times \cdots \times U(n_{k-1}) \times SO(2n_k + 1). \tag{2.2}$$

Here, we note that $(G, L_{\Pi'})$ forms a symmetric pair if and only if $(\Pi')^c = \Pi \setminus \Pi' = \{\alpha_1\}$, and that $G / L_{\{\alpha_n\}^c}$ is a weakly symmetric space in the sense of Selberg. For later purposes, we give explicitly an involution τ_1 and an automorphism μ satisfying $\mu^4 = \mathrm{id}$ of which the connected components of fixed point subgroups are $L_{\{\alpha_1\}^c}$ and $L_{\{\alpha_n\}^c}$ respectively:

$$L_{\{\alpha_1\}^c} = (G^{\tau_1})_0, \quad \tau_1 : G \to G, \quad g \mapsto I_{1,2(n-1)+1,1}\, g\, I_{1,2(n-1)+1,1},$$

$$L_{\{\alpha_n\}^c} = G^{\mu}, \quad \mu : G \to G, \quad g \mapsto I_{\sqrt{-1}}\, g\, I_{\sqrt{-1}}, \tag{2.3}$$

where K_0 denotes the connected component of K containing the identity element for a Lie group K, and $I_{1,2(n-1)+1,1}$, $I_{\sqrt{-1}}$ are defined by

$$I_{1,2(n-1)+1,1} := \mathrm{diag}(-1, \overbrace{1, \ldots, 1}^{2(n-1)+1}, -1),$$

$$I_{\sqrt{-1}} := \mathrm{diag}(\overbrace{\sqrt{-1}, \ldots, \sqrt{-1}}^{n}, 1, \overbrace{-\sqrt{-1}, \ldots, -\sqrt{-1}}^{n}).$$

To obtain a generalised Cartan decomposition by the herringbone stitch method, we will use an involutive automorphism τ_p of G $(1 \leq p \leq n)$ given by

$$\tau_p : G \to G, \quad g \mapsto I_{p,2(n-p)+1,p}\, g\, I_{p,2(n-p)+1,p},$$

where

$$I_{p,2(n-p)+1,p} := \mathrm{diag}(\overbrace{-1, \ldots, -1}^{p}, \overbrace{1, \ldots, 1}^{2(n-p)+1}, \overbrace{-1, \ldots, -1}^{p}).$$

Then $(G^{\tau_p})_0$ is given by

$$\mathrm{SO}(2p) \times \mathrm{SO}(2n - 2p + 1)$$

$$= \left\{ \left(\begin{array}{c|c|c} A & & B \\ \hline & S & \\ \hline C & & D \end{array} \right) : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SO}(2p), S \in \mathrm{SO}(2n - 2p + 1) \right\}.$$

3. Generalised Cartan decomposition

In this section we give a proof of the implication (ii) $\Rightarrow$ (i) in Theorem 1.1. The idea is to use the herringbone stitch method that reduces unknown decompositions for nonsymmetric pairs to the known Cartan decomposition for symmetric pairs.

3.1. Cartan decomposition for symmetric pairs. In this subsection we recall a well-known fact on the Cartan decomposition for the symmetric case ([Ho, Theorem 6.10], [Ma2, Theorem 1]).

FACT 3.1. *Let K be a connected compact Lie group with Lie algebra $\mathfrak{k}$ and two involutions τ, τ' $(\tau^2 = (\tau')^2 = \mathrm{id})$. Let H and L be subgroups of K such that*

$$(K^\tau)_0 \subset L \subset K^\tau \quad and \quad (K^{\tau'})_0 \subset H \subset K^{\tau'}.$$

We take a maximal abelian subspace $\mathfrak{b}$ in

$$\mathfrak{k}^{-\tau, -\tau'} := \{X \in \mathfrak{k} : \tau(X) = \tau'(X) = -X\},$$

and write B for the connected abelian subgroup with Lie algebra $\mathfrak{b}$. Suppose that $\tau\tau'$ is semisimple on the center $\mathfrak{z}$ of $\mathfrak{k}$. Then

$$K = LBH.$$

3.2. Decomposition for case I. This subsection is devoted to showing the following proposition.

PROPOSITION 3.2 (Generalised Cartan decomposition for case I). *Let $G = \mathrm{SO}(2n + 1)$ and $(\Pi')^c = (\Pi'')^c = \{\alpha_n\}$. Then $G = \mathrm{L}_{\Pi'} \exp(\mathfrak{a} \oplus \mathfrak{q}) \mathrm{L}_{\Pi''}$ where $\mathfrak{a}$ and $\mathfrak{q}$ are defined by*

$$\mathfrak{a} := \bigoplus_{i=1}^{[n/2]} \mathbb{R}(\mathrm{E}_{2i-1,2n-2i+2} - \mathrm{E}_{2i,2n-2i+3} - \mathrm{E}_{2n-2i+2,2i-1} + \mathrm{E}_{2n-2i+3,2i}),$$

$$\mathfrak{q} := \bigoplus_{i=1}^{[(n+1)/2]} \mathbb{R}(\mathrm{E}_{2i-1,n+1} - \mathrm{E}_{n+1,2n+3-2i} - \mathrm{E}_{n+1,2i-1} + \mathrm{E}_{2n+3-2i,n+1}).$$

PROOF. Since an automorphism μ of $\mathfrak{g}$ is an involution of $\mathfrak{g}^{\mu^2}$ (see (2.3) for the definition of μ) and $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{g}^{-\mu}$,

$$\mathfrak{g} = \mathfrak{g}^\mu \oplus \left(\bigcup_{g \in G^\mu} \mathrm{Ad}(g)\mathfrak{a} \right) \oplus \mathfrak{g}^{-\mu^2}. \tag{3.1}$$

Let $Z_{\mathfrak{g}^\mu}(\mathfrak{a})$ denote the centraliser of $\mathfrak{a}$ in $\mathfrak{g}^\mu$. Then $M := \exp(Z_{\mathfrak{g}^\mu}(\mathfrak{a}))$ is given by

$$M = \begin{cases} \mathrm{SU}(2)^m & (n = 2m), \\ \mathrm{SU}(2)^m \times \mathrm{U}(1) & (n = 2m + 1). \end{cases}$$

By using this block diagonal matrix group M, we rewrite the third factor $\mathfrak{g}^{-\mu^2}$ of the decomposition (3.1) as

$$\mathfrak{g}^{-\mu^2} = \bigcup_{g \in M} \mathrm{Ad}(g)\mathfrak{q}. \tag{3.2}$$

We omit details since this can be verified by a simple matrix computation. Equation (3.2) yields

$$\left(\bigcup_{g \in G^\mu} \mathrm{Ad}(g)\mathfrak{a}\right) \oplus \mathfrak{g}^{-\mu^2} = \bigcup_{g \in G^\mu} \mathrm{Ad}(g)(\mathfrak{a} \oplus \mathfrak{q}). \tag{3.3}$$

Let us verify (3.3). It is clear that the left-hand side contains the right-hand side. We show the converse inclusive relation. From (3.2), for any $l \in G^\mu$, $X \in \mathfrak{a}$ and $Z \in \mathfrak{g}^{-\mu^2}$, there exist $h \in M$ and $Y \in \mathfrak{q}$ satisfying

$$\mathrm{Ad}(h)Y = \mathrm{Ad}(l)^{-1}Z.$$

Then

$$\mathrm{Ad}(l)X + Z = \mathrm{Ad}(l)(\mathrm{Ad}(h)X) + \mathrm{Ad}(lh)(Y)$$
$$= \mathrm{Ad}(lh)(X + Y).$$

Thus $\mathrm{Ad}(l)X + Z$ belongs to $\bigcup_{g \in G^\mu} \mathrm{Ad}(g)(\mathfrak{a} \oplus \mathfrak{q})$, and we have shown (3.3).

We are ready to give a generalised Cartan decomposition for case I. We continue the decomposition (3.1) as follows:

$$\mathfrak{g} = \mathfrak{g}^\mu \oplus \left(\bigcup_{g \in G^\mu} \mathrm{Ad}(g)\mathfrak{a}\right) \oplus \mathfrak{g}^{-\mu^2}$$
$$= \mathfrak{g}^\mu \oplus \left(\bigcup_{g \in G^\mu} \mathrm{Ad}(g)(\mathfrak{a} \oplus \mathfrak{q})\right) \quad \text{by (3.3).}$$

Hence we can find that the exponential mapping

$$\exp : \bigcup_{g \in G^\mu} \mathrm{Ad}(g)(\mathfrak{a} \oplus \mathfrak{q}) \to G/G^\mu$$

is surjective [He]. Consequently,

$$G = \exp\left(\bigcup_{g \in G^\mu} \mathrm{Ad}(g)(\mathfrak{a} \oplus \mathfrak{q})\right)G^\mu$$
$$= G^\mu \exp(\mathfrak{a} \oplus \mathfrak{q})G^\mu$$
$$= L_{\Pi'} \exp(\mathfrak{a} \oplus \mathfrak{q}) L_{\Pi''}. \qquad \square$$

3.3. Decomposition for case II. The aim of this subsection is to show the following proposition.
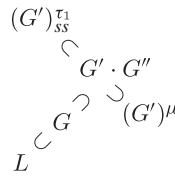
FIGURE 2. Herringbone stitch used for $L\backslash G/H$ in case II.

PROPOSITION 3.3 (Generalised Cartan decomposition for case II). *Let $G = \mathrm{SO}(2n + 1)$,* $(\Pi')^c = \{\alpha_1\}$ *and* $(\Pi'')^c = \{\alpha_j\}$ $(1 \le j \le n)$. *We define abelian subspaces* $\mathfrak{b}_1$ *and* $\mathfrak{b}_2$ *of* $\mathfrak{g}$ *by*

$$\mathfrak{b}_1 := \mathbb{R}(E_{1,j+1} - E_{2n-j+1,2n+1} - E_{j+1,1} + E_{2n+1,2n-j+1})$$
$$+ \mathbb{R}(E_{1,2n-j+1} - E_{j+1,2n+1} - E_{2n-j+1,1} + E_{2n+1,j+1}),$$
$$\mathfrak{b}_2 := \mathbb{R}(E_{1,2n-j+2} - E_{j,2n+1} - E_{2n-j+2,1} + E_{2n+1,j}).$$

Then $G = \mathrm{L}_{\Pi'} \exp(\mathfrak{b}_1) \exp(\mathfrak{b}_2) \mathrm{L}_{\Pi''}$.

PROOF. We put $L = \mathrm{L}_{\Pi'}$, $H = \mathrm{L}_{\Pi''}$ for simplicity. Let us take a symmetric subgroup $G'G'' = (G^{\tau_j})_0$ containing H where G' and G'' are given by $G' := \mathrm{SO}(2j) \times I_{2n-2j+1}$ and $G'' := I_{2j} \times \mathrm{SO}(2n - 2j + 1)$. In light of the fact that $\mathfrak{b}_1$ is a maximal abelian subspace of $\mathfrak{g}^{-\tau_1,-\tau_j}$, we can see from Fact 3.1 that

$$G = L \exp(\mathfrak{b}_1)G'G''. \tag{3.4}$$

We take a symmetric subgroup $(G')^\mu = \mathrm{U}(j) \times I_{2n-2j+1}$ of G'. We again use Fact 3.1 as follows:

$$G' = (G')^{\tau_1}_0 \exp(\mathfrak{b}_2)(G')^\mu. \tag{3.5}$$

Further, (3.5) can be rewritten as

$$G' = (G')^{\tau_1}_{ss} \exp(\mathfrak{b}_2)(G')^\mu, \tag{3.6}$$

where $(G')^{\tau_1}_{ss}$ denotes the analytic subgroup of $(G')^{\tau_1}$ with Lie algebra the semisimple part of the Lie algebra of $(G')^{\tau_1}$. Then we continue the decomposition (3.4) as follows:

$$\begin{aligned}
G &= L \exp(\mathfrak{b}_1)G'G'' \quad \text{by (3.4)} \\
&= L \exp(\mathfrak{b}_1)((G')^{\tau_1}_{ss} \exp(\mathfrak{b}_2)(G')^\mu)G'' \quad \text{by (3.6)} \\
&= L(G')^{\tau_1}_{ss} \exp(\mathfrak{b}_1) \exp(\mathfrak{b}_2)(G')^\mu G'' \quad \text{by } (G')^{\tau_1}_{ss} \subset Z_G(\mathfrak{b}_1) \\
&= L \exp(\mathfrak{b}_1) \exp(\mathfrak{b}_2)H \quad \text{by } (G')^\mu G'' = H. \qquad \square
\end{aligned}$$

Figure 2 shows a herringbone stitch which we have used for $L\backslash G/H$ in case II.

4. Application of invariant theory of quivers

The aim of this section is to prove the implication (i) $\Rightarrow$ (ii) in Theorem 1.1. We shall use invariants of quivers for the proof as in [Ko4]. This section could be read independently of Section 3 which gives a proof of the opposite implication (ii) $\Rightarrow$ (i) in Theorem 1.1.

4.1. Invariants of quivers. In the following, the proofs of Lemmas 4.1–4.3 are essentially the same as [Ko4, Lemmas 6.1–6.3], respectively. We therefore give necessary changes and precise statements, but omit the proofs.

Let $\sigma : \mathrm{M}(N, \mathbb{C}) \to \mathrm{M}(N, \mathbb{C})$ be the complex conjugation with respect to $\mathrm{M}(N, \mathbb{R})$.

Lemma 4.1 (see [Ko4, Lemma 6.1]). *Let $G \subset \mathrm{GL}(N, \mathbb{C})$ be a σ-stable subgroup, $R \in \mathrm{M}(N, \mathbb{R})$, and L a subgroup of G. If there exists $g \in G$ such that*

$$\mathrm{Ad}(L)(\mathrm{Ad}(g)R) \cap \mathrm{M}(N, \mathbb{R}) = \varnothing,$$

then $G \neq LG^\sigma G_R$. Here $G_R := \{h \in G : hRh^{-1} = R\}$.

We return to the case $G = \mathrm{SO}(2n + 1)$. We fix a partition $n = n_1 + \cdots + n_k$ with $n_i > 0$ $(1 \leq i \leq n - 1)$, $n_k \geq 0$, and a positive integer $r \geq 2$. We consider the following loop:

$$i_0 \to i_1 \to \cdots \to i_r, \quad i_s \in \{1, \ldots, 2k - 1\}, i_0 = i_r, i_{s-1} \neq i_s \ (1 \leq s \leq r).$$

Correspondingly, we define a nonlinear mapping

$$A_{i_0 \cdots i_r} : \mathrm{M}(2n + 1, \mathbb{C}) \to \begin{cases} \mathrm{M}(n_{i_0}, \mathbb{C}) & (i_0 = i_r \neq k), \\ \mathrm{M}(2n_{i_0} + 1, \mathbb{C}) & (i_0 = i_r = k), \end{cases}$$

as follows: let $P \in \mathrm{M}(2n, \mathbb{C})$, and write P as $(P_{ij})_{1 \leq i, j \leq 2k-1}$ in block matrix form corresponding to the partition

$$2n + 1 = n_1 + \cdots + n_{k-1} + (2n_k + 1) + n_{k-1} + \cdots + n_1$$

such that

$$P_{ij} \in \begin{cases} \mathrm{M}(n_i, n_j; \mathbb{C}) & (i, j \neq k), \\ \mathrm{M}(2n_k + 1, n_j; \mathbb{C}) & (i = k, j \neq k), \\ \mathrm{M}(n_i, 2n_k + 1; \mathbb{C}) & (i \neq k, j = k), \\ \mathrm{M}(2n_k + 1, \mathbb{C}) & (i = j = k), \end{cases} \tag{4.1}$$

where $n_{2k-i} := n_i$ $(1 \leq i \leq k)$. Then we define $(\tilde{P})_{1 \leq i, j \leq 2k-1}$ and $A_{i_0 \cdots i_k}(P)$ by

$$\tilde{P}_{ij} := \begin{cases} P_{ij} & (i + j \leq 2k), \\ J_{n_i}{}^t P_{2k-j, 2k-i} J_{n_j} & (i + j > 2k, \ i, j \neq k), \\ J_{2n_k+1}{}^t P_{2k-j, k} J_{n_j} & (i = k, j > k), \\ J_{n_i}{}^t P_{k, 2k-i} J_{2n_k+1} & (i > k, j = k), \end{cases}$$

and

$$A_{i_0 \cdots i_r}(P) := \tilde{P}_{i_0 i_1} \tilde{P}_{i_1 i_2} \cdots \tilde{P}_{i_{r-1} i_r}.$$

The point here is that for any $l = (l_1, \ldots, l_{k-1}, l_k) \in L := \mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{k-1}) \times \mathrm{SO}(2n_k + 1)$ (see (2.2) in Section 2 for the realisation as a matrix), the following equality holds:

$$(\widetilde{\mathrm{Ad}(l)}P)_{ij} = l_i \tilde{P}_{ij} l_j^{-1}. \tag{4.2}$$

We omit details since (4.2) can be verified by a simple matrix computation. This equality leads us to the following lemma (see [Ko4, Lemma 6.2]).

LEMMA 4.2. *If there exists a loop $i_0 \to i_1 \to \cdots \to i_r$ such that at least one of the coefficients of the characteristic polynomial $\det(\lambda I_{n_{i_0}} - A_{i_0 \cdots i_r}(P))$ is not real, then*

$$\mathrm{Ad}(L)P \cap \mathrm{M}(2n, \mathbb{R}) = \varnothing.$$

Combining Lemma 4.1 with Lemma 4.2, we obtain the next lemma (see [Ko4, Lemma 6.3]).

LEMMA 4.3. *Let $n = n_1 + \cdots + n_k$ be a partition with $n_i > 0$ $(1 \le i \le n - 1)$, $n_k \ge 0$, and $L = \mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{k-1}) \times \mathrm{SO}(2n_k + 1)$ which is the corresponding Levi subgroup of $\mathrm{SO}(2n + 1)$. Let us suppose that R is a block diagonal matrix*

$$R := \begin{pmatrix} R_1 & & & \\ & R_2 & & \\ & & \ddots & \\ & & & R_{2k-1} \end{pmatrix},$$

where $R_s, R_{2k-s} \in \mathrm{M}(n_s, \mathbb{R})$ $(1 \le s \le k - 1)$, $R_k \in \mathrm{M}(2n_k + 1, \mathbb{R})$.
 If there exist $X \in \mathfrak{so}(2n + 1)$ and a loop $i_0 \to \cdots \to i_r$ such that

$$\det(\lambda I_{n_{i_0}} - A_{i_0 \cdots i_r}([X, R])) \notin \mathbb{R}[\lambda],$$

then the multiplication map $L \times G^\sigma \times G_R \to G$ is not surjective. Here, $[X, R] := XR - RX$.

We shall use Lemma 4.3 in each of the subsequent Propositions 4.4–4.6.

4.2. Necessary conditions for $G = LG^\sigma H$. Throughout this subsection we set $G = \mathrm{SO}(2n + 1)$ and

$$(L, H) = (\mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{k-1}) \times \mathrm{SO}(2n_k + 1),$$
$$\mathrm{U}(m_1) \times \cdots \times \mathrm{U}(m_{l-1}) \times \mathrm{SO}(2m_l + 1)),$$

where $n = n_1 + \cdots + n_k = m_1 + \cdots + m_l$ with $n_i, m_j > 0$ $(1 \le i \le k - 1,\ 1 \le j \le l - 1)$, and $n_k, m_l \ge 0$. We give necessary conditions on (L, H) under which $G = LG^\sigma H$ holds. We divide the proof into three cases (Propositions 4.4–4.6).

PROPOSITION 4.4. *If $k = 3$, $l = 2$, $m_1 = 1$, then $G \neq LG^\sigma H$.*

PROPOSITION 4.5. *If $k = l = 2$, $n_1, m_1 \geq 2$, $n_2, m_2 \neq 0$, then $G \neq LG^\sigma H$.*

PROPOSITION 4.6. *If $k = l = 2$, $n_1 \geq 2$, $n_2 \neq 0$, $m_2 = 0$, then $G \neq LG^\sigma H$.*

PROOF OF PROPOSITION 4.4. Let $1 \to 3 \to 5 \to 2 \to 1$ be a loop, and R a diagonal matrix $R = \mathrm{diag}(1, 0, \ldots, 0, -1)$ of size $(2n + 1) \times (2n + 1)$. Then G_R coincides with H. Let us fix $u \in \mathbb{C}$ and define $X = (X_{ij})_{1 \leq i, j \leq 5} \in \mathfrak{so}(2n + 1)$ in block matrix form corresponding to the partition $2n + 1 = n_1 + n_2 + (2n_3 + 1) + n_2 + n_1$ as (4.1):

$$X_{13} := \mathrm{E}_{1, n_3 + 1} = \begin{pmatrix} O & \begin{matrix} 1 \\ \\ 0 \end{matrix} & O \end{pmatrix} \in \mathrm{M}(n_1, 2n_3 + 1; \mathbb{C}),$$

$$X_{41} := \mathrm{E}_{n_2, 1} = \begin{pmatrix} & O \\ 1 & \end{pmatrix}, \quad X_{21} = u\, \mathrm{E}_{1,1} = \begin{pmatrix} u & \\ & O \end{pmatrix} \in \mathrm{M}(n_2, n_1; \mathbb{C}).$$

We define the block entries $X_{11}, X_{15}, X_{22}, X_{23}, X_{24}, X_{32}, X_{33}, X_{34}, X_{42}, X_{43}, X_{44}, X_{51}$ and X_{55} to be zero matrices. The remaining block entries are automatically determined by the definition (2.1) of $\mathfrak{so}(2n + 1)$. Then $Q := [X, R]$ has the following block entries:

$$Q_{13} = - \mathrm{E}_{1, n_3 + 1}, \quad Q_{41} = \mathrm{E}_{n_2, 1}, \quad Q_{21} = u\, \mathrm{E}_{1,1}\,.$$

By a simple matrix computation (here we recall that $k = 3$),

$$A_{13521}(Q) = Q_{13} J_{2n_3 + 1} {}^t Q_{13} J_{n_1} J_{n_1} {}^t Q_{41} J_{n_2} Q_{21} = u\, \mathrm{E}_{1,1} \in \mathrm{M}(n_1, \mathbb{C}).$$

Therefore

$$\det(\lambda I_{n_1} - A_{13521}(Q)) = \lambda^{n_1} - u\lambda^{n_1 - 1} \notin \mathbb{R}[\lambda] \quad \text{if } u \notin \mathbb{R}.$$

By Lemma 4.3, we have shown that $G \neq LG^\sigma H$. □

PROOF OF PROPOSITION 4.5. We may and do assume without loss of generality that $m_1 \geq n_1 \geq 2$. Let $1 \to 2 \to 3 \to 1$ be a loop, and $R \in \mathrm{M}(2n + 1, \mathbb{R})$ a diagonal matrix with the following entries:

$$R := \mathrm{diag}(\overbrace{1, \ldots, 1}^{m_1}, \overbrace{2, \ldots, 2}^{2m_2 + 1}, \overbrace{-1, \ldots, -1}^{m_1}).$$

Then $G_R = H$. We fix $u \in \mathbb{C}$ and define $X = (X_{ij})_{1 \leq i, j \leq 3} \in \mathfrak{so}(2n + 1)$ in block matrix

form corresponding to the partition $2n + 1 = n_1 + (2n_2 + 1) + n_1$ as (4.1):

$$X_{12} := \mathrm{E}_{1,n_2} + u\,\mathrm{E}_{1,n_2+1} + \mathrm{E}_{n_1,n_2+1} + \mathrm{E}_{n_1,n_2+2}$$

$$= \begin{pmatrix} O & \begin{matrix} 1 & u & 0 \end{matrix} & O \\ & O & \\ O & \begin{matrix} 0 & 1 & 1 \end{matrix} & O \end{pmatrix} \in \mathrm{M}(n_1, 2n_2 + 1; \mathbb{C}),$$

$$X_{31} := -\mathrm{E}_{1,1} + \mathrm{E}_{n_1,n_1}$$

$$= \begin{pmatrix} -1 & & \\ & O & \\ & & 1 \end{pmatrix} \in \mathrm{M}(n_1, \mathbb{C}).$$

We define the block entries X_{11}, X_{22} and X_{33} to be zero matrices. The remaining block entries of X are determined automatically by (2.1). Then $Q := [X, R]$ has the following block entries:

$$Q_{12} = \mathrm{E}_{1,n_2} + u\,\mathrm{E}_{1,n_2+1} + \mathrm{E}_{n_1,n_2+1} + \mathrm{E}_{n_1,n_2+2}, \qquad Q_{31} = -2\,\mathrm{E}_{1,1} + 2\,\mathrm{E}_{n_1,n_1}.$$

By a simple matrix computation (here we recall that $k = 2$),

$$\begin{aligned} A_{1231}(Q) &= Q_{12} J_{2n_2+1} \, {}^{t}Q_{12} J_{n_1} Q_{31} \\ &= -2(1 + u)\,\mathrm{E}_{1,1} + 2u^2\,\mathrm{E}_{1,n_1} - 2\,\mathrm{E}_{n_1,1} + 2(1 + u)\,\mathrm{E}_{n_1,n_1} \in \mathrm{M}(n_1, \mathbb{C}). \end{aligned}$$

Consequently,

$$\det(\lambda I_{n_1} - A_{1231}(Q)) = \lambda^{n_1} - 4(1 + 2u)\lambda^{n_1-2} \notin \mathbb{R}[\lambda] \quad \text{if } u \notin \mathbb{R}.$$

Using Lemma 4.3, we have $G \neq LG^{\sigma}H$. $\qquad\square$

PROOF OF PROPOSITION 4.6. We consider the loop $1 \to 2 \to 1 \to 3 \to 2 \to 1$, and a diagonal matrix $R \in \mathrm{M}(2n + 1, \mathbb{R})$ with the following entries:

$$R := \mathrm{diag}(\overbrace{1, \ldots, 1}^{n-1}, -1, 0, 1, \overbrace{-1, \ldots, -1}^{n-1}).$$

Then G_R is conjugate to H by an element of G^{σ}. We fix $u \in \mathbb{C}$ and define $X = (X_{ij})_{1 \leq i,j \leq 3} \in \mathfrak{so}(2n + 1)$ in block matrix form corresponding to the partition $2n + 1 = n_1 + (2n_2 + 1) + n_1$ as (4.1):

$$X_{12} := \mathrm{E}_{1,n_2} + \mathrm{E}_{1,n_2+1} - \mathrm{E}_{n_1,n_2+1}$$

$$= \begin{pmatrix} O & \begin{matrix} 1 & 1 & 0 \end{matrix} & O \\ & O & \\ O & \begin{matrix} 0 & -1 & 0 \end{matrix} & O \end{pmatrix} \in \mathrm{M}(n_1, 2n_2 + 1; \mathbb{C}),$$

$$X_{13} := u\,\mathrm{E}_{1,1} - u\,\mathrm{E}_{n_1,n_1}$$

$$= \begin{pmatrix} u & & \\ & O & \\ & & -u \end{pmatrix} \in \mathrm{M}(n_1, \mathbb{C}).$$

We define the block entries X_{11}, X_{22} and X_{33} to be zero matrices. The remaining block entries are automatically determined by (2.1). Then $Q := [X, R]$ has the following block entries:

$$Q_{12} = -2\,\mathrm{E}_{1,n_2} - \mathrm{E}_{1,n_2+1} + \mathrm{E}_{n_1,n_2+1}, \quad Q_{13} = -2u\,\mathrm{E}_{1,1} + 2u\,\mathrm{E}_{n_1,n_1},$$
$$Q_{21} = -2\,\mathrm{E}_{n_2,1} - \mathrm{E}_{n_2+1,1} + \mathrm{E}_{n_2+1,n_1}.$$

By a simple matrix computation (here we recall that $k = 2$),

$$A_{121321}(Q) = Q_{12}Q_{21}Q_{13}J_{n_1}\,{}^tQ_{21}J_{2n_2+1}Q_{21} = 8u\,\mathrm{E}_{1,1} - 8u\,\mathrm{E}_{1,n_1} \in \mathrm{M}(n_1, \mathbb{C}),$$

and thus

$$\det(\lambda I_n - A_{121321}(Q)) = \lambda^{n_1} - 8u\lambda^{n_1-1} \notin \mathbb{R}[\lambda] \quad \text{if } u \notin \mathbb{R}.$$

By Lemma 4.3, we have shown that $G \neq LG^\sigma H$.   □

4.3. Completion of the proof of Theorem 1.1. We complete the proof of the implication (i) $\Rightarrow$ (ii) in Theorem 1.1 (Proposition 4.7) by using Propositions 4.4–4.6. We recall that for a given partition $n = n_1 + \cdots + n_k$ with $n_1, \ldots, n_{k-1} > 0$ and $n_k \geq 0$, we have the corresponding Levi subgroup $\mathrm{L}_{\Pi'} = \mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{k-1}) \times \mathrm{SO}(2n_k + 1)$ of $\mathrm{SO}(2n + 1)$, which is associated to the subset

$$\Pi' := \Pi \setminus \left\{ \alpha_i \in \Pi : i = \sum_{s=1}^{j} n_s, 1 \leq j \leq k-1 \right\}$$

of the simple system Π (see Figure 1 for the labelling of the Dynkin diagram).

PROPOSITION 4.7. *Let G be the special orthogonal group $\mathrm{SO}(2n + 1)$, σ a Chevalley–Weyl involution, Π', Π'' subsets of Π, and $\mathrm{L}_{\Pi'}$, $\mathrm{L}_{\Pi''}$ the corresponding Levi subgroups. Then*

$$G \neq \mathrm{L}_{\Pi'}\,G^\sigma\,\mathrm{L}_{\Pi''}, \tag{4.3}$$

if, for $1 \leq i, j, k \leq n$, one of the following conditions up to switching of Π' and Π'' is satisfied:

(I)   *either $(\Pi')^c$ or $(\Pi'')^c$ contains more than one element;*
(II)   *$(\Pi')^c = \{\alpha_i\}$, $(\Pi'')^c = \{\alpha_j\}$ and $i, j \notin \{1, n\}$;*
(III)   *$(\Pi')^c = \{\alpha_i\}$, $(\Pi'')^c = \{\alpha_n\}$ and $i \notin \{1, n\}$.*

PROOF. Let

$$(\mathrm{L}_{\Pi'}, \mathrm{L}_{\Pi''}) = (\mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_{k-1}) \times \mathrm{SO}(2n_k + 1),$$
$$\mathrm{U}(m_1) \times \cdots \times \mathrm{U}(m_{l-1}) \times \mathrm{SO}(2m_l + 1)).$$

First, let us show that the condition (I) implies (4.3). Without loss of generality, we may and do assume that $n_1 \geq \cdots \geq n_{k-1}$, $m_1 \geq \cdots \geq m_{l-1}$ and that $(\Pi')^c$ contains more than one element since the roles of Π' and Π'' are symmetric.

Case (I)(1): $m_1 = 1$. Since L and H are contained in

$$U(n_1) \times U(n_2) \times SO(2(n_3 + \cdots + n_k) + 1)$$

and

$$U(1) \times SO(2(m_2 + \cdots + m_l) + 1)$$

respectively, we can see that (4.3) holds by Proposition 4.4.

Case (I)(2): $m_1 \geq 2, n_k \neq 0$. Since L and H are contained in

$$U(n_1 + n_2) \times SO(2(n_3 + \cdots + n_k) + 1)$$

and

$$U(m_1) \times SO(2(m_2 + \cdots + m_l) + 1)$$

with $m_1 \geq 2$ respectively, we find that (4.3) holds by using Propositions 4.5 and 4.6.

Case (I)(3): $m_1 \geq 2, n_k = 0$. In this case n_1 is greater than one, and thus (4.3) follows from Propositions 4.5 and 4.6. Here, we note that L and H are contained in

$$U(n_1) \times SO(2(n_2 + \cdots + n_k) + 1) \quad \text{with } n_2 \neq 0$$

and

$$U(m_1) \times SO(2(m_2 + \cdots + m_l) + 1),$$

respectively.

Next, let us treat the conditions (II) and (III). We immediately find that each of conditions (II) and (III) implies (4.3) by using Propositions 4.5 and 4.6, respectively.

Therefore the proof is complete. □

By Propositions 3.2, 3.3 and 4.7, we have finished the proof of Theorem 1.1.

5. Application of visible actions to representation theory

As an application of Theorem 1.1, we obtain some multiplicity-free theorems by using Kobayashi's theory of visible actions. Here we recall the definition of strong visibility [Ko2].

DEFINITION 5.1. We say that a biholomorphic action of a Lie group G on a complex manifold D is *strongly visible* if the following two conditions are satisfied.

(1) There exists a real submanifold S (called a *slice*) such that $D' := G \cdot S$ is an open subset of D.

(2) There exists an antiholomorphic diffeomorphism σ of D' such that

$$\sigma|_S = \mathrm{id}_S,$$
$$\sigma(G \cdot x) = G \cdot x \quad \text{for any } x \in S.$$

DEFINITION 5.2. In the above setting, we say the action of G on D is S-visible. This terminology will also be used if S is just a subset of D.

Let G be a compact Lie group and L, H its Levi subgroups. Then $G/L, G/H$ and $(G \times G)/(L \times H)$ are complex manifolds. If the triple (G, L, H) satisfies $G = LG^\sigma H$, then the following three group actions are all strongly visible:

$$L \curvearrowright G/H,$$
$$H \curvearrowright G/L,$$
$$\Delta(G) \curvearrowright (G \times G)/(L \times H).$$

Here, $\Delta(G)$ is defined by $\Delta(G) := \{(x, y) \in G \times G : x = y\}$. The following fact [Ko3, Theorem 4.3] constructs a family of multiplicity-free representations from visible actions.

FACT 5.3. *Let G be a Lie group and $\mathcal{V}$ a G-equivariant Hermitian holomorphic vector bundle on a connected complex manifold D. If the following three conditions are satisfied, then any unitary representation that can be embedded in the vector space $\mathcal{O}(D, \mathcal{V})$ of holomorphic sections of $\mathcal{V}$ decomposes multiplicity-freely.*

(1) *The action of G on D is S-visible. That is, there exists a subset $S \subset D$ satisfying the conditions given in Definition 5.1. Further, there exists an automorphism $\hat{\sigma}$ of G such that $\sigma(g \cdot x) = \hat{\sigma}(g) \cdot \sigma(x)$ for any $g \in G$ and $x \in D'$.*

(2) *For any $x \in S$, the fibre $\mathcal{V}_x$ at x decomposes as the multiplicity-free sum of irreducible unitary representations of the isotropy subgroup G_x. Let $\mathcal{V}_x = \bigoplus_{1 \le i \le n(x)} \mathcal{V}_x^{(i)}$ denote the irreducible decomposition of $\mathcal{V}_x$.*

(3) *σ lifts to an antiholomorphic automorphism $\tilde{\sigma}$ of $\mathcal{V}$ and satisfies $\tilde{\sigma}(\mathcal{V}_x^{(i)}) = \mathcal{V}_x^{(i)}$ $(1 \le i \le n(x))$ for each $x \in S$.*

We return to the case where $G = \mathrm{SO}(2n + 1)$. The fundamental weights $\omega_1, \ldots, \omega_n$ with respect to the simple roots $\alpha_1, \ldots, \alpha_n$ are given as follows (see Figure 1 for the labelling of the Dynkin diagram):

$$\omega_i = \alpha_1 + 2\alpha_2 + \cdots + (i - 1)\alpha_{i-1}$$
$$+ i(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{n-1} + \tfrac{1}{2}\alpha_n) \quad (1 \le i \le n).$$

By using the Borel–Weil theory together with Fact 5.3 and our generalised Cartan decompositions, we obtain the following two corollaries of Theorem 1.1.

COROLLARY 5.4. *If the pair (L, λ) is an entry in Table 1, then the restriction $\pi_\lambda|_L$ of the irreducible representation π_λ of $\mathrm{SO}(2n + 1)$ with highest weight λ to L decomposes multiplicity-freely. Here, $1 \le i \le n$ and a is an arbitrary nonnegative integer.*

COROLLARY 5.5. *The tensor product representation $\pi_{a\omega_1} \otimes \pi_{b\omega_i}$ decomposes as a multiplicity-free sum of irreducible representations of $\mathrm{SO}(2n + 1)$ for $1 \le i \le n$ and arbitrary nonnegative integers a, b. Likewise, the tensor product $\pi_{a\omega_n} \otimes \pi_{b\omega_n}$ is also multiplicity-free for any $a, b \in \mathbb{N}$.*

TABLE 1. Pairs L, λ such that restrictions $\pi_\lambda|_L$ are multiplicity-free.

Levi subgroup L	Highest weight λ
$U(n)$	$a\omega_n$
$U(1) \times SO(2n-1)$	$a\omega_i$
$U(i) \times SO(2n-2i+1)$	$a\omega_1$

REMARK 5.6. The above representations have been shown to be multiplicity-free by Littelmann [Li] by checking the sphericity of the product of flag varieties associated to maximal parabolic subgroups, and by Stembridge [St] by a combinatorial method using the Weyl character to analyse the tensor product multiplicities. Our approach is different from these two methods, and uses the notion of visible actions.

We have listed an application of Fact 5.3 only for the line bundle case. Let us give a simple example of that in the vector bundle setting. Let G be the spin group $\mathrm{Spin}(2n+1)$ and T a maximal torus of G. We let π_λ denote any irreducible representation of G with highest weight λ and π_{ω_n} as above. Since π_{ω_n} is weight multiplicity-free, that is, π_{ω_n} decomposes multiplicity-freely as a representation of T, we can apply Fact 5.3 to the tensor product representation of π_λ and π_{ω_n} by setting $\mathcal{V} := G \times_T (\mathbb{C}_\lambda \otimes \pi_{\omega_n})$, $D := G/T$, $S := \{o\}$, and then conclude that $\pi_\lambda \otimes \pi_{\omega_n}$ is multiplicity-free as a representation of G (the irreducible decomposition may be thought of a Pierri rule for a type B group). Here, we note that $\mathcal{V}_x$ and G_x for $x = o$ are given by $\mathbb{C}_\lambda \otimes \pi_{\omega_n}$ and T respectively in this setting. We hope that further applications of Theorem 1.1 and Fact 5.3 to representation theory will be discussed in a future paper.

Acknowledgements

References

[He] S. Helgason, 'Differential geometry, lie groups and symmetric spaces', in: *Pure and Applied Mathematics* (Academic Press, New York, 1978).

[Ho] B. Hoogenboom, *Intertwining Functions on Compact Lie Groups*, CWI Tract 5 (Stichting Mathematisch Centrum, Centrum voor Wiskunde en Informatica, Amsterdam, 1984).

[Ko1] T. Kobayashi, 'Geometry of multiplicity-free representations of $GL(n)$, visible actions on flag varieties, and triunity', *Acta Appl. Math.* **81** (2004), 129–146.

[Ko2] T. Kobayashi, 'Multiplicity-free representations and visible actions on complex manifolds', *Publ. Res. Inst. Math. Sci.* **41** (2005), 497–549.

[Ko3] T. Kobayashi, *Propagation of Multiplicity-freeness Property for Holomorphic Vector Bundles*, Progress in Mathematics (Birkhäuser, Boston, 2012), (in press), math. RT/0607004.

[Ko4]   T. Kobayashi, 'A generalized Cartan decomposition for the double coset space $(U(n_1) \times U(n_2) \times U(n_3)) \backslash U(n)/(U(p) \times U(q))$', *J. Math. Soc. Japan* **59** (2007), 669–691.

[Ko5]   T. Kobayashi, 'Visible actions on symmetric spaces', *Transform. Groups* **12** (2007), 671–694.

[Ko6]   T. Kobayashi, 'Multiplicity-free theorems of the restriction of unitary highest weight modules with respect to reductive symmetric pairs', in: *Representation Theory and Automorphic Forms*, Progress in Mathematics, 255 (Birkhäuser, Boston, 2007), pp. 45–109, math. RT/0607002.

[Kr]   M. Krämer, 'Sphärische Untergruppen in kompakten zusammenhängenden Liegruppen', *Compositio Math.* **38**(2) (1979), 129–153.

[Li]   P. Littelmann, 'On spherical double cones', *J. Algebra* **166** (1994), 142–157.

[Ma1]   T. Matsuki, 'Double coset decomposition of algebraic groups arising from two involutions. I', *J. Algebra* **175** (1995), 865–925.

[Ma2]   T. Matsuki, 'Double coset decompositions of reductive Lie groups arising from two involutions', *J. Algebra* **197** (1997), 49–91.

[Sa1]   A. Sasaki, 'A characterization of nontube type Hermitian symmetric spaces by visible actions', *Geom. Dedicata* **145**(3) (2010), 151–158.

[Sa2]   A. Sasaki, 'A generalized Cartan decomposition for the double coset space $SU(2n + 1) \backslash SL(2n + 1, \mathbb{C})/Sp(n, \mathbb{C})$', *J. Math. Sci. Univ. Tokyo* **17** (2010), 201–215.

[St]   J. R. Stembridge, 'Multiplicity-free products and restrictions of Weyl characters', *Represent. Theory* **7** (2003), 404–439.

[Ta]   Y. Tanaka, Visible actions on flag varieties of type D and a generalization of the Cartan decomposition (submitted).

[Wo]   J. A. Wolf, *Harmonic Analysis on Commutative Spaces*, Mathematical Surveys and Monographs (American Mathematical Society, Providence, RI, 2007).

YUICHIRO TANAKA, Graduate School of Mathematical Sciences,
The University of Tokyo, 3-8-1, Meguro-ku, Tokyo 153-8914, Japan
e-mail: yuichiro@ms.u-tokyo.ac.jp