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Local analytic structure in certain commutative topological algebras

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Let B be a topological algebra with Fréchet space topology, A an algebra with locally convex topology and $\underline{\underline{A}}$ an algebra of formal power series over A in n commuting indeterminates which carries a Fréchet space topology. In a previous paper the author showed, for the case $n = 1$, that a homomorphism of B into $\underline{\underline{A}}$ whose range contains polynomials is necessarily continuous provided the coordinate projections of $\underline{\underline{A}}$ into A satisfy a certain equicontinuity condition. This result is here extended to the case of general n, and also to weaker topological assumptions.

An application to the case $\underline{\underline{A}} = {}_n0$, the stalk at zero of the sheaf of germs of holomorphic functions on $\mathbb{C}^n$ yields a local condition, in terms of the sheaf of germs of B-holomorphic functions, which is sufficient for the existence of an analytic polydisc in the spectrum of a commutative F-algebra B. The proof requires very little analytic function theory, and for this reason is not amenable to extension to a corresponding result for analytic subvarieties.

Preliminaries

An F-algebra is a complete topological algebra whose topology is given by a countable sequence of algebraic seminorms.

Let A be a commutative complex F-algebra, Φ the spectrum of A

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(the set of *continuous* multiplicative linear functionals of A with the weak topology induced by the algebra $\hat{A}$ of Gelfand transforms of elements of A). It is known that Φ is Hausdorff and hemicompact. (For fuller details regarding F-algebras see [*6*].) Following [*1*], for each open set $U \subseteq \Phi$, let $A(U)$ denote the completion of $\hat{A}|U$ under the topology of uniform convergence on the sets $\{U \cap M : M \subseteq \Phi, M \text{ compact}\}$. Since Φ is hemicompact, $A(U)$ is itself an F-algebra. Extending the notation of [*7*], a function f on Φ will be termed A-holomorphic if each point $\phi \in \Phi$ has an open neighbourhood U_ϕ such that $f|U_\phi \in A(U_\phi)$. The family $\{A(U) : U \subseteq \Phi, U \text{ open}\}$ together with the appropriate restriction maps is a presheaf over Φ (see, for example, [*4*], Chapter IV); the associated sheaf H is the sheaf of germs of A-holomorphic functions. If $\phi \in \Phi$ and $\mathcal{U}_\phi$ is the set of open neighbourhoods of ϕ partially ordered by reverse inclusion, then the stalk H_ϕ of H at ϕ is given by $\lim\{A(U) : U \in \mathcal{U}_\phi\}$ algebraically, and H_ϕ is a topological algebra with the inductive limit topology (see, for example, [*3*], §6.3). For example, taking A as the algebra of entire functions with the compact-open topology, $\Phi = \mathbb{C}$, $A(U)$ is the algebra of holomorphic functions on U which are bounded on bounded subsets of U , and H is the sheaf of germs of holomorphic functions. In this case H_0 is denoted $\mathcal{O}$, and more generally the stalk at zero of the sheaf of germs of holomorphic functions on $\mathbb{C}^n$ is denoted ${}_n\mathcal{O}$.

When A is a commutative F-algebra which is singly-generated it is known ([*1*], Theorem 3.4) that if H_ϕ is algebraically isomorphic with $\mathcal{O}$ then there is an analytic disc in Φ centred at ϕ , that is, there is a homeomorphism Γ from an open disc Δ in $\mathbb{C}$ into Φ such that $\Gamma(0) = \phi$ and, for each $x \in A$, $\hat{x} \circ \Gamma$ is holomorphic on Δ . We show below that this result can be extended to general commutative F-algebras with ${}_n\mathcal{O}$ rather than just $\mathcal{O}$.

For the sheaf S of germs of locally-A functions on Φ (which is a subsheaf of H , generally proper) similar results have been obtained in [*2*] for the Banach algebra case. Namely suppose there is an epimorphism

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ψ of S_ϕ onto $H_0(W)$, the stalk at zero of the sheaf of holomorphic functions on a subvariety W of an open set in $\mathbb{C}^n$, and that zero is an interior point of W. Then there is a continuous injection Γ of a neighbourhood W' of zero in W into Φ such that $\Gamma(0) = \phi$ and for each $x \in A$, $\hat{x} \circ \Gamma$ is holomorphic on W'. If ψ is an isomorphism and $\{\phi\}$ a G_δ then Γ is a homeomorphism. The proof of these results makes considerable use of properties of holomorphic functions in contrast to our proof below which, however, only considers the case of ${}_n0$ rather than the more general $H_0(W)$ to which it is not applicable, as will become apparent.

Results

Our method of proof is to extend a result of [5] to such an extent that the desired result is almost an immediate corollary.

Let A be an algebra over $\mathbb{C}$ with identity which has a locally convex topology as a vector space. Let $t_1, \ldots, t_n$ be commutative indeterminates and $\underline{A}$ an algebra of formal power series in $t_1, \ldots, t_n$ over A. We will suppose that $A[t_i] \subseteq \underline{\underline{A}}$ for at least one i, and choose $i = 1$ with no loss of generality. Elements of $\underline{\underline{A}}$ will be denoted $\sum a_K t^K$ where $K = (k_1, \ldots, k_n) \in \mathbb{N}^n$ is a multi-index; as usual $|K| = k_1 + \ldots + k_n$. Ordering and addition in $\mathbb{N}^n$ will always be component-wise. We suppose finally that $\underline{A}$ has a Fréchet space topology such that there is a sequence $\{\gamma_K : K \in \mathbb{N}^n\}$ of positive reals with the property that $\{\gamma_K^{-1} p_K\}$ is an equicontinuous family, where $p_K : \sum a_J t^J \mapsto a_K$ is the K-th coordinate projection.

THEOREM 1. *Let B be a topological algebra with Fréchet space topology, and $\phi : B \to \underline{A}$ a homomorphism with $t_1 \in \phi(B)$. Then ϕ is continuous.*

Proof. The proof is similar to that of [5], Theorem 1, and will be outlined only. Supposing ϕ discontinuous, there is a seminorm $\|\cdot\|_N$ on A and a multi-index K such that $p_J\phi : B \to (A, \|\cdot\|_N)$ is continuous for $J < K$ but discontinuous for $J = K$. Take neighbourhoods U, V in B, $\underline{\underline{A}}$ repectively, and $\{|\cdot|_m\}$, $\{M_m\}$, $\{\delta_m\}$ all as in [5], and let $\{\mu_J\}$ be a sequence of positive reals with $\mu_J^{-1}\gamma_J \to 0$ as $|J| \to \infty$. Let $\mathcal{i} = (i, 0, \ldots, 0) \in \mathbb{N}^n$, and let $s \in B$ with $\phi(s) = t_1$. Define $\{x_m\} \subseteq B$ inductively such that

(i) $x_m \in U$,

(ii) $|x_m|_i \le 2^{-m}\delta_m \min\limits_{\substack{1\le j\le m \\ 1\le l\le m}} \left\{1+|s^j|_l\right\}^{-1}$ for $1 \le i \le M_m$,

(iii) $\left\|p_K\phi(x_m)\right\|_N \ge \mu_{K+m} + k_1 + \sum\limits_{i=1}^{m-1} \left\|p_{K+\mathcal{i}}\phi(x_{m-1})\right\|_N$.

Setting $y = \sum\limits_{m\ge 1} s^m x_m \in B$ we have

$$p_M\phi(y) = p_M\left\{\sum_{i=1}^{m_1} t_1^i\phi(x_i)\right\}$$
$$= \sum_{i=1}^{m_1} \sum_{0\le L\le M} p_L\left(t_1^i\right)p_{M-L}\phi(x_i)$$
$$= \sum_{i=0}^{m_1-1} p_{(i,m_2,\ldots,m_n)}\phi\left(x_{m_1-i}\right),$$

and taking $M = K + \mathcal{i}$,

$$\left\|p_{K+\mathcal{i}}\phi(y)\right\|_N \ge \mu_{K+\mathcal{i}},$$

giving a contradiction as in [5].

Having extended the result of [5] to several indeterminates we now weaken the topological assumptions.

THEOREM 2. *Suppose* $\underline{\underline{A}}$, B, ϕ *are as above except that the topologies*

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on $\underline{\underline{A}}$ *and* B *satisfy:*

(i) $\underline{\underline{A}}$ *is the countable internal inductive limit of a sequence* $\{\underline{\underline{A}}_m\}$ *of algebras each with Fréchet space topology,*

(ii) B *is the internal inductive limit of a directed family* $\{B_\alpha\}$ *of topological algebras each with Fréchet space topology* τ_α *such that if* $\alpha \leq \beta$ *then* $B_\alpha \subseteq B_\beta$ *and* $\tau_\alpha \geq \tau_\beta|_{B_\alpha}$.

Then ϕ *is continuous.*

Proof. Suppose firstly that B has Fréchet space topology. Then the argument of Theorem 1, using [3] Theorem 6.7.1 in place of the closed graph theorem, gives the required result.

In the general case note that the hypothesis *(ii)* entails that $B = \lim_{\alpha\geq\beta} B_\alpha$ for each fixed β (this crucially uses the fact that $\{B_\alpha\}$ is directed rather than just partially ordered). Choosing β such that $t_1 \in \phi(B_\beta)$, it follows by the first part that $\phi : B_\alpha \to \underline{\underline{A}}$ is continuous for each $\alpha \geq \beta$, so that $\phi : B \to \underline{\underline{A}}$ is continuous.

COROLLARY. *If* $\underline{\underline{A}} = \bigcup_{i=1}^{\infty} A_i$ *where each* A_i *is a closed set in* $\underline{\underline{A}}$ *, then for each* α *there is an integer* i_α *such that* $\phi(B_\alpha) \subseteq \operatorname{span}\left(A_{i_\alpha}\right)$.

Proof. A category argument as in [1], Lemma 3.3.

We now observe that ${}_n0$ satisfies the conditions on $\underline{\underline{A}}$ of Theorem 2 and its corollary. Firstly ${}_n0$ can clearly be considered as an algebra of formal power series over C , for if $x \in {}_n0$, there is a unique power series of which x is the germ, and henceforth we identify x with this power series. Also, if

$$\Delta(r) = \left\{z = \left(z_1, \ldots, z_n\right) : |z_i| < r, i = 1, 2, \ldots, n\right\}$$

and $\bar{\Delta}(r)$ is its closure in C^n , taking $\underline{\underline{A}}_m = \operatorname{hol}\left(\Delta\left(\frac{1}{m}\right)\right)$, the algebra of

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functions holomorphic on $\Delta\left(\frac{1}{m}\right)$ and continuous on $\overline{\Delta}\left(\frac{1}{m}\right)$ with supremum norm $\|\cdot\|_m$, we have ${}_n0 = \lim \underline{\underline{A}}_m$. Cauchy's inequalities show that for each $x \in {}_n0$, $|p_J(x)| \leq \|x\|_m \cdot m^{|J|}$ for m sufficiently large, and hence the equicontinuity condition is satisfied with $\gamma_K = |K|^{|K|}$. Finally, letting $A_m = \left\{x \in {}_n0 : |p_J(x)| \leq m^{|J|}\right\}$ we have ${}_n0 = \cup A_m$ and each A_m is closed in ${}_n0$.

We also note at this point that the reason our methods are of no avail for $H_0(W)$ is that this is not an algebra of power series like ${}_n0$, but rather a quotient of ${}_n0$.

The desired result is the following, which, incidentally, does not use the full strength of Theorem 2.

THEOREM 3. *Let* B *be a commutative* F*-algebra with spectrum* Φ. *Suppose that for some* $\phi \in \Phi$ *and for some integer* n *there is an epimorphism* $\psi : H_\phi \to {}_n0$. *Then there is a homeomorphism* Γ *of a polydisc* Δ *in* $\mathbb{C}^n$ *into* Φ *such that* $\Gamma(0) = \phi$ *and for each* $x \in B$, $\hat{x} \circ \Gamma$ *is holomorphic on* Δ; *that is, there is an analytic polydisc in* Φ *centered at* ϕ.

Proof. (*Cf.* [*1*] Theorem 3.4) Let U be an open neighbourhood of ϕ such that $t_i \in \psi(B(U))$, $i = 1, 2, \ldots, n$. Then the corollary shows that $\psi : B(U) \to \text{hol}(\Delta)$ for some open polydisc Δ, since $\underline{\underline{A}}_m \subseteq \text{span}(A_m) \subseteq \underline{\underline{A}}_{m+1}$ for each integer m. If $\theta : B \to B(U)$ is the inclusion map then θ is a continuous homomorphism and so $\psi\theta : B \to \text{hol}(\Delta)$ is a continuous homomorphism with range dense in $\text{hol}(\Delta)$. It follows that the adjoint map $(\psi\theta)^* : \overline{\Delta} \to \Phi$ gives the required homeomorphism Γ.

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