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CANONICAL EXTENSIONS OF HARISH-CHANDRA MODULES TO REPRESENTATIONS OF G

W. CASSELMAN

Introduction. Let G be the group of $\mathbf{R}$-rational points on a reductive, Zariski-connected, algebraic group defined over $\mathbf{R}$, let K be a maximal compact subgroup, and let $\mathfrak{g}$ be the corresponding complexified Lie algebra of G. It is a curious fault of the current representation theory of G that for technical reasons one very rarely works with representations of G itself, but rather with a certain category of simultaneous representations of $\mathfrak{g}$ and K. The reasons for this are, roughly speaking, that for a given $(\mathfrak{g}, K)$-module of finite length there are clearly any number of overlying rather distinct continuous G-representations, whose 'essence' is captured by the $(\mathfrak{g}, K)$-module alone. At any rate, this paper will propose a remedy for this inconvenience, and define a category of smooth representations of G of finite length which will, I hope, turn out to be as easy to work with as representations of $(\mathfrak{g}, K)$ and occasionally much more convenient. It is to be considered a report on what has been to a great extent joint work with Nolan Wallach, and is essentially a sequel to [**38**].

Assume G to be algebraically embedded as a closed subset of a finite-dimensional matrix algebra, and let $\|g\|$ be the associated Banach norm. A continuous representation π of G on a topological vector space V will be called *of moderate growth* if V is a Fréchet space and for every semi-norm ρ on V there exist a positive integer N and semi-norm ν such that

$$\|\pi(g)v\|_\rho \leqq \|g\|^N \|v\|_\nu$$

for all $v \in V, g \in G$. In contrast, say, to the tempered representations introduced by Harish-Chandra, these representations are ubiquitous; the great majority of frequently occurring continuous representations or their topological duals are of moderate growth. For example, any continuous representation of G on a Banach space is of moderate growth. If V is the space of any reasonable continuous representation of G, then the subspace $V^{\mathrm{sm}} \cap V^K$ of smooth K-finite vectors in V is a module over $(\mathfrak{g}, K)$, and the representation of G on V is said to *extend* the one of $(\mathfrak{g}, K)$ on $V^{\mathrm{sm}} \cap V^K$. If each K-isotypic constituent of V^K (or, equivalently, V) has finite dimension then every K-finite vector is already smooth and the structures of V as a G-module and of V^K as a $(\mathfrak{g}, K)$-module are known to be closely related; for example, one is irreducible if and only if the other is. Trivial examples show that the representation of G on V itself, even with this assumption on the K-isotypic constituents, is not determined by that of $(\mathfrak{g}, K)$ on V^K. What I shall prove in this paper, however, is that the representation of G

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on the subspace of smooth vectors in a representation of moderate growth is in fact determined uniquely by its underlying $(\mathfrak{g}, K)$-module, or more completely: given any finitely generated Harish-Chandra module (π, V) over $(\mathfrak{g}, K)$, there is up to canonical topological isomorphism exactly one smooth representation of G of moderate growth whose underlying $(\mathfrak{g}, K)$-module is isomorphic to V. The assignment taking V to this smooth representation of G is functorial and exact. I propose to call the G-spaces obtained in this way *Harish-Chandra representations* of G. In the course of the proof of the fundamental result I shall construct several functors from the category of finitely generated Harish-Chandra modules to that of smooth representations of G of moderate growth, among them the smallest possible one and the largest possible one, and then show that all these constructions agree.

One consequence of the main result is that if U and V are the spaces of smooth representations of G of moderate growth, then any continuous G-covariant map from U to V whose image has the property that its underlying $(\mathfrak{g}, K)$-subspace is a Harish-Chandra module of finite length has closed range. This can be used easily in several situations otherwise unapproachable, for example to carry out the meromorphic continuation of Eisenstein series (in the theory of automorphic forms) which are not necessarily K-finite.

This paper will be somewhat verbose. Much of it presents mild extensions of arguments contained already in [**38**], but which I have in many cases explained from scratch, because of the slightly different perspective here. Also, because in the literature the relationship between representations of G and those of $(\mathfrak{g}, K)$ is not covered in exactly the form I need, I have included in the first few sections what amounts to a review (for the most part without proofs) of what will be relevant. I hope that this will make the main results of this joint work with Wallach less technical in appearance. Although the final results seem to me relatively easy to comprehend and exceptionally easy to apply, and although there are few, if any, places where the arguments are hard to follow on a close scale, the assembly of all the bits and pieces is admittedly rather elaborate. This is, I hope, an artifact of the way that Nolan and I found our proofs (painfully, over a long period of time, and with many errors dogging our path) rather than a reflection of the true nature of the material, and I imagine that more natural proofs will ultimately be found.

Here is a more detailed account of the paper: It is in Sections 1–5 that I review relations between representations of G and of $(\mathfrak{g}, K)$. In Section 6 I illustrate some of the phenomena included in this review, as well as some of the results to be explained subsequently, in the relatively simple case $G = SL_2(\mathbf{R})$. The new material begins in Section 7, where I introduce two functorial constructions of smooth representations of G from Harish-Chandra modules associated to embeddings of Harish-Chandra modules in representations induced from parabolic subgroups of G. More precisely, let P be a minimal parabolic subgroup with unipotent radical N, whose complexified Lie algebra is $\mathfrak{n}$. Then if (π, V) is a finitely generated Harish-Chandra module over $(\mathfrak{g}, K)$, it is well known that V

embeds into $\mathrm{Ind}(V/\mathfrak{n}^k V|P,G)$ for k sufficiently large. It is relatively elementary to show that assigning to V its closure $\overline{V}$ in this induced representation is independent of k (for k large), functorial, left exact. The representation of G on $\overline{V}$ is a Harish-Chandra representation since that on every $\mathrm{Ind}^{\mathrm{sm}}(\sigma|P,G)$ is. If V is itself some induced representation $\mathrm{Ind}(\sigma|P,G)$ then $\overline{V}$ turns out to be the same as the smoothly induced representation $\mathrm{Ind}^{\mathrm{sm}}(\sigma|P,G)$; this amounts to what Wallach calls the 'automatic continuity theorem', which asserts that any $(\mathfrak{g},K)$-map from $\mathrm{Ind}(\sigma|P,G)$ to $\mathrm{Ind}(\tau|P,G)$ extends to a unique continuous G-map between the associated smoothly induced space $\mathrm{Ind}^{\mathrm{sm}}(\sigma|P,G)$ and $\mathrm{Ind}^{\mathrm{sm}}(\tau|P,G)$. A second functorial G-extension $\underline{V}$ is defined by representing V as a quotient of representations induced from P, and the automatic continuity theorem implies a canonical continuous injection of $\underline{V}$ into $\overline{V}$. I define V to be *regular* if this canonical injection is surjective, hence an isomorphism. The main result of this paper may be formulated in two halves, the first of which is the assertion that every finitely generated Harish-Chandra is regular. This is equivalent to the more concretely formulated assertion that any continuous G-map between two Harish-Chandra representations induced from P has closed range. It is also equivalent to the assertion that the assignment taking V to $\overline{V}$ is exact. The argument for this itself breaks further into two pieces: the first part amounts to showing that every essentially square-integrable representation is regular, and the second part then applies Langlands' classification theorem and a peculiar argument about what I call 'thickened principal series' to conclude. This last step is carried out in Sections 9–10. The argument that essentially square-integrable representations are regular is part of an argument of [**38**] which I reproduce in Section 8. This argument at the same time introduces two more G-extensions $\underline{\underline{V}}$ and $\overline{\overline{V}}$, this time associated to matrix coefficients of a given Harish-Chandra module. Essentially from the definition one deduces a sequence of inclusions

$$\underline{\underline{V}} \subseteq \underline{V} \subseteq \overline{V} \subseteq \overline{\overline{V}}.$$

The first part of the argument in the proof of the main result of this paper shows that the inner inclusion is an equality. The second part generalizes slightly the main result of [**38**], and shows that the outer two inclusions are equalities. The proofs in Section 8 are by and large just a rearrangement of Wallach's proofs. Finally, to conclude I give in Section 11, as an application, a very useful description of the asymptotic behaviour of the matrix coefficients of Harish-Chandra representations. The technique illustrated there might turn out to be a typical way to apply the main theorem.

A few more remarks about the argument may be useful. The different constructions in Sections 7–8 reflect the general principle that representations of G which occur in nature arise usually in one of two ways: very roughly speaking, associated to functions (or, rather, to sections of certain vector bundles) on either G/K or on G/P. The proofs of equivalence reflect then the principle that the second group of realizations are in some sense boundary values of the first. These points, incidentally, appear also in work of various people, most recently [**34**] and earlier, most notably, [**23**], exhibiting another group of functorial extensions of $(\mathfrak{g}, K)$-modules to representations of G. The new point, again speaking roughly, is that the boundary values examined in the work of Wallach and myself are distributions while those of Schmid (and earlier Kashiwara and his colleagues) are hyperfunctions. The argument in Section 9 is in fact reminiscent of Schmid's proof of his analogous result, which depends on a result from [**19**], which in turn applies Langlands' classification.

I have already mentioned that in the final analysis the construction given here as well as its important properties depend on two results which of themselves might not seem so important: the first is that every $(\mathfrak{g}, K)$-map from one K-finite principal series representation to another extends to a (unique) continuous map between the corresponding C^∞ principal series, and the second is that any such intertwining operator from one C^∞ principal series representation to another has closed range. The first is dealt with (but not, perhaps, definitively) in [**38**], and I shall say little about it here. It is, essentially, the second point that this paper is concerned with. It should be pointed out that in spite of the relatively down-to-earth nature of these assertions, no direct proof in terms of, say, the order of growth of the coefficients of intertwining operators is known. In the case of certain groups, for example SL_2, drastic simplifications in the proof I give below are possible, but even here no direct attack is known. (At any rate, in the section on SL_2 I do not exhibit the possible simplifications, although I considered doing it.)

In subsequent papers I hope to show how the results given here allow one to refine Bruhat's thesis and describe in more detail the structure of the C^∞ principal series; sharpen results of [**26**], [**38**], and [**35**] on Whittaker models for Harish-Chandra modules; give a new proof of the automatic continuity theorem (the first of the two assertions in the previous paragraph); give relatively explicit formulas for the asymptotic expansions of matrix coefficients of principal series representations; and improve the Paley-Wiener theorem of [**1**] to deal with functions which are not necessarily K-finite. It seems likely to me also that the representations described here could also allow one to construct a decent analytical theory of characters; to the point, for example, where one could prove the main result of [**19**] (once known as Osborne's conjecture) as a special case of a fixed-point theorem.

Incidentally, this is the second paper of mine to be produced in $\TeX$. As in the first, I wish to thank the National Science and Engineering Research Council of Canada for financial support in this project. I also wish to thank D. Miličić and P. Delorme for suggestions.

Notation. Fix the group K throughout the paper. Whenever P is a parabolic subgroup of G set

$N_P :=$ the unipotent radical of P

$M_P :=$ the reductive quotient P/N_P

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A_P: = the elements of the topologically connected component of the maximal $\mathbf{R}$-split torus in the centre of M_P which are also in the kernel of every character of G

$(K)P := K \cap M_P$

$K(M) :=$ the image of $K(P)$ modulo N_P

$\delta_P :=$ the modulus character of $P\colon p \longmapsto |\det(Ad_n(p))|$

$\Delta_P :=$ the basis of positive roots for P, the eigencharacters of the adjoint action of A_P acting on $\mathfrak{n}_P$

$A_P^{--} := \{a \in A_P | \gamma(a) \leqq 1$ for all $\gamma \in \Delta_P\}$.

When confusion is unlikely I shall abandon the subscripts. Note that if P is minimal and G is semi-simple then A_P^{--} is the multiplicative negative Weyl chamber.

1. Representations of G**.** Let V be a locally convex Hausdorff topological vector space, $C(G,V)$ the space of continuous functions from G to V. Suppose that π is a linear representation of G on V. Define the embedding $\Phi = \Phi_\pi$ of V in the vector space of all V-valued functions on G: $v \in V$ corresponds to the function Φ_v where

$$\Phi_v(g) = \pi(g)v.$$

Thus Φ is a linear G-covariant map from V into the right regular representation, and the image of V consists precisely of the functions φ satisfying

$$(1.1) \qquad \varphi(gh) = \pi(g)\varphi(h).$$

A representation π of G on V is said to be *separately continuous* if each $\pi(g)$ is a continuous map from V to itself and if for each v in V the map Φ_v lies in $C(G,V)$. It is said to be *continuous* if the map from $G \times V$ to V taking (g,v) to $\pi(g)v$ is continuous. If π is separately continuous then the space of functions φ satisfying (1.1) (that is to say, the image of V under Φ) is a closed subspace of $C(G,V)$.

LEMMA 1.1. *If (π, V) is a separately continuous representation of G, then the following are equivalent:*

(a) *It is continuous;*

(b) *The embedding Φ_π is a continuous map from V into $C(G,V)$;*

(c) *The topology of V agrees with that induced from its embedding as a closed subspace of $C(G,V)$.*

If V is barreled (in particular if it is a Fréchet space or an LF space) then these conditions are implied automatically by separate continuity.

Proof. Conditions (b) and (c) are equivalent since the evaluation map from $C(G,V)$ to V taking φ to $\varphi(1)$ is a continuous splitting of Φ. Since the topology on $C(G,V)$ is that defined by the semi-norms

$$\|f\|_{\Omega,\rho} = \sup_{g\in\Omega} \|f(g)\|_\rho$$

where ρ is a semi-norm on V and Ω is a compact subset of G, the equivalence of (a) and (c) as well as the last remark are straightforward (and amount to a rephrasing of [**4**, 3.2], to which I refer in general for an efficient summary).

If the topological vector space V is quasi-complete, then the convex hull of any compact subset of V is again compact [**6**, III.2.5], so that one may integrate functions in $C(G, V)$ with respect to compactly supported measures on G. Any continuous representation of G therefore gives rise to a representation of the convolution algebra $M_c(G)$ of such measures, and in particular, after a choice of Haar measure, one of $C_c(G)$, the sub-algebra of compactly supported continuous functions on G, according to the formula

$$\pi(f)v = \int f(g)\pi(g)v dg.$$

Associated to any continuous representation (π, V) of G is the transpose representation of G on the continuous dual of V. Under a mild assumption on V this is again continuous in the weak or compact-open topology, but not generally in the strong topology (see [**39**, 4.1.2] for a discussion of this point). Eventually we shall be dealing with the case when V is Fréchet as well as nuclear, in which case the compact-open and strong topologies on the dual are the same by [**36**, Proposition 34.5, Corollary 3 on p. 520], so that these technicalities will cause no trouble. Note that if V is quasi-complete and barreled then according to the Banach-Steinhaus Theorem [**36**, Theorem 34.2, Corollary 2 on p. 356], the continuous dual of V is again quasi-complete with respect to the weak, compact-open, and strong topologies.

A vector in V is said to be G-*smooth* (or, usually, just smooth) if the function Φ_v is smooth. The subspace V^{sm} of smooth vectors is G-stable and embeds under Φ_π as a closed subspace of the topological space $C^\infty(G, V)$ of smooth functions on G with values in V. It inherits from this space a canonical topology in which G again acts continuously. If $V = V^{\mathrm{sm}}$ with the canonical topology then π itself is said to be smooth. The space V^{sm} inherits also from $C^\infty(G, V)$ the structure of a representation of $\mathfrak{g}$ [**4**, 3.8], which acts by right derivations. If V is a Fréchet space then so is $C^\infty(G, V)$, hence also V^{sm} with the canonical topology.

The following point seems not to be clear in the literature:

LEMMA 1.2. *If (π, V) is any continuous representation of G then the canonical topology on the subspace V^{sm} is the same as the topology assigned by the semi-norms*

$$\|v\|_{X,\nu} = \|\pi(X)v\|_\nu$$

where ν ranges over the semi-norms of V and X over the elements of $U(\mathfrak{g})$.

Proof. This is a direct corollary of Lemma 1.1.

According to [**7**, VIII.2.1, Rem. (3)] a continuous representation on the locally convex topological vector space V extends uniquely to a continuous representation on the completion of V. It is therefore no serious restriction to impose from now on the condition that V be complete, and in particular quasi-complete.

Since V^{sm} is identified with a closed subspace of $C^{\infty}(G, V)$, the representation of G on it may be extended, as is explained in [**4**, 3.8], to include operators $\pi(D)$, where D is a distribution on G of compact support. It is relatively elementary (refer to [**8**], for example) to see that the subspace V^{sm} is the same as that of all vectors v which can be expressed as linear combinations of the form $\pi(f)u$, where f is of compact support on G and of arbitrarily high differentiability. It contains in turn the *Gårding subspace* of V, which is that spanned by the vectors in V of the form $\pi(f)v$ with $f \in C_c^{\infty}(G)$. It is dense in V^{sm} and of course G-stable as well. In subsequent arguments I shall require the strong main result [**15**, Théorème 3.3], according to which in any Fréchet space on which G acts continuously the spaces V^{sm} and the Gårding subspace coincide.

The subspace V^{sm} also contains the subspace V^{an}, that of *analytic vectors* of the representation, those for which Φ_v is an analytic function on G. It too is stable under the operators $\pi(D)$.

A vector in V is said to be K-finite if it is contained in a finite-dimensional K-stable subspace. The subspace V^K of such vectors is of course stable under K, and it is the algebraic direct sum of its K-isotypic constituents. The intersection $V^{\mathrm{sm}} \cap V^K$ is dense in V and stable under both $\mathfrak{g}$ and K.

Any embedding of G as a Zariski-closed algebraic subset of real affine space determines an algebraic norm $\|g\| = \sup(1, |g_i|)$ on G. Any two different such embeddings determine norms which are equivalent in the sense that each is bounded by some multiple of a power of the other.

I shall say that the continuous representation (π, V) is *of moderate growth* if

(T1) The topological space V is a Fréchet space;

(T2) For every semi-norm ρ on V there exist a positive integer N and semi-norm ν such that

$$\|\pi(g)v\|_\rho \leqq \|g\|^N \|v\|_\nu$$

for every $v \in V, g \in G$.

If these conditions hold, then it follows from Lemma 1.2 that V^{sm} is also of moderate growth, with semi-norms

$$\|v\|_{\rho,X} = \|Xv\|_\rho$$

where ρ ranges over the semi-norms on V and X over the elements of $U(\mathfrak{g})$. Of course any unitary representation of G is of moderate growth, and it is also known that more generally every continuous representation of G on a Banach space is of moderate growth according to [**38**, 2.2] (following [**39**, Example on p. 282]).

Define the *Schwartz space* of G to be that of all smooth functions on G which are, along with all their derivatives, of rapid decrease at infinity, or more precisely:

$$\begin{aligned}\mathcal{S}(G) := \{f \in C^\infty(G) | R_X f(g) \\ = O(\|g\|^{-N}) \text{ for all } X \in U(\mathfrak{g}), \text{all } N > 0\}.\end{aligned}$$

The condition on the right regular representation R is equivalent to the analogous one on the left regular regular representation. This space (defined also in [**38**, 2.5]) becomes a Fréchet space with the semi-norms

$$\|f\|_{X,N} = \sup_{g \in G} |R_X f(g)| \, \|g\|^N.$$

and becomes a smooth representation of $G \times G$ of moderate growth, by means of the right and left regular representations. It is even a nuclear space, but this seems less trivial; see [**11**]. Incidentally, the Schwartz space defined here is not by any means the same as that defined by [**18**]. The simplest case where the two differ is that of the group $\mathbf{R}^\times$, where the Schwartz functions defined here are the restrictions to the multiplicative group of those functions in $\mathcal{S}(\mathbf{R})$ which vanish of infinite order at zero, while Harish-Chandra's are those lifted back from $\mathcal{S}(\mathbf{R})$ by means of the logarithm. The Schwartz space I define here is attached intrinsically to the structure of G as a real algebraic manifold. Harish-Chandra's Schwartz functions will play no role in this paper, so there will be no confusion in terminology. It is curious that there is any event no confusion of notation, since Harish-Chandra's Schwartz space is ordinarily written as $\mathcal{C}(G)$.

The representation of the algebra $C_c(G)$ on any representation (π, V) of moderate growth extends uniquely and continuously to one of $\mathcal{S}(G)$, since for any $n > 0$

$$\begin{aligned}\left\| \int_\Omega f(g)\pi(g)v\,dg \right\|_\rho &\leqq \int_\Omega |f(g)| \, \|\pi(g)v\|_\rho dg \\ &\leqq C_{f,v,n} \int_\Omega \|g\|^{-n} dg.\end{aligned}$$

The subspace of smooth vectors is stable under $\mathcal{S}(G)$. The corresponding map from the algebraic tensor product of $\mathcal{S}(G)$ and V to V extends continuously to one defined on the topological tensor product $\mathcal{S}(G) \hat{\otimes} V$ (which makes unambiguous sense since $\mathcal{S}(G)$ is nuclear). This representation has occurred before in representation theory, but not until recently by name; among other places, in Gårding's proof that the analytic vectors in a continuous Banach representation are dense, where $\pi(f)$ is used for f equal to $\exp(-t\Delta)\delta$ for $t > 0$. Schwartz functions in the sense defined here are just about as good as ones in $C_c^\infty(G)$, and often more convenient to work with. As Wallach has pointed out, for example, they may be used in the Selberg trace formula where it is usually much simpler to find a Schwartz function to do the job one wants than it is to find a compactly supported one.

2. Representations of $(\mathfrak{g}, K)$. Let for the moment H be an arbitrary closed subgroup of G. A $(\mathfrak{g}, H)$-module is a vector space on which $\mathfrak{g}$ and H act simultaneously, satisfying these conditions of H-finiteness and compatibility:

(C1) Every vector is H-finite and H-continuous; i.e., contained in an H-stable finite-dimensional subspace of V on which H acts continuously (and therefore smoothly as well);

(C2) The two representations of $\mathfrak{h}$ one obtains, as a subalgebra of $\mathfrak{g}$ and as the Lie algebra of H, are the same;

(C3) For any $h \in H$ and $X \in \mathfrak{g}$,

$$\pi(\mathrm{Ad}(h)X) = \pi(h)\pi(X)\pi(h^{-1}).$$

If H is compact, condition (C1) is equivalent to:

(C4) The restriction of π to H is a direct sum of irreducible, continuous, finite-dimensional representations.

If (π, V) is a continuous representation of G then the (dense) subspace $V^{\mathrm{sm}} \cap V^K$ which is stable under $\mathfrak{g}$ and K, as I have already recalled, clearly satisfies these conditions with $H = K$. Conversely, if (π, V) is a $(\mathfrak{g}, K)$-module satisfying these conditions then a G-extension of π consists of a continuous representation of G on a topological vector space V_* and an isomorphism of π with the associated $(\mathfrak{g}, K)$-module $V_*^{\mathrm{sm}} \cap V_*^K$. A $(\mathfrak{g}, K)$-module is usually called a *Harish-Chandra module* if it satisfies in addition

(H1) Any irreducible K-representation occurs with finite multiplicity.

There are several more or less imediate consequences of this assumption:

PROPOSITION 2.1. *Let* (π, V) *be a* $(\mathfrak{g}, K)$*-module satisfying* (H1). *Then*

(a) *Every* $v \in V$ *is annihilated by some ideal of finite codimension in* $Z(\mathfrak{g})$ *(the centre of* $U(\mathfrak{g})$*);*

(b) *The associated representation of* $(\mathfrak{z}_G, K \cap Z_G)$ *extends to a unique representation of* Z_G *on* V.

Proof. Assertion (a) is true because K commutes with $Z(\mathfrak{g})$ and hence each K-isotypic constituent is $Z(\mathfrak{g})$-stable. Assertion (b) holds since Z_G is the direct product of a connected, simply-connected subgroup and a compact subgroup in the centre of K.

If π is irreducible then Z_G must act by a character which I shall refer to as $\zeta(\pi)$.

PROPOSITION 2.2. *Let* (π_*, V_*) *be a continuous representation of* G *satisfying* (H1) *and let* V *be the subspace of its* K*-finite vectors. Then*

(a) *Every vector in V is smooth;*

(b) *Every K-finite linear functional on V may be extended to a unique, continuous, K-finite linear functional on V_* (so that the K-finite elements in the algebraic dual of V may be identified with the K-finite continuous linear functionals on V_*);*

(c) *For every v in V there exists $f \in C_c^\infty(G)$ with $\pi(f)v = v$;*

(d) *The correspondence which takes a $(\mathfrak{g}, K)$-stable subspace of V to its closure in V_* is a bijection between the set of all $(\mathfrak{g}, K)$-stable subspaces of V and the closed G-stable subspaces of V_*.*

Proof. Of course these assertions are elementary and well known. Property (a) holds because $V^{\mathrm{sm}} \cap V^K$ is dense in V. Assertion (b) is true because the projection onto a K-isotypic constituent is continuous, and any linear functional on a finite-dimensional space is continuous. To prove assertion (c), Let $\mathcal{P}$ be the projection onto a sum of isotypic K-components containing v, and consider the sequence $\mathcal{P}\pi(f_n)(v)$ where (f_n) is a Dirac sequence on G, keeping in mind that the image of $C_c^\infty(G)$ in the finite-dimensional ring $\mathrm{End}(\mathcal{P}(V))$ must be closed. For (d), I refer to the proof of [**4**, 3.17]. Incidentally, the inverse map takes the closed G-stable subspace U_* to its intersection with V.

It is a result of Harish-Chandra that if π is irreducible and unitary then it satisfies (H1), but it does not seem to be known whether (H1) holds, say, for an arbitrary irreducible Banach representation.

If (π, V) is a $(\mathfrak{g}, K)$-module satisfying (H1) then its *contragredient* $(\tilde{\pi}, \tilde{V})$ is defined to be the contragredient representation of $(\mathfrak{g}, K)$ on the subspace of K-finite elements in the algebraic dual of V. This again satisfies (H1).

I will impose further, as part of the definition of a Harish-Chandra module, the condition

(H2) The space V is annihilated by some ideal of finite codimension in $Z(\mathfrak{g})$.

This is implied by this apparently stronger condition, which I do not assume:

(H3) The space V is finitely generated over $U(\mathfrak{g})$, which is in turn implied by
(H4) The $(\mathfrak{g}, K)$-module V has finite length.

In fact these conditions (H2), (H3), and (H4) are all equivalent, as is well known and will later be recalled in more detail (at the end of Section 5).

The contragredient of a Harish-Chandra module is again a Harish-Chandra module: if V is annihilated by I in $Z(\mathfrak{g})$ then $\tilde{V}$ is annihilated by I^ι, where ι is the involution of $U(\mathfrak{g})$ taking X into $\mathfrak{g}$ to $-X$.

I shall call a continuous representation (π, V) of G a *Harish-Chandra representation* if it is smooth and of moderate growth, and the underlying representation of $(\mathfrak{g}, K)$ on the subspace of smooth K-finite vectors is a Harish-Chandra module. This definition is at least independent of the choice of K since all maximal compact subgroups are conjugate in G.

Many of the definitions above can be made slightly more general. Suppose that P is any Zariski-connected algebraic group defined over $\mathbf{R}$ with unipotent radical N, reductive quotient $M = P/N$, and maximal compact subgroup $K(P)$ whose image in M is $K(M)$. A Harish-Chandra representation of P is defined to be a smooth representation (σ, U) of P such that

(P1) For some positive integer k, $\mathfrak{n}^k U = 0$;

(P2) the representation of M on any quotient $U[\mathfrak{n}^{m+1}]/U[\mathfrak{n}^m]$ is a Harish-Chandra representation.

Here $U[\mathfrak{n}^k]$ means the subspace of U annihilated by elements of $U(\mathfrak{n})$ of the form $\nu_1 \dots \nu_k$ with each ν_i in $\mathfrak{n}$. Similarly a $(p, K(P))$-module (σ, U) is called a Harish-Chandra module if it satisfies the first condition above and

(P3) Each quotient $U[\mathfrak{n}^{m+1}]/U[\mathfrak{n}^m]$ is a Harish-Chandra module over $(m, K(M))$.

LEMMA 2.3. *Every finite-dimensional Harish-Chandra module over* $(\mathfrak{p}, K(P))$ *extends uniquely to a continuous representation of* P *on the same space.*

Proof. Every finite-dimensional representation of $\mathfrak{p}$ extends uniquely to a continuous representation of the universal covering group of the connected identity component of P. Since it is already a representation of $K(P)$ which has the same fundamental group as P, it extends to a unique representation of the connected component of P. But since according to the classical result of [**30**] the group $K(P)$ meets all components of P and the connected component is normal in P, it extends as well uniquely to all of P.

3. Matrix coefficients and realizability. I shall call a Harish-Chandra module (π, V) over $(\mathfrak{g}, K)$ *realizable* if it is the $(\mathfrak{g}, K)$-module underlying some continuous representation (π_*, V_*) of G. Of course V will be realizable if and only if its contragredient $\tilde{V}$ is. When π is realizable, given $v \in V$ and $\lambda \in \tilde{V}$ one defines the corresponding *matrix coefficient* $c(v, \lambda)$ to be the function on G determined by the formula

$$c(v, \lambda)(g) := \langle \pi_*(g) v, \lambda \rangle.$$

It is right- and left-K-finite and annihilated by any ideal I in $Z(\mathfrak{g})$ which annihilates V, so by [**4**, 3.12] it is an analytic function. The map $c\colon V \otimes \tilde{V} \longrightarrow C^\omega(G)$ has these properties:

(M1) $\quad c(\pi(k)v, \lambda) = R_k c(v, \lambda)$;

(M2) $\quad c(v, \tilde{\pi}(k)\lambda) = L_k c(v, \lambda)$;

(M3) $\quad c(\pi(X)v, \lambda) = R_X c(v, \lambda)$;

(M4) $\quad c(v, \lambda)(1) = \langle v, \lambda \rangle$

for all $v \in V, \lambda \in V, X \in U(\mathfrak{g}), k \in K$. Conditions (M1)–(M3) say, essentially, that c is a $(\mathfrak{g}, K) \times (\mathfrak{g}, K)$-map from the tensor product of V and $\tilde{V}$ to $C^\omega(G)$.

All the conditions together imply that the function $c(v, \lambda)$ is unique with these properties: on the one hand the function $c(v, \lambda)$ is analytic so that it is determined on the connected component of G by its Taylor series at 1, which is specified by (M3)–(M4); while on the other (M1)–(M2) assert that it is determined on the other components by what it is on the identity component, since K meets all components. In particular the matrix coefficient $c(v, \lambda)$ doesn't depend on the choice of G-extension, so writing $\langle \pi(g)v, \lambda\rangle$ for $c(v, \lambda)(g)$, as I shall do, is not to be ambiguous.

If (π, V) is a Harish-Chandra module I shall say that it *has matrix coefficients* if there exists a map

$$c : V \otimes \tilde{V} \longrightarrow C^{\omega}(G)$$

satisfying the conditions (M).

LEMMA 3.1. *A finitely generated Harish-Chandra module is realizable if and only if it has matrix coefficients.*

Proof. It suffices to show that the contragredient of a finitely generated Harish-Chandra module having matrix coefficients is realizable. But if $v_1, \ldots, v_m$ generate V then the map taking each λ in $\tilde{V}$ to the m-tuple of matrix coefficients $\langle \pi(g)v_i, \lambda\rangle$ gives an embedding of $\tilde{V}$ in $C^{\omega}(G)^m$.

THEOREM 3.2. *Every Harish-Chandra module has matrix coefficients.*

In the form which asserts that every finitely generated Harish-Chandra module is realizable, this result is apparently due originally to [**33**]. Several proofs are now known, some more illuminating than others (see also the remarks just before Corollary 5.7).

I shall give no details, but only make a few remarks about a possible direct proof. As we have seen above, every matrix coefficient defines a right- and left-K-finite, $Z(\mathfrak{g})$-finite formal power series at the identity of G, and the problem is to see that this is in fact the Taylor series of an analytic function globally defined on G. The system of differential equations satisfied by K-finite matrix coefficients annihilated by an ideal I of finite codimension in $Z(\mathfrak{g})$, expressed in radial coordinates according to the Cartan factorization of G, is described in [**12**] and shown there to have regular singularities at infinity. In fact (just like the Poisson equation in polar coordinates on Euclidean space) it has a regular singularity at the identity. A theorem of [**22**], generalizing a classical result [**14**, 4.3] about ordinary differential equations, asserts precisely that a formal power series solution of such a system is the Taylor series of a locally convergent solution. An elementary monodromy argument then shows that it extends to all of G. (This argument appeared originally in the manuscript [**9**], and can also be found in [**2**].)

This result may be enhanced by the further observation:

COROLLARY 3.3. *Every finitely generated Harish-Chandra module over* $(\mathfrak{g}, K)$ *is realizable in a Harish-Chandra representation of* G.

Proof. The corollary may be proved in any of several ways, given the Theorem, but I shall specify a realization which is itself of considerable interest and will play a role in several places later on. Therefore I will take the opportunity to recall here at some length basic facts about the asymptotic behaviour of matrix coefficients of a Harish-Chandra module at infinity on G.

Nearly all matters concerning the behaviour of functions at infinity on G become clearer if one introduces a partial compactification $\overline{G}$ of G which is attributed to Oshima but for which I know only the partial reference [**32**]. In certain cases it is derived from a compactification of the complex adjoint group which is one of a class constructed by DeConcini and Procesi, but their description is not quite adequate for my purposes. It is also related to certain compactifications of complex semi-simple groups constructed by Luna and Vust, and another version appears in the recent preprint [**16**]. Borel has dealt with the real compactifications associated to these complex algebraic completions in a seminar at the Institute for Advanced Study (spring, 1987) and I imagine that what I need will be covered in his written notes. Here I shall describe the completion I require, I am afraid, without justification. Proofs are not so different from those used in [**31**] to deal with a similar compactification of the symmetric space G/K.

The *Oshima completion* $\overline{G}$ is defined for every linear reductive group G. It arises as the fibre product of G and the completion $\overline{G^{\mathrm{der}}}$ of the derived group G^{der} of G. Its simplest properties are these two:

(1) The topological space $\overline{G}$ is a real analytic (Hausdorff) manifold with corners, whose interior is G.

(2) The canonical regular action of the group $G \times G$ on G extends to an analytic action of $\overline{G}$.

If G is the union of several connected components then $\overline{G}$ is the disjoint union of the closures of these connected components.

Let P, P^{opp} be an *opposing pair* of parabolic subgroups. Thus the intersection $P \cap P^{\mathrm{opp}}$ is a reductive component of each; or in other words the intersections $N \cap P^{\mathrm{opp}}$ and $N^{\mathrm{opp}} \cap P$ are trivial. I will in fact identify the intersection with M_P, which in particular contains a unique copy of A_P. The basis $\Delta = \Delta_P$ of positive roots determines a covariant embedding of A_P into $\mathbf{R}^{\Delta}$: $a \longmapsto (\gamma(a))$. Define A_P^* to be its closure, the closed positive quadrant. It is well to keep in mind that A_P^* depends on the choice of P containing A_P (that is to say, on the direction in which one passes off to infinity) and does not depend intrinsically just on the group A_P itself. Roughly speaking, all corners of $\overline{G}$ look like the corner of some A_P^*. The starting point of a precise description of the situation is this:

(3) The inclusion of A_P into G extends to an analytic embedding of A_P^* into $\overline{G}$.

Let M_P^* be the fibre product of M_P, the quotient of the product of M_P and A_P^* by the inner A_P-action: (ma^{-1}, ax) is equivalent to (m, x). The embedding of A_P^* into $\overline{G}$ extends by $G \times G$-covariance to an $M_P \times M_P$-covariant analytic map from M_P^* into $\overline{G}$ as well, and in fact:

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(4) The inclusion of M_P into G extends to an analytic embedding of M_P^* in $\overline{G}$.

I will identify M_P^* with its image. Let M_P^P be the unique closed M_P-orbit in M_P^*, which is isomorphic to M_P/A_P.

(5) The closure of M_P^P is the subset of all points of $\overline{G}$ fixed by the product $N_P^{\mathrm{opp}} \times N_P$ in $G \times G$.

The subgroup of all elements in $G \times G$ leaving each point in M_P^P fixed is the product of $N_P^{\mathrm{opp}} \times N_P$ and $A_P \times A_P$.

Of course $P^{\mathrm{opp}} \times P$ stabilizes M_P^P. Thus the inclusion of M_P^P in $\overline{G}$ induces an analytic $G \times G$-covariant map from the fibre product G_P of $G \times G/P^{\mathrm{opp}} \times P$ with M_P/A_P into $\overline{G}$.

(6) This is an embedding of G_P into $\overline{G}$.

Two subsets G_P and G_Q will coincide if and only if P and Q are conjugate. The set M_P^P will lie in the closure of M_Q^Q, and G_P will lie in the closure of G_Q, precisely when P is contained in Q. The closure of M_P^P in $\overline{G}$ is in fact isomorphic to the Oshima completion of M_P/A_P.

(7) The space $\overline{G}$ is the disjoint union of the G_P, one for each conjugacy class of parabolic subgroups of G.

Incidentally, if P is a parabolic subgroup of G then the set of all triples (R, Q, M) where R is a conjugate of P, Q is opposite to R, and m is an element of $R \cap Q$ form an open orbit of the image of G in $G \times G$ under the diagonal embedding.

Any manifold with corners possesses a canonical stratification by the union of coordinate planes of equal dimension at the corners. In our case, this stratification coincides with the one we already have:

(8) The stratification of $\overline{G}$ by corner components is the same as that by the union of pieces G_P corresponding to parabolic subgroups of equal rank.

The topology of $\overline{G}$ is controlled by $G \times G$-covariance if one knows an open set intersecting each $G \times G$-orbit. It therefore suffices to describe a neighborhood of each M_P^*:

(9) The embedding of M_P^* into $\overline{G}$ extends via the product map to one of the product $N_P \times M_P^* \times N_P^{\mathrm{opp}}$, whose image is an open neighborhood of M_P^*.

This is at least plausible since $N_P M_P N_P^{\mathrm{opp}}$ is open in G.

The effect of several of these properties can be roughly summarized as saying that the manifold with corners $\overline{G}$ is stratified by the disjoint union of submanifolds G_P indexed by the conjugacy classes of parabolic subgroups of G, and locally along the stratum G_P the tranverse slice may be identified canonically up to A_P-translation with a neighborhood of the origin in A_P^*.

Suppose for the moment that U is a neighborhood of the origin in the product, which I call temporarily X, of $\mathbf{R}^q$ with the closed positive quadrant in $\mathbf{R}^n$. For every s in $\mathbf{C}^n$ and m in $\mathbf{N}^n$ define the functions.

$$x^s = \prod x_i^{s_i}, \quad \log^m(x) = \prod \log^{m_i}(x_i)$$

on the positive quadrant of $\mathbf{R}^n$. Define further the space of functions $A_{s,m}(U)$ to be that of smooth functions on the interior of U which are finite sums of functions of the form

$$x^s \log^k(x) f_{s,k}(x, y)$$

with each $k_i \leqq m_i$, where x defines the corner and y the transverse coordinates, and each $f_{s,k}$ is the restriction to U of a smooth function on $\mathbf{R}^{n+m}$. Now I introduce slightly inconsistent notation: if S is a subset of $\mathbf{C}^n$, call it *finitely generated,* $\mathbf{N}^n$*-stable* if it is a finite union of sets $s + \mathbf{N}^n$. For such a subset S, define $A_{S,m}(U)$ to be the sum of $A_{s,m}(U)$ with s ranging over a finite generating subset of S. Since each such $s = (s_1, \dots, s_n)$ determines also an element (s_i) of $\mathbf{C}^I$ where I is a subset of $1, \dots, n$ and similarly m determines an element of $\mathbf{N}^I$, the pair (s, m) determines also a space $A_{s,m}(U)$ whenever U is a neighborhood of any point in X. Let $\mathcal{A}_{S,m}$ be the sheaf on X associated to the presheaf which assigns to the open set U the space $A_{S,m}(U \cap X^{\text{int}})$. Since any change of coordinates on X involves a change in the X-variables of the form

$$x_i \longmapsto x_i c_i(y) f_i(x, y)$$

where $f_i(0, y) = 1$ for all y, this sheaf is independent of the choice of coordinates, since under such a coordinate change $\log(x)$ changes to the sum

$$\log(x) + \log(c(y)) + \text{something which is 0 along the corner.}$$

Fix for the moment a minimal parabolic subgroup P, and let $\Delta = \Delta_P$. Then the transverse slices to the closed orbit G_P may be identified with A_P^*, so that we obtain for every finitely generated $\mathfrak{n}^\Delta$-stable $S \subseteq \mathbf{C}^\Delta$ and m in $\mathfrak{n}^\Delta$ a sheaf $\mathcal{A}_{S,m}$ defined globally on $\overline{G}$.

The main results of [**12**] (following Harish-Chandra) imply that for every finitely generated Harish-Chandra module (π, V) there exist a finite set S in $\mathbf{C}$ and a non-negative integer m such that all matrix coefficients $\langle \pi(g)v, \lambda \rangle$ for $v \in V, \lambda \in V$ are global sections of $\mathcal{A}^\omega_{S,m}$ over $\overline{G}$. Hence there exists an integer $N > 0$ such that by means of matrix coefficients π may be embedded in a sum of copies of the Banach space of all functions f on G such that $|f(g)| = O(\|g\|^N)$. From this the Corollary, at least, is clear.

This construction will occur again later, where among other things I shall show how to put a topology on the space of global sections of each $\mathcal{A}_{S,m}$.

Now let Q be a parabolic subgroup. I have remarked already that the subset M_Q^Q is the interior of the fixed-point set of $N_Q^{\mathrm{opp}} \times N_Q$. In fact, this group acts so that along M_Q^Q elements of its Lie algebra take $\mathcal{A}_{S,m}$ into $I\mathcal{A}_{S,m}$, where I is the ideal of smooth function in the neighborhood of M_Q^Q vanishing on M_Q^Q. In other words, the space of global sections of the quotient sheaf $\mathcal{A}_{S,m}/I\mathcal{A}_{S,m}$, which has support on the closure of M_Q^Q, becomes naturally the space of a representation of M_Q. It is not difficult to see that it is a subspace of $A_{S,m}(M_Q)$. It is slightly technical to describe in general but if $m = 0$ and S has a single element s, then it is the subspace which transforms with respect to A_Q according to the character of A_Q induced by s. This plays an implicit role in various results later on about embeddings of representations into representations induced from parabolic subgroups.

Continue to let P be a minimal parabolic subgroup. Then $\mathbf{C}^\Delta$ may be identified with

$$X(A_P) := \text{the group of } \mathbf{C}^\times\text{-valued characters of } A_P.$$

Define

$$X_{\mathbf{R}}^-(A_P) := \{\chi \in X_{\mathbf{R}}(A_P) | \chi = \prod\nolimits_{\alpha\in\Delta} \alpha^{\sigma_\alpha} \quad (\sigma_\alpha > 0)\}.$$

Since this definition depends on the choice of P containing the torus A_P the notation is slightly faulty, but it should not cause confusion. If χ and ξ lie in $X(P)$, the group of complex characters of P, they define by restriction elements of $X(P_P)$. Define $\chi \leqq \xi$ to mean that the restriction of $\xi\chi^{-1}$ lies in the closure of $X_{\mathbf{R}}^-(A_P)$. In terms of the identification of $X(A_P)$ with $\mathbf{C}^\Delta$ this translates to the condition that $s \leqq t$ when the different $t - s$ has non-negative coordinates. Define χ and ξ in $X(P)$ to be *integrally equivalent* if the restriction of $\chi\xi^{-1}$ to A_P is of the form $\prod \alpha^{s_\alpha}$ with the s_α integral. This translates to the condition that $s - t$ have integral coordinates.

The result of **[12]** mentioned above asserts that in the neighborhood of any point of M_P^P a K-finite matrix coefficient of a Harish-Chandra module π may be expressed as an (infinite) sum of terms $c_{s,k}x^s \log^k(x)$. Those s which are minimal in their integral equivalence class with the property that some $c_{s,k}$ is not zero I call *dominant asymptotic characters* of the Harish-Chandra module π.

An irreducible Harish-Chandra module whose central character is unitary is *square-integrable* if the absolute value of every (or, equivalently, some) matrix coefficient is square-integrable on G/Z_G, and *tempered* if the absolute value of every matrix coefficient lies in $L^{2+\epsilon}(G/Z_G)$ for every $\epsilon > 0$. It is called *essentially* square-integrable (essentially tempered) if it is the tensor product of a square-integrable (tempered) module with a character of G.

Of course these conditions can be read as conditions local on $\overline{G}$. In the open subset PP^{opp} the volume on G may be expressed as the form $dg = \delta_P(m)^{-1} dn\, dm\, dn^{\mathrm{opp}}$. If P is contained in the parabolic subgroup Q then the restriction of $\delta - P$ to A_Q is the same as δ_Q. Let ρ_P be the square root of δ_P. Then since χ lies in $X_{\mathbf{R}}^-(A_P)$ if and only if $\chi(a) < 1$ for all a in A_P^{--}, the conditions local on $\overline{G}$ translate to:

PROPOSITION 3.4. *Let P be a minimal parabolic subgroup of G. The Harish-Chandra module (π, V) is essentially square-integrable precisely when the absolute values of the dominant asymptotic characters of π all lie in $X_{\mathbf{R}}^-(A_P)\rho_P$. It is essentially tempered when they lie in the closure of $X_{\mathbf{R}}^-(A_P)\rho_P$ in $X_{\mathbf{R}}(A_P)$.*

A representation of $(\mathfrak{g}, K)$ is sometimes called *admissible* if it satisfies condition (H1) and in addition is compatibly a representation of what is called the Hecke algebra $\mathcal{H}(G)$ (in analogy with a p-adic device), which is the convolution algebra obtained as the direct sum of the right- and left-K-finite functions in $C_c^\infty(G)$ and the K-finite functions on K embedded as distributions on G. It can be shown that such a representation possesses matrix coefficients, which means that this notion of admissibility is directly equivalent to what I call realizability. Perhaps it was introduced (by Jacquet and Langlands) exactly in order to avoid the possibility of Harish-Chandra modules which were not realizable.

4. Representations induced from parabolic subgroups. Given a smooth representation (σ_*, U_*) of P, the smooth representation of G *induced* by it is the right regular representation of G on the space

$$\begin{aligned}\mathrm{Ind}^{\mathrm{sm}}(\sigma_*|P, G) := \{f \in C^\infty(G, U_*) | f(pg) \\ = \sigma_*(p)f(g) \text{ for all } p \in P, g \in G\}.\end{aligned}$$

Since $G = PK$, if σ_0 is the restriction of σ_* to K then restricting a function to K induces a K-isomorphism of this with

$$\mathrm{Ind}^{\mathrm{sm}}(\sigma_0|K(P), K),$$

so that Frobenius reciprocity in the elementary form

$$\mathrm{Hom}_K(\tau, \mathrm{Ind}^{\mathrm{sm}}(\sigma_0|K(P), K)) \cong \mathrm{Hom}_{K(P)}(\tau, U_*)$$

implies that if condition (P1) is satisfied then the induced representation satisfies (H1). The space of the representation may be identified with the space of $\mathrm{Ind}^{\mathrm{sm}}(\sigma_0|K(P), K)$ in which case it is realized by certain operators on the latter space corresponding to elements of $U(\mathfrak{g})$. Two representations σ_* and $\sigma_*\chi$ will be thus realized on the same space if χ is a character of P trivial on $K(P)$ (that is to say, an *unramified* character of P) since then the restrictions of σ_* and $\sigma_*\chi$ to K are the same. Let φ_{σ_*} be the inverse of the map defined by restriction to K, the K-isomorphism

$$\varphi_{\sigma_*} : \mathrm{Ind}(\sigma_0|K(P), K) \longrightarrow \mathrm{Ind}(\sigma_*|P, G).$$

The representation $\mathrm{Ind}^{\mathrm{sm}}(\sigma_*|P, G)$ varies analytically with σ_* in the sense that the operators $R_{\sigma_*\chi}(X)$ corresponding to $X \in U(\mathfrak{g})$, interpreted as acting on the fixed space $\mathrm{Ind}(\sigma_0|K(P), K)$, vary analytically with χ.

The topology on the space of the induced representation is that given by the semi-norms

$$\|f\|_{X,\rho} := \sup_{k \in K} \|R_X f(k)\|_\rho$$

where $X \in U(\mathfrak{g})$ and ρ is a semi-norm on U_*.

PROPOSITION 4.1. *If σ_* is a representation of P of moderate growth then the smooth representation induced by it is a representation of G of moderate growth, and if σ_* is a Harish-Chandra representation of P then the induced representation is one of G.*

Proof. The first is true since in the factorization of kg as $p_* k_*$, the norm $\|g\|$ is the same as $\|p_*\|$.

The second is well known, but because I will need relevant notation later on I recall the proof. Let $p = m + \mathfrak{n}$ be the Lie algebra of P, and let

$$\rho = \rho_p : Z(\mathfrak{g}) \longrightarrow Z(m)$$

be the Harish-Chandra map, characterized by the property that for any $X \in Z(\mathfrak{g})$,

$$X - \rho(X) \in \mathfrak{n} U(g).$$

This map is an injective ring homomorphism, and the algebra $Z(m)$ is a finite module over $\rho(Z(\mathfrak{g}))$ (according to [**39**, 2.3.3.4–5], for example). It is easy to see that if σ_* is trivial on N and I is an ideal of $Z(m)$ annihilating U then $\rho^{-1}(I)$ is an ideal of finite codimension in $Z(\mathfrak{g})$ annihilating $\mathrm{Ind}^{\mathrm{sm}}(\sigma_*|P, G)$. Since any Harish-Chandra representation of P posseses a finite P-stable filtration whose quotients are annihilated by $\mathfrak{n}$, the proposition follows.

A trivial form of Frobenius reciprocity holds for this induced representation. Let

$$\Lambda : \mathrm{Ind}^{\mathrm{sm}}(\sigma_*|P, G) \longrightarrow U_*$$

be the continuous P-covariant map taking f to $f(1)$. The following is straightforward:

LEMMA 4.2. *If (π, V) is any smooth representation of G, then compositions with Λ induces an isomorphism*

$$\mathrm{Hom}_{G,\mathrm{cont}}(V, \mathrm{Ind}^{\mathrm{sm}}(\sigma_*|P, G)) \cong \mathrm{Hom}_{P,\mathrm{cont}}(V, U_*).$$

The $(\mathfrak{g}, K)$-module underlying $\mathrm{Ind}^{sm}(\sigma_*|P, G)$ depends only on the $(\mathfrak{p}, K(P))$-module (σ, U) underlying σ_*. This can be shown by constructing the induced $(\mathfrak{g}, K)$-module algebraically. In explaining this I follow the treatment in [**5**, II.2] who, however, dealt with a slightly different set-up. Let (σ, U) be any Harish-Chandra module over $(\mathfrak{p}, K(P))$, let H be the $U(\mathfrak{g})$-module $\mathrm{Hom}_{U(\mathfrak{p})}(U(\mathfrak{g}), U)$, and define $I = \mathrm{Ind}(\sigma|P, G))$ to be the subspace of elements φ of H such that

(a) φ is $\mathfrak{k}$-finite;

(b) φ is invariant with respect to the kernel of the canonical projection from $\tilde{K}$ to K, where $\tilde{K}$ is the simply-connected covering of the connected component K^0 of K;

(c) for any $\kappa \in K(P) \cap K^0$ and $X \in U(\mathfrak{g})$,

$$(\kappa\varphi)(X) = \sigma(\kappa)\varphi(\mathrm{Ad}(\kappa^{-1})X).$$

These conditions require explanation. The subspace $I_{(a)}$ of elements of H satisfying (a) is clearly a $(\mathfrak{g}, \mathfrak{k})$-module which as a $\mathfrak{k}$-module is a direct sum of finite-dimensional subspaces. Hence it may be considered also a representation of $(\mathfrak{g}, \tilde{K})$. (Since the inclusion of K into G is a homotopy equivalence, $\tilde{K}$ may be identified with a subgroup of the universal covering $\tilde{G}$ of G.) The subspace $I_{(b)}$ of elements of $I_{(a)}$ satisfying (b) is then a representation of $(\mathfrak{g}, K^0)$. But since $K(P)$ intersects all connected components of K, the group K is the product of $K(P)$ and its normal subgroup K^0, and condition (c) is exactly what is needed to define $I_{(c)}$ as a $(\mathfrak{g}, K)$-module.

One crucial property of the space $\mathrm{Ind}(\sigma|P, G))$ is that a form of Frobenius reciprocity holds for it also. Let Λ be the $\mathfrak{p}$-map from H to U taking φ to $\varphi(1)$. When restricted to $\mathrm{Ind}(\sigma|P, G)$, it is a $(\mathfrak{p}, K(P))$-morphism. For trivial reasons the $U(\mathfrak{g})$-module $\mathrm{Hom}_{U(\mathfrak{p})}(U(\mathfrak{g}), U)$ satisfies a simple version of Frobenius reciprocity, and directly from the construction above it follows that:

Lemma 4.3. *If (π, V) is a Harish-Chandra module over $(\mathfrak{g}, K)$, then composition with Λ induces a bijection:*

$$\mathrm{Hom}_{(\mathfrak{g},K)}(V, \mathrm{Ind}(\sigma|P, G)) \cong \mathrm{Hom}_{(\mathfrak{p},K(P))}(V, U).$$

If σ_* is an extension of σ to P, then when restricted to the subspace of K-finite vectors in $\mathrm{Ind}(\sigma_*|P, G)^K$ the map Λ is a $(\mathfrak{p}, K(P))$-morphism which therefore induces a canonical $(\mathfrak{g}, K)$-morphism into $\mathrm{Ind}(\sigma|P, G)$.

Proposition 4.4. *If σ_* is an extension to P of the Harish-Chandra module σ over $(\mathfrak{p}, K(P))$, then the canonical $(\mathfrak{g}, K)$-morphism from the subspace of K-finite functions in* $\mathrm{Ind}^{\mathrm{sm}}(\sigma_*|P, G)$ *to* $\mathrm{Ind}(\sigma|P, G)$ *is an isomorphism.*

The proof is a little technical (see [**5**, III.2] for details of a similar result). Basically, the conditions (a)–(c) defining the algebraically induced representation guarantee exactly that its restriction to K is isomorphic to the subspace of K-finite elements of $\mathrm{Ind}^{\mathrm{sm}}(\sigma_*|P, G)$.

The contragredient of $\mathrm{Ind}(\sigma|P,G)$ may be identified with $\mathrm{Ind}(\tilde{\sigma}\delta_P|P,G)$, by means of the canonical pairing into $\mathrm{Ind}(\delta_P|P,G)$, which may be identified with the space of one-densities on $P\backslash G$.

In the rest of this section I will recall the results and definitions leading up to Langlands' classification of irreducible Harish-Chandra modules. All my subsequent references will be to [**5**] since it is convenient, although the original results are due to Harish-Chandra, [**27**], and [**29**]. Fix throughout the discussion a minimal parabolic subgroup R.

Recall that for any parabolic subgroup $P, X_{\mathbf{R}}(A_P)$ is the real vector space of positive real characters of A_P, and that

$$X_{\mathbf{R}}^{-}(A_P) := \left\{\chi \in X_{\mathbf{R}}(A_P) \mid \chi = \prod\nolimits_{\alpha\in\Delta_P} \sigma^{\sigma_\alpha}, \sigma_\alpha > 0\right\}.$$

Fix for the rest of this section a minimal parabolic subgroup R. If (π, V) is a tempered Harish-Chandra module over $(\mathfrak{g}, K)$, then all the dominant asymptotic characters χ of π lie in the closure of $X_{\mathbf{R}}^{-}(A_R)$. Choose such a character χ such that the set

$$\Theta(\chi) = \{\alpha \mid \mathrm{Re}(s_\alpha) = 0\}$$

is maximal, where the s_α are defined by the formula $\chi = \prod \alpha^{s_\alpha}$, and let P be the parabolic subgroup containing R such that

$$A_P = \bigcap_{\mathrm{Re}(s_\alpha)=0} \mathrm{Ker}(\alpha).$$

Then the χ-primary component σ of $V/\mathfrak{n}_P V$ has the property that $\sigma\rho_P^{-1}$ is a square-integrable representation of M_P, so that by Frobenius reciprocity:

PROPOSITION 4.5. [**5** IV.3.5]. *If (π, V) is a tempered representation of $(\mathfrak{g}, K)$ then there exists a parabolic subgroup P of G and a square-integrable Harish-Chandra module σ of M_P such that π embeds as a summand of* $\mathrm{Ind}(\sigma\rho_P|P,G)$.

Continue to let P be a parabolic subgroup of G. Define $X_{\mathbf{C}}(P)$ to be the set of all unramified characters of P. It possesses a canonical structure as a complex vector space of dimension equal to the sum of the split rank of G and the dimension of A_P. Fix for the moment a tempered Harish-Chandra module (σ_0, U) of $(\mathfrak{m}_P, K(P))$. The integral

$$\Lambda_\sigma(f) = \int_{N^{\mathrm{opp}}} f(x)dx,$$

with values in U, converges for every f in the space $\mathrm{Ind}^{\mathrm{sm}}(\sigma\rho_P|P,G)$ with support in the P^{opp}-stable open subset PN^{opp} of G and defines a P^{opp}-covariant map from this subspace to the P^{opp}-representation $\sigma\rho_{\mathrm{opp}}$. Here P^{opp} is a parabolic subgroup opposite to P, so that σ may be identified with a module over

$(\mathfrak{p}^{\mathrm{opp}}, K(P^{\mathrm{opp}}))$ since the intersection of P and P^{opp} is isomorphic to the Levi components of both. For convenience I have written as earlier

$$\rho = \delta_P^{1/2}$$
$$\rho_{\mathrm{opp}} = \delta_{P^{\mathrm{opp}}}^{1/2}.$$

For any essentially tempered Harish-Chandra module over $(\mathfrak{m}, K(M))$ let

$$\langle\sigma\rangle := |\zeta(\sigma)| \,\Big|\, A_P.$$

It thus lies in $X_{\mathbf{R}}(A_P)$. If $\sigma = \sigma_0\chi$ with $\chi \in X_{\mathbf{C}}(P)$ then $\langle\sigma\rangle$ is the same as the restriction of $|\chi|$ to A_P. Define

$$X_{\mathbf{R}}^{--}(P) := \{\chi \in X_{\mathbf{R}}(A_P) \mid \chi = \prod\nolimits_{Q\supset P} \delta_Q^{s_Q}, \mathrm{Re}(s_Q) > 0\}.$$

It is well known (see [**24** VIII.10] for some of this) that:

PROPOSITION 4.6. *Let σ be a Harish-Chandra module over $(\mathfrak{m}, K(M))$ which is esentially tempered. Then*

(a) *If $\langle\sigma\rangle$ lies in $X_{\mathbf{R}}^{--}(P)$ then the integral defining Λ_σ converges for all f in* $\mathrm{Ind}(\sigma\rho|P, G)$, *defining by Frobenius reciprocity a G-map T_σ taking f to the function $\Lambda_\sigma(R_g f)$:*

$$\mathrm{Ind}(\sigma\rho|P, G) \longrightarrow \mathrm{Ind}(\sigma\rho_{\mathrm{opp}}|P^{\mathrm{opp}}, G).$$

(b) *The map T_σ is an analytic function of $\sigma = \sigma_0\chi$ in the sense that the composition of T_σ with φ_σ is an analytic function of χ when $|\chi|$ lies in $X_{\mathbf{R}}^{--}(P)$, and continues meromorphically to all of $X_{\mathbf{C}}(P)$.*

(c) *The map T_σ is generically an isomorphism.*

As a consequence of a simple formula for the asymptotic behaviour of matrix coefficients in certain directions at infinity:

PROPOSITION 4.7. [**5**, IV.4.5–6]. *Let σ be an essentially tempered representation of M with $\langle\sigma\rangle$ in $X_{\mathbf{R}}^{--}(P)$. Then*

(a) *The image of T_σ is the unique irreducible* $(\mathfrak{g}, K)$*-submodule of* $\mathrm{Ind}(\sigma\rho_{\mathrm{opp}}|P^{\mathrm{opp}}, G)$*;*

(b) *Equivalently, the kernel of T_σ is the unique proper maximal* $(\mathfrak{g}, K)$*-stable subspace of* $\mathrm{Ind}(\sigma\rho|P, G)$.

In these circumstances define

$$\pi(P, \sigma) := \text{this unique irreducible } (\mathfrak{g}, K)\text{-submodule of } \mathrm{Ind}(\sigma_{\mathrm{opp}}|P^{\mathrm{opp}}, G).$$

An analysis of the dominant asymptotic characters of a representation and the relationship with embeddings into representations induced from parabolic subgroups shows that every irreducible Harish-Chandra module can be embedded into some $\mathrm{Ind}(\sigma\rho_{\mathrm{opp}}|P^{\mathrm{opp}}, G)$ with $\langle\sigma\rangle \in X_{\mathbf{R}}^{--}(P)$, and in fact:

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PROPOSITION 4.8. *Every irreducible Harish-Chandra module is isomorphic to $\pi(P,\sigma)$ for some parabolic subgroup P and σ an irreducible essentially tempered Harish-Chandra module over $(\mathfrak{p}, K(P))$ with $\langle\sigma\rangle$ in $X_{\mathbf{R}}^{--}(P)$. The subgroup P and module σ are unique up to conjugation by an element of G.*

Any character χ in $X_{\mathbf{R}}(A_P)$ may be identified with an element of $X_{\mathbf{R}}(A_R)$ since the characters δ_Q for maximal proper Q containing P span $X_{\mathbf{R}}(A_P)$ and may be identified also as characters of A_R. The closure of $X_{\mathbf{R}}^{--}(R)$ is in particular the union of the images of the $X_{\mathbf{R}}^{--}(P)$ as P ranges over the parabolic subgroups containing R. Thus $\langle\sigma\rangle$ may also be considered as an element of $X_{\mathbf{R}}(A_R)$.

PROPOSITION 4.9. [**5**, IV.4.11–13]. *Suppose that σ is an essentially tempered Harish-Chandra module over $(\mathfrak{m}, K(M))$ with $\langle\sigma\rangle$ in $X_{\mathbf{R}}^{--}(P)$. If π is an irreducible constituent of the maximal proper $(\mathfrak{g}, K)$-stable subspace of* $\mathrm{Ind}(\sigma\rho|P,G)$, *with π isomorphic to $\pi(Q,\tau)$, then $\langle\tau\rangle < \langle\sigma\rangle$.*

It follows immediately from Proposition 4.7 that for σ essentially tempered with $\langle\sigma\rangle$ in $X_{\mathbf{R}}^{--}(P)$ the Harish-Chandra module $\mathrm{Ind}(\sigma\rho|P,G)$ is finitely generated. Something more precise will be needed later on. If σ is a character of R trivial on $K(R)$, the representation $\mathrm{Ind}(\sigma\rho|R,G)$ may be identified as a K-representation with the space of K-finite functions on $K(R)\backslash K$, so that the subspace of vectors fixed by K may be identified with the space of constants on K. The intertwining operator T_σ when it converges, acts therefore essentially as a scalar on the subspace of K-fixed vectors. This scalar can be calculated explicitly (see, for example [**37**]), and it turns out that that when $|\sigma|$ lies in $X_{\mathbf{R}}^{--}(A_R)$ it is not zero. Since Proposition 4.7(b) asserts that any element of $\mathrm{Ind}(\sigma\rho|R,G)$ not in the kernel of T_σ generates $\mathrm{Ind}(\sigma\rho|R,G)$, the K-fixed vectors in $V = \mathrm{Ind}(\sigma\rho|R,G)$ for $\langle\sigma\rangle$ in $X_{\mathbf{R}}^{--}(R)$ generate V as a $(\mathfrak{g}, K)$-module, which is a well known result of [**20**] and, in a stronger form, [**25**].

A weaker result than the following will do for the moment, but the strong version given will play a role further on:

LEMMA 4.10. *Every finite-dimensional (continuous) representation of R is a quotient of a sum of representations of the form*

$$\chi \otimes \omega \otimes E$$

where E is a finite-dimensional representation of G, $|\chi|$ lies in $X_{\mathbf{R}}^{--}(R)$, and ω is a finite-dimensional, indecomposable, unipotent representation of A.

This relatively elementary result is [**38**, 4.11].

As a consequence of this and the remark made just earlier:

PROPOSITION 4.11. *If P is a minimal parabolic subgroup and (σ, U) any finite-dimensional representation of P, then* $\mathrm{Ind}(\sigma|P,G)$ *is a Harish-Chandra module over $(\mathfrak{g}, K)$ of finite length.*

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Proof. This representation and its contragredient are both finitely generated, so that since $U(\mathfrak{g})$ is Noetherian it satisfies both the descending and ascending chain conditions.

5. Completions with respect to $\mathfrak{n}$. Let (π, V) be a Harish-Chandra module, P a parabolic subgroup. Define

$$\mathfrak{n}^k V := \text{the subspace of } V \text{ spanned by vectors of the form}$$
$$\pi(\nu_1)\dots\pi(\nu_k)v,$$

with the ν_i in $\mathfrak{n}$ and v in V. If V is finitely generated over $U(\mathfrak{g})$, then it is a relatively elementary result (perhaps first pointed out in [**13**, 2.3]) that V is also finitely generated over $U(\mathfrak{n})$ when P is minimal.

PROPOSITION 5.1. *If P is a parabolic subgroup of G and (π, V) a finitely generated Harish-Chandra module, then $V/\mathfrak{n}^k V$ is a Harish-Chandra module over $(\mathfrak{p}, K(P))$.*

Proof. If Q is a minimal parabolic subgroup contained in P then the image of Q modulo N_P is a minimal parabolic subgroup R of M_P, and from the remark above $V/\mathfrak{n}_P V$ is a Harish-Chandra module over $(\mathfrak{m}_P, K(M_P))$ finitely generated over $U(\mathfrak{b})$. But then $\mathfrak{n}^k V/\mathfrak{n}^{k+1} V$ is the surjective image of the tensor product of this and the finite-dimensional M_P-module $\mathfrak{n}^k/\mathfrak{n}^{k+1}$.

Now let P be minimal. Define

$$V_{[\mathfrak{n}]} := \text{the projective limit of the quotients} V/\mathfrak{n}^k V,$$

the completion of V with respect to the $\mathfrak{n}$-adic topology. Note that each $V/\mathfrak{n}^k V$ is finite-dimensional since V is finitely generated over $U(\mathfrak{n})$. If

$$\mathfrak{R} := \text{the completion of } U(\mathfrak{n}) \text{ with respect to powers of the augmentation ideal } \mathfrak{n}U(\mathfrak{n}),$$

then $V_{[\mathfrak{n}]}$ may also be identified with the tensor product $V \otimes_{U(\mathfrak{n})} \mathfrak{R}$. As explained in [**10**], there exists for each X in $U(\mathfrak{g})$ an integer $d > 0$ such that $X\mathfrak{n}^k V$ is contained in $\mathfrak{n}^{k-d} V$ for all $k \geqq 0$, so that although each $V/\mathfrak{n}^k V$ is only a $(\mathfrak{p}, K(P))$-module, the projective limit $V_{[\mathfrak{n}]}$ inherits from V the structure of a module over $\mathfrak{g}$. Incidentally, since each $V/\mathfrak{n}^k V$ is finite-dimensional, by Lemma 2.3 it possesses the structure of a P-module. The space $V_{[\mathfrak{n}]}$ is therefore simultaneously a module over $U(\mathfrak{g})$, P, and the completion $\mathfrak{R}$. I will not make explicit the compatibility conditions, but point out instead relations with a better known type of $\mathfrak{g}$-module. Define

$$\begin{aligned} \hat{V}^{[\mathfrak{n}]} &:= \mathfrak{n}\text{-adic dual of } V_{[\mathfrak{n}]} \\ &= \text{elements of } \hat{V} \text{ trivial on some } \mathfrak{n}^k V \\ &= \text{the union of the subspaces } \hat{V}[\mathfrak{n}^k] \\ &= \text{the subspace of } \mathfrak{n}\text{-torsion in } \hat{V}. \end{aligned}$$

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This is a $(\mathfrak{g}, P)$-module, annihilated by the ideal I^ι if V is annihilated by $I \subseteq Z(\mathfrak{g})$.

Note that each quotient $\mathfrak{n}^k V / \mathfrak{n}^{k+1} V$ is canonically a module over m (or even M). Recall that $\rho: Z(\mathfrak{g}) \longrightarrow Z(m)$ is the Harish-Chandra homomorphism.

LEMMA 5.2. *If V is a finitely generated Harish-Chandra module over $(\mathfrak{g}, K)$ annihilated by the ideal I of finite codimension in $Z(\mathfrak{g})$, then for k sufficiently large the $\rho(I)$-primary constituent of $\mathfrak{n}^k V / \mathfrak{n}^{k+1} V$ is null.*

Proof. Since $Z(m)$ is a finite module over $\rho(Z(\mathfrak{g}))$, the ideal of $Z(m)$ generated by $\rho(I)$ has finite codimension also. The lemma is therefore a consequence of these observations: (a) the space $\mathfrak{n}^k V / \mathfrak{n}^{k+1} V$ is the canonical image of the tensor product of $V / \mathfrak{n} V$ and the k-fold tensor product $T^k(\mathfrak{n})$ and (b) since the positive roots lie in a non-degenerate cone, the k-fold sums of positive roots pass off to infinity as k gets large.

LEMMA 5.3. *Any $(\mathfrak{g}, P)$-module X annihilated by an ideal I of finite codimension in $Z(\mathfrak{g})$ such that each subspace $X[\mathfrak{n}^k]$ has finite dimension is of finite length.*

Proof. Any $(\mathfrak{g}, P)$-subquotient Y has the property that $Y[\mathfrak{n}]$ is non-trivial and annihilated by $\rho(I)$. Apply Lemma 5.2.

PROPOSITION 5.4. *There exists a finite-dimensional P-space W such that the $(\mathfrak{g}, P)$-module $\hat{V}^{[\mathfrak{n}]}$ is a quotient of the Verma module $U(\mathfrak{g}) \otimes_{U(p)} W$.*

Proof. If W is a finite-dimensional P-stable subspace generating $\hat{V}^{[\mathfrak{n}]}$, then the canonical map from $U(\mathfrak{g}) \otimes_{U(p)} W$ into $\hat{V}^{[\mathfrak{n}]}$ is surjective.

If $f : U \longrightarrow V$ is a morphism of finitely generated Harish-Chandra modules, then induces for each k a map $U / \mathfrak{n}^k U \longrightarrow V / \mathfrak{n}^k V$ hence in the limit a $(\mathfrak{g}, P)$-map

$$f_{[\mathfrak{n}]} : U_{[\mathfrak{n}]} \longrightarrow V_{[\mathfrak{n}]}.$$

The assignment taking V to $V_{[\mathfrak{n}]}$ (V a finitely generated Harish-Chandra module) is therefore functorial, and similarly for $\hat{V}^{[\mathfrak{n}]}$ (which is, however, contravariant). Just as for completions of finitely generated modules over Noetherian commutative rings:

PROPOSITION 5.5. *The functors assigning $V_{[\mathfrak{n}]}$ and $\hat{V}^{[\mathfrak{n}]}$ to V are exact.*

As in the commutative case, this follows from the Artin-Rees Lemma (as proven, say, in greatest generality in [**28**]).

Much less trivial:

PROPOSITION 5.6. *If V is a non-trivial finitely generated Harish-Chandra module, then the canonical map from V into $V_{[\mathfrak{n}]}$ is an injection.*

If U is the kernel of the canonical map, then it is clearly $\mathfrak{g}$-stable, and since $K(P)$ meets all topological components of G it is in fact $(\mathfrak{g}, K)$-stable. By Proposition 5.6 $U_{[\mathfrak{n}]}$ injects into $V_{[\mathfrak{n}]}$ so in fact $U_{[\mathfrak{n}]}$ must be trivial. The result therefore reduces to the assertion that if V is a non-trivial Harish-Chandra module then $V_{[\mathfrak{n}]}$ is also non-trivial. By the $U(\mathfrak{n})$-version of Nakayama's Lemma, for this it will suffice to know only that $V/\mathfrak{n}V$ is non-trivial. There are now a number of proofs of this: the first argument can be found in [**12**], but perhaps the most elegant one is in [**3**].

COROLLARY 5.7. *If (π, V) is a finitely generated Harish-Chandra module then for k sufficiently large the canonical $(p, K(P))$-projection from V to $V/\mathfrak{n}^k V$ induces an embedding of V into* $\mathrm{Ind}(V/\mathfrak{n}^k V|P, G)$. *If (π, V) is irreducible then there exists a $(\mathfrak{g}, K)$-covariant embedding of π in* $\mathrm{Ind}(V/\mathfrak{n}V|P, G)$.

COROLLARY 5.8. *Any Harish-Chandra module has finite length.*

Proof. It is elementary (Lemma 5.3) that any Verma module has finite $\mathfrak{g}$-length. The module $V_{[\mathfrak{n}]}$ is the algebraic dual of $\hat{V}^{[\mathfrak{n}]}$. Subspaces of $V_{[\mathfrak{n}]}$ which are closed in the trivial topology correspond bijectively to subspaces of $\hat{V}^{[\mathfrak{n}]}$. Therefore if V is finitely generated the Corollary follows directly from Proposition 5.5.

Since for any ideal I of finite codimension in $Z(\mathfrak{g})$ the ideal of $Z(\mathfrak{m})$ generated by $\rho(I)$ has finite codimension in $Z(\mathfrak{m})$, so that for a given ideal I only finitely many induced representations are annihilated by I. Therefore Corollary 5.7 implies that any increasing chain of finitely generated Harish-Chandra modules must be stationary.

The proof of this sketched here is rather intricate when laid out in detail, and Harish-Chandra's argument in terms of possible characters has by comparison a good deal to be said for it.

COROLLARY 5.9. *If V is finitely generated Harish-Chandra module then its contragredient is also finitely generated.*

COROLLARY 5.10. *If U is a finite-dimensional subspace of the Harish-Chandra module (π, V) generating V over $(\mathfrak{g}, K)$ then every v in V can be expressed as a sum $\sum \pi(f_i)u_i$ with the u_i in U and f a function in $C_c^\infty(G)$ which is K-finite on right and left.*

6. The group $SL_2(\mathbf{R})$. Some simple examples with the group $G = SL_2(\mathbf{R})$ illustrate nicely some of the problems this paper is concerned with. Choose for the maximal compact subgroup SO_2. The group G acts by projective transformations on the upper half plane in $\mathbf{C}$, and through conjugation by the matrix

$$\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

on both the open unit disc $\mathbf{D}$ in $\mathbf{C}$ and its boundary $\mathbf{S}$. The group K may then be identified with rotations around the origin. The disc may therefore be identified with G/K, and as algebraic norm on G one may use the right-K-invariant norm induced from the norm $(1 - r^2)^{-1}$ on the disc. The boundary is the projective real line, and may be identified with G/P where P is the parabolic subgroup of upper triangular matrices, the stabilizer of 1.

I am going to be concerned with several different representations of both G and $(\mathfrak{g}, K)$ arising naturally in this context. To start, let (π, V) be the representation of $(\mathfrak{g}, K)$ on the space of K-finite functions on $\mathbf{S}$ and (σ, U) that on the space of K-finite one-forms on $\mathbf{S}$. The first may be extended to several continuous representations of G, on the spaces:

$$
\begin{aligned}
V^{\omega} &:= \text{real analytic functions on } \mathbf{S} \\
V^{\infty} &:= \text{smooth functions on } \mathbf{S} \\
V^{-\infty} &:= \text{distributions on } \mathbf{S} \\
V^{-\omega} &:= \text{hyperfunctions on } \mathbf{S}
\end{aligned}
$$

Of course we have inclusions

$$V^{\omega} \subseteq V^{\infty} \subseteq V^{-\infty} \subseteq V^{-\omega}.$$

Similarly the second may be extended successively to analytic one-forms U^{ω}, smooth one-forms U^{∞}, one-currents $U^{-\infty}$, and hyperforms $U^{-\omega}$. Integration of the product gives a canonical duality between the two representations of $(\mathfrak{g}, K)$, so that one is the contragredient of the other. Incidentally, in between the analytic functions and the smooth ones lie the Gevrey classes, which are also interesting (see [**21**, Sections 1.3, 8.4] for an elementary account and [**17**] for use in representation theory).

Let ϵ be this character of K:

$$\begin{pmatrix} c & -s \\ s & c \end{pmatrix} \longmapsto c + is.$$

Thus K acts on the tangent space at 0 in $\mathbf{D}$ according to ϵ^{-2}, the function z on $\mathbf{C}$ is an eigenfunction with character ϵ^2, and the one-form dz on $\mathbf{C}$ is an eigenform also with character ϵ^2. The restrictions to $\mathbf{S}$ of the powers of z form a basis v_n of V, and similarly the restrictions to $\mathbf{S}$ of the one-forms $u_n = z^{n-2}dz$ form a basis of U. In other words, the space V is made up of sums

$$\sum_{n \in \mathbf{Z}} c_n v_n \tag{6.1}$$

where all but a finite number of coefficients are non-zero, and similarly for U. The elements of the extensions of V (or U) as well as their topologies may also be characterized in terms of the Fourier series expansions (6.1):

(1) The elements of V^{ω} are those infinite sums (6.1) with

$$c_n = O(r^{-|n|})$$

for some $r > 1$. The space V^{ω} is as a topological vector space the inductive limit [**36**, Exercises 13.6, 13.7 on p. 134 and Remark (ii) on p. 514] of the spaces of such sums for a fixed r, or equivalently the inductive limit of the spaces of functions holomorphic on annuli around **S**.

(2) Those of V^{∞} are such that with

$$c_n = O(|n|^{-k})$$

for all positive integers k, which is thus a Fréchet space with norms $\|\varphi\|_k := \sup_n |c_n n^k|$.

(3) Those of $V^{-\infty}$ are such that

$$c_n = O(|n|^k)$$

for some positive integer k.

(4) The space of hyperfunctions is by definition the topological dual of that of analytic one-forms, so that according to [**36**, Exercise 13.6, p. 134] those of $V^{-\omega}$ are such that

$$c_n = O(r^{|n|})$$

for all $r > 1$. This also is a Fréchet space, with norms

$$\|F\|_r := \sup_n |c_n| r^{|n|},$$

where now $r < 1$.

All of these topological spaces are nuclear [**36**, Chapter 50], in particular reflexive. Similar remarks apply to one-forms as well. The representation of G on any of these spaces is smooth.

Any function $v = \sum c_n v_n$ in V extends uniquely to a harmonic function defined on all of **C**: it is the restriction to **S** of the function

$$F_{\varphi} = \sum c_n f_n \tag{6.2}$$

where now f_n is the monomial z^n if n is non-negative but $\bar{z}^{|n|}$, the conjugate of z^{-n}, if n is negative. Restriction to **S** in fact identifies V with the space of all K-finite harmonic functions on **D**. This identification extends to the spaces of the extensions to G:

PROPOSITION 6.1. *The map taking φ in V to F_{φ} extends to a unique continuous $(\mathfrak{g}, K)$-map from $V^{-\omega}$ to the space of all harmonic functions on* **D**. *Furthermore, it induces continuous isomorphisms of:*

(a) *V^{ω} with the space of all functions harmonic in a neighborhood of* $\overline{\mathbf{D}}$;

(b) *V^{∞} with the space of all smooth harmonic functions on* $\overline{\mathbf{D}}$;

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(c) $V^{-\infty}$ *with the space of all harmonic functions on* $\mathbf{D}$ *of moderate growth (with respect to the norm* $(1-r^2)^{-1}$).

These results are classical, but it may be instructive if I recall the proofs. The first assertion is straightforward, since if $\sup_n |c_n|r^{|n|} < \infty$ then the series $\sum c_n f_n$ clearly converges on the disc $|z| < r$. The space of all harmonic functions on $\mathbf{D}$ is given the topology determined by the norms

$$\|F\|_r := \sup_{|z|<r} |F(z)|$$

where $r < 1$. The map taking φ to F_φ is continuous since for $r < \rho < 1$.

$$|F(z)| \leqq \sum |c_n| r^{|n|} \leqq \frac{2}{1-(r/\rho)}\|\varphi\|.$$

Conversely, a similar estimate shows that any harmonic function on $\mathbf{D}$ is the image of some φ.

The arguments for the remaining assertions are not very different.

Incidentally, by means of this isomorphism it is easy to see that the representation of G on the Fréchet space $V^{-\omega}$ is not of moderate growth: if

$$F(z) = e^{-\frac{1}{(1-z)}}$$

then

$$\|\pi(g)F\|_r \sim e^{\frac{1-r}{1+r}}/2a^2$$

where

$$g = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

and a tends to 0.

The disc $\mathbf{D}$ is isomorphic to the quotient G/K, and the extension of φ to F_φ amounts to associating to φ its matrix coefficient $\langle \pi(g^{-1})\varphi, u_0\rangle$, as is shown by the classical Poisson (or Cauchy) integral formula. Now if (π_*, V_*) is any smooth representation of G extending π and v lies in V_* then the matrix coefficient $\langle \pi_*(g^{-1})v, u_0\rangle$ may be identified with a harmonic functon on $\mathbf{D}$. Therefore:

PROPOSITION 6.2. *The representation* $\pi^{-\omega}$ *of* G *on* $V^{-\omega}$ *is canonically the largest smooth* G*-extension of the Harish-Chandra module* π.

The representation σ^∞ of G on U^∞ is of moderate growth, and $\pi^{-\infty}$ is its dual. In fact, if σ_* is any Harish-Chandra representation extending σ then for any continuous linear functional λ on U_* the matrix coefficient $\langle \lambda, \sigma(g)u_0\rangle$ will be a harmonic function on $\mathbf{D}$ of moderate growth. In other words the dual of the Fréchet space U_* embeds canonically into $V^{-\infty}$. This is a continuous embedding. If U were reflexive, this would imply that U^∞ embeds into U_*, which would mean that U^∞ would be (canonically) the smallest Harish-Chandra representation of G. A variation of this argument will prove this minimality: the point is, as will be used later on, that there is always a minimal Harish-Chandra extension of any Harish-Chandra module, defined in the following manner. If σ_* is any Harish-Chandra representation of G extending a given Harish-Chandra module σ, then, as we have seen already, it is in a natural way a module over $\mathcal{S}(G)$. If W is a finite-dimensional K-stable space generating U over $(\mathfrak{g}, K)$ then integration defines a continuous map $\mathcal{S}(G) \otimes W \longrightarrow U_*$. The kernel is closed, and the image contains all of U itself (according to 5.10). The image, assigned the quotient topology, will be the minimal Harish-Chandra extension of σ, and since $\mathcal{S}(G)$ is nuclear so is this quotient, hence reflexive. The argument above then shows:

PROPOSITION 6.3. *The representation σ^∞ is canonically the smallest Harish-Chandra representation of G extending the Harish-Chandra module σ.*

Incidentally, a similar argument using $C_c^\infty(G)$ instead of the Schwartz space, together with Proposition 6.2, shows that σ^ω is canonically the smallest smooth extension of σ to G, and equal to the image of $C_c^\infty(G) \otimes W$. This is the simplest case of the result of Schmid mentioned in the Introduction, whereas Proposition 6.3 is the simplest case of one of the main results of this paper. It is not so easy to prove that π^∞ is likewise the smallest Harish-Chandra extension of π. One possible argument uses some relatively intricate analysis realizing U as one-forms on $\mathbf{D}$. However, as compensation:

PROPOSITION 6.4. *The Harish-Chandra representation π^∞ is the largest Harish-Chandra extension of π.*

The proof goes like this: if π_* is a Harish-Chandra extension then for every v in V_* the matrix coefficient $F_v(g) = \langle \pi(g^{-1})v, u_0 \rangle$ lies at least in $V^{-\infty}$, so that we have an identification of V with a subspace of the space of distributions on $\mathbf{S}$. According to [**15**] the vector v can be represented as a sum $\sum \pi(f_i)v_i$ with each v_i in V, f_i in $C_c^\infty(G)$. But the v_i can be identified with distributions on $\mathbf{S}$, so the $\pi(f_i)v_i$ are smooth functions on $\mathbf{S}$, and therefore v lies in V^∞.

The upshot of these arguments is that explicit realizations of smooth principal series and associated spaces lead to strong characterizations of these principal series. In the case of $SL_2(\mathbf{R})$ one can use various accidents to carry out these arguments, and it is conceivable that arguments close to the ones given above might work for more general groups. Of course it is known already (from, say, [**23**] and earlier results of Helgason and others) that some of these realizations can be carried out more generally. Nonetheless, the arguments given in the rest of this paper will be somewhat different.

There is another phenomenon occurring here which is important more generally. On the disc the G-invariant non-Euclidean metric is

$$(1 - r^2)^{-2}(dx^2 + dy^2)$$

and the corresponding Laplacian

$$\Delta_{\text{non-Euclidean}} = (1 - r^2)^2 \Delta_{\text{Euclidean}}.$$

Expressing the Euclidean Laplacian in radial coordinates, one can see explicitly that the singularity at the origin which arises in radial coordinates is regular. The nature of this singularity is implicit in the assertion that every function on the disc which is harmonic and K-finite is the restriction of a harmonic polynomial. In [**23**] much use is made of the explicit form of the Laplacian at infinity on G/K, and it is again conceivable that a more detailed analysis of differential equations associated to elements of $Z(\mathfrak{g})$ could replace arguments from this paper. But instead I shall use a relatively elementary idea due to Wallach, incorporating simple facts about solutions to ordinary differential equations with constant coefficients.

7. Functorial G-extensions associated to principal series embeddings. From Lemma 5.2 it follows directly that for $k \leqq l$ sufficiently large the canonical G-surjection

$$\operatorname{Ind}^{\text{sm}}(V/\mathfrak{n}^l V|P, G) \longrightarrow \operatorname{Ind}^{\text{sm}}(V/\mathfrak{n}^k V|P, G)$$

induces an isomorphism of their I-primary constituents. Hence:

PROPOSITION 7.1. (Wallach). *For $k \leqq l$ sufficiently large the canonical surjection*

$$\operatorname{Ind}^{\text{sm}}(V/\mathfrak{n}^l V|P, G) \longrightarrow \operatorname{Ind}^{\text{sm}}(V/\mathfrak{n}^k V|P, G)$$

induces an isomorphism between the closures of the canonical copies of V in each.

In other words, it makes sense to define the smooth representation $\overline{\pi}$, which I shall call the large G-extension associated to principal series embeddings, to be that of G on

$$\overline{V} := \text{closure of the canonical image of } V \text{ in } \operatorname{Ind}^{\text{sm}}(V/\mathfrak{n}^k V|P, G) \text{ for } k \text{ large.}$$

PROPOSITION 7.2. *The assignment taking (π, V) to $(\overline{\pi}, \overline{V})$ is functorial and left exact.*

Proof. Let U and V be two Harish-Chandra modules. A $(\mathfrak{g}, K)$-map $f: U \longrightarrow V$ induces a P-map from $U/\mathfrak{n}^k U$ to $V/\mathfrak{n}^k V$, hence a continuous G-map from $\operatorname{Ind}^{\text{sm}}(U/\mathfrak{n}^k U|P, G)$ to $\operatorname{Ind}^{\text{sm}}(V/\mathfrak{n}^k V|P, G)$ such that the following diagram commutes:

$$\begin{array}{ccc}
U & \longrightarrow & V \\
\downarrow & & \downarrow \\
\operatorname{Ind}^{\text{sm}}(U/\mathfrak{n}^k U|P, G) & \longrightarrow & \operatorname{Ind}^{\text{sm}}(V/\mathfrak{n}^k V|P, G)
\end{array}$$

For k sufficiently large this clearly induces a map from $\overline{U}$ to $\overline{V}$, thus giving functoriality.

Given an exact sequence

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

we get also an exact sequence

$$0 \longrightarrow U/U \cap \mathfrak{n}^k V \longrightarrow V/\mathfrak{n}^k V \longrightarrow W/\mathfrak{n}^k W \longrightarrow 0.$$

Since all these spaces are finite-dimensional, of course this sequence splits linearly. Therefore the corresponding sequence of smoothly induced G-representations splits linearly (and continuously) as well, and in particular the image of $\mathrm{Ind}^{\mathrm{sm}}(U/U \cap \mathfrak{n}^k V|P, G)$ in $\mathrm{Ind}^{\mathrm{sm}}(V/\mathfrak{n}^k V|P, G)$ is closed. Now by the Artin-Rees Lemma again the subspaces $U \cap \mathfrak{n}^k V$ are cofinal with the $\mathfrak{n}^k U$ for large k, so this gives also a diagram

$$\overline{U} \longrightarrow \overline{V} \longrightarrow \overline{W}.$$

Injectivity of the first map is clear. Exactness in the middle means that the image of $\overline{U}$ in $\overline{V}$ is the same as the intersection of $\overline{V}$ with $\mathrm{Ind}^{\mathrm{sm}}(U/U \cap \mathfrak{n}^k V|P, G)$ in $\mathrm{Ind}^{\mathrm{sm}}(V/\mathfrak{n}^k V|P, G)$, or that the image of $\overline{U}$ in $\overline{V}$ is closed.

Corollary 7.3. *If $U \hookrightarrow V$ is an injection of Harish-Chandra modules, then $\overline{U}$ is a topological linear summand of V.*

Proof. This follows immediately from the result just proven together with the observation that any closed K-stable subspace of $\mathrm{Ind}^{\mathrm{sm}}(\sigma|P, G)$ with σ finite-dimensional possesses a closed, K-stable, linear complement (the orthogonal complement with respect to the usual K-invariant Hilbert norm on $\mathrm{Ind}^{\mathrm{sm}}(\sigma)$).

If (π, V) is a finitely generated Harish-Chandra module and (π_*, V_*) a smooth extension to G, I shall call π_* a *Frobenius extension* if the continuous G-morphisms from V_* into the representations smoothly induced from a minimal parabolic subgroup are the same as those from (π, V). More precisely, it is required that the inclusion of V in V_* induces an isomorphism of each $V/\mathfrak{n}^k V$ with the quotient of V_* by the closure of $\mathfrak{n}^k V_*$, or in other words it is required that $V_{[\mathfrak{n}]}$ be isomorphic to the topological $\mathfrak{n}$-adic completion of V_*, the projective limit $V^{\mathrm{sm}}_{[\mathfrak{n}],\mathrm{top}}$ of the quotients of V_* by the closures of the $\mathfrak{n}^k V_*$. If V_* is such an extension then, by Frobenius reciprocity, for each k the canonical map

$$V \longrightarrow \mathrm{Ind}(V/\mathfrak{n}^k V|P, G)$$

extends (uniquely) to a continuous map

$$V_* \longrightarrow \mathrm{Ind}^{\mathrm{sm}}(V/\mathfrak{n}^k V|P, G),$$

and hence V_* has a canonical image in $\overline{V}$. Furthermore:

PROPOSITION 7.4. *The representation* $(\overline{\pi}, \overline{V})$ *is itself a Frobenius extension of* (π, V).

This follows immediately from the definition. In other words, in a precise sense:

COROLLARY 7.5. *The smooth extension* $(\overline{\pi}, \overline{V})$ *is the largest Frobenius extension of* (π, V).

The following is a fundamental result:

THEOREM 7.6. *If* σ *is a finite-dimensional representation of the minimal parabolic subgroup* P *then the representation* $\mathrm{Ind}^{\mathrm{sm}}(\sigma|P, G)$ *is a Frobenius extension of* $\mathrm{Ind}(\sigma|P, G)$.

This seems to be highly non-trivial. There are several proofs of this known, but the only one written down, apparently, is that in [**38**], about which I shall say something in a moment. Another way of expressing this result is this:

COROLLARY 7.7. *If* σ *and* τ *are two finite-dimensional* $(\mathfrak{p}, K(P))$*-modules, then every algebraic* $(\mathfrak{g}, K)$*-morphism from* $\mathrm{Ind}(\sigma|P, G)$ *to* $\mathrm{Ind}(\tau|P, G)$ *extends to a continuous one from* $\mathrm{Ind}^{\mathrm{sm}}(\sigma|P, G)$ *to* $\mathrm{Ind}^{\mathrm{sm}}(\tau|P, G)$.

I am not sure to whom the first proof of this is to be credited. It had been conjectured in the course of correspondence between myself and M. S. Osborne as early as 1975, but I did not find a proof until the spring of 1978, at the beginning of the collaboration between myself and Noland Wallach. The proof found then is rather close to the one presented in [**38**]. Independently of this development, it was proven in the special case where σ and τ are $\mathfrak{n}$-trivial by Kashiwara (who has not, as far as I know, published his result). His proof (explained to me rapidly in conversation several years ago) begins with an observation of C. Rader (thus anticipating W. Schmid) that the given $(\mathfrak{g}, K)$-morphism could be extended to a map between analytic vectors. By Frobenius reciprocity this morphism therefore amounts to an $\mathfrak{n}$-invariant, P-covariant hyperfunction on G. The condition on this hyperfunction amounts to asserting that it is the solution to a system of differential equations on G which Kashiwara was able to show regular and holonomic, and he concludes with a general result that hyperfunction solutions of such systems are in fact distributions. This argument seems to me very natural. It puts the result in an illuminating context, whereas the proofs found by Wallach and myself are rather technical. I imagine that many of the similarly technical proofs of other results in this paper will eventually possess elegant replacements.

I outline the proof ot the Proposition found in [**38**]. It must be shown that if V is $\mathrm{Ind}(\sigma|P, G)$ then the injection of V into $V^{\mathrm{sm}} = \mathrm{Ind}^{\mathrm{sm}}(\sigma|P, G)$ induces an isomorphism of $V_{[\mathfrak{n}]}$ with $V^{\mathrm{sm}}_{[\mathfrak{n}],\mathrm{top}}$. Since V is dense in V^{sm}, it is clearly a surjection, so it remains only to show that it is an injection. There are two steps in the proof of this. Step (1) According to Lemma 4.10, the P-representation σ may be expressed as a quotient of a sum ρ of P-representation $\chi \otimes E$, where E is a finite-dimensional representation of G and χ is the tensor product of an unramified character in $X_{\mathbf{R}}^{--}(A_P)$ and an indecomposable nilpotent representation. Let τ be the kernel. Thus we have an exact sequence of finite-dimensional representations of P:

$$0 \longrightarrow \tau \longrightarrow \rho \longrightarrow \sigma \longrightarrow 0$$

which induces two exact sequences in a diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I(\tau) & \longrightarrow & I(\rho) & \longrightarrow & V & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^{\mathrm{sm}}(\tau) & \longrightarrow & I^{\mathrm{sm}}(\rho) & \longrightarrow & V^{\mathrm{sm}} & \longrightarrow & 0 \end{array}$$

where $I(\tau)$ means $\operatorname{Ind}(\tau|P,G)$ etc. Consider the corresponding diagram of $\mathfrak{n}$-completions in the top row and topological $\mathfrak{n}$-completions in the bottom row. It is not exact, but it is sufficiently exact that a simple diagram chase shows that in order for the canonical map from $V_{[\mathfrak{n}]}$ to $V^{\mathrm{sm}}_{[\mathfrak{n}],\mathrm{top}}$ to be an injection, it suffices for the associated map for $\operatorname{Ind}(\rho)$ to be one. But on the one hand $\operatorname{Ind}(\xi \otimes E)$ is canonically isomorphic to $\operatorname{Ind}(\xi) \otimes E$, and on the other $\mathfrak{n}$-completion commutes with tensoring by finite-dimensional representations. Hence, because of the stucture of $I(\rho)$, it suffices now to prove that the result is true when σ is of the form $\chi \otimes \omega$ with $|\chi|$ in $X_{\mathbf{R}}^{--}(A_P)$ and ω indecomposable, nilpotent. An easy argument on filtrations reduces this in turn to the case $\sigma = \chi$. Thus the end of the first step is that it remains only to show that the result is true for unramified principal series in the positive cone. Step (2). As I have recalled in Section 5, if $|\chi|$ lies in $X_{\mathbf{R}}^{--}(A_P)$ then the one-dimensional space of vectors in $V = \operatorname{Ind}(\chi|P,G)$ fixed by K generates V. This means that the canonical map from $U = U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \mathbf{C}$ to V is surjective, and in fact the finer formulation of [**20**] and [**25**] asserts that if I is the maximal ideal of $Z(\mathfrak{g})$ annihilating V then the map induced by this one on the quotient U/IU is an isomorphism (for this, the theory of harmonic polynomials on $\mathfrak{g}/\mathfrak{k}$ is used to show that the restrictions to K are the same). But it is relatively simple (see [**38**, 4.1]) to see that U/IU is free over $U(\mathfrak{n})$ of rank equal to the cardinality of the real Weyl group, so that V must be free over $U(\mathfrak{n})$ as well. This is reasonable since one knows already for V the existence of intertwining operators into other principal series defined by convergent integrals and parametized by $W_{\mathbf{R}}$. For one conclusion to the argument from this point on I refer to [**38**, 4.8 ff.]. Another possibility would be to show that the topological completion of $\operatorname{Ind}^{\mathrm{sm}}(\chi|P,G)$ is free of rank equal to the cardinality of $W_{\mathbf{R}}$ over the completed ring $\mathfrak{R}$. This amounts to a refinement of Bruhat's thesis which I hope to write up soon. As I have already remarked several times, this proof is rather elaborate. But I believe that it actually gives a result perhaps stronger than Theorem 7.6: it must imply that, at least in the negative Weyl chamber of characters, all $\mathfrak{n}$-nilpotent functionals are obtained from the standard ones by the two processes of applying elements of $U(\mathfrak{g})$ to them and also taking derivatives in the parameter s.

Yet another formulation of Theorem 7.6:

COROLLARY 7.8. *If* (σ, U) *is a finite-dimensional representation of* P *and* π *is the induced representaton* $\mathrm{Ind}(\sigma|P, G)$ *then* $\overline{\pi}$ *may be identified with* $\mathrm{Ind}^{\mathrm{sm}}(\sigma|P, G)$.

Proof. Here is an argument I find curious. Consider, for k large, this diagram:

$$\begin{tikzcd} & V = \mathrm{Ind}(\sigma|P,G) \arrow[dl, "\iota_k"'] \arrow[dr, "\text{Identity}"] & \\ \mathrm{Ind}(V/\mathfrak{n}^k V|P,G) \arrow[rr, "\Pi"] & & \mathrm{Ind}(\sigma|P,G) \end{tikzcd}$$

where the lower map Π is induced by Frobenius reciprocity from the canonical projection from V to U taking f to $f(1)$. This diagram commutes, and thus V is a summand of each $\mathrm{Ind}(V/\mathfrak{n}^k V|P, G)$. According to Corollary 7.7, the algebraic map ι_k extends continuously to one of the smooth induced representations, and the horizontal ones does also for trivial reasons, $\mathrm{Ind}^{\mathrm{sm}}(\sigma|P, G)$ becomes a continuous summand of $\mathrm{Ind}^{\mathrm{sm}}(V/\mathfrak{n}^k V|P, G)$.

COROLLARY 7.9. *If* (π, V) *is any Harish-Chandra module embedded in some* $\mathrm{Ind}(\sigma|P, G)$, *then the closure of* V *in* $\mathrm{Ind}^{\mathrm{sm}}(\sigma|P, G)$ *is isomorphic to* $\overline{V}$.

This follows from left-exactness and Corollary 7.8.

As observed in Corollary 5.9, the contragredient of a Harish-Chandra module is also finitely generated. If one embeds $\tilde{V}$ in the representation $\tilde{I}_k = \mathrm{Ind}(\tilde{V}/\mathfrak{n}^k\tilde{V}|P, G)$ for large k, then dually we may express V as a quotient of the contragredient I_k of $\tilde{I}_k$. Define the *small principal series completion* $\underline{V}$ of V to be the topological quotient of the corresponding smooth induced representation I_k^{sm} by the closure of the kernel of the map from I_k to V. Since the assignment taking V to $\overline{V}$ is a contravariant functor, the assignment taking V to $\underline{V}$ is easily seen also to be functorial (covariant). If V is represented as a quotient of some $\mathrm{Ind}(\sigma \mid P, G)$ and V is embedded into some $\mathrm{Ind}(\tau \mid P, G)$, then the algebraic $(\mathfrak{g}, K)$-composition

$$\mathrm{Ind}(\sigma \mid P, G) \longrightarrow V \longrightarrow \mathrm{Ind}(\tau \mid P, G)$$

extends continuously to a smooth map between the associated smooth induced representations by Corollary 7.7, and this induces a smooth injection which I shall call $\kappa_V : \underline{V} \hookrightarrow \overline{V}$. In other words, $\underline{V}$ is a Frobenius extension. Since V is dense in $\underline{V}$, the injection κ_V is the unique continuous extension from $\underline{V}$ to $\overline{V}$ of

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the identity on V itself, hence is canonical and in particular does not depend on the choice of σ or τ. I recall that in the Introduction I defined V to be a *regular* Harish-Chandra module if κ_V is a surjection or, equivalently (since both spaces are Fréchet), a topological isomorphism. I recall also that eventually it will be shown that all Harish-Chandra modules are regular.

PROPOSITION 7.10. *If σ is any finite-dimensional $(\mathfrak{p}, K(P))$-module and* $V = \operatorname{Ind}(\sigma|P, G)$ *then* $\underline{V} = \operatorname{Ind}^{\mathrm{sm}}(\sigma|P, G)$. *In other words* $\operatorname{Ind}(\sigma|P, G)$ *is regular.*

The proof of this claim is almost exactly the same as that of Corollary 7.8.

The functor taking V to $\underline{V}$ is not obviously either left or right exact. The following elementary result shows that its exactness is intimately related to that of the earlier functor taking V to $\overline{V}$ (which, I recall, is left exact):

LEMMA 7.11. *Suppose given a commutative diagram*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & Y_1 & \hookrightarrow & Y_2 & \longrightarrow & Y_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where all vertical sequences, the top row, and the middle row are exact. Then $Y_1 \longrightarrow Y_2$ *is injective,* $Y_2 \longrightarrow Y_3$ *is surjective, and*

$$\frac{\operatorname{Ker}(Y_2 \longrightarrow Y_3)}{\operatorname{Im}(Y_1 \longrightarrow Y_2)} \cong \frac{A_3}{\operatorname{Im}(A_2)}.$$

I leave this as an exercise. Taking the middle row to be representations induced from P, one sees first of all:

PROPOSITION 7.12. *If* $f : U \longrightarrow V$ *is a* $(\mathfrak{g}, K)$*-map of Harish-Chandra modules and* $\underline{f}$: $\underline{U} \longrightarrow \underline{V}$ *is the map induced by it then (a)* $\underline{f}$ *is injective if* f *is injective; (b)* $\underline{f}$ *is surjective if* f *is surjective.*

But one also sees that the functor taking V to $\underline{V}$ is exact if and only if that taking V to $\overline{V}$ is exact, and indeed the extent to which one functor is not exact measures the non-exactness of the other. This observation, I am afraid, does not seem to be other than a curiosity.

Corollary 7.13. *If σ is finite-dimensional and* $\mathrm{Ind}(\sigma|P,G) \longrightarrow V$ *is a surjection of Harish-Chandra modules, then* $\underline{V}$ *is isomorphic to the corresponding quotient of the induced representation* $\mathrm{Ind}^{\mathrm{sm}}(\sigma|P,G)$.

If U and V are submodules of a third Harish-Chandra module X, their sum $U+V$ in X is of course again a Harish-Chandra module, and there exist canonical embeddings of $\underline{U}$ and $\underline{V}$ in $\underline{U+V}$.

The following is proven by a straightforward chase around this diagram:

$$\begin{array}{ccccccccc} & & \underline{U} & \hookrightarrow & \underline{V} & \longrightarrow & \underline{W} & \longrightarrow & 0 \\ & & \Downarrow & & \downarrow & & \Downarrow & & \\ 0 & \longrightarrow & \overline{U} & \longrightarrow & \overline{V} & \longrightarrow & \overline{W} & & \end{array}$$

Lemma 7.15. *Let*

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

be an exact sequence of Harish-Chandra modules. If U and W are regular then so is V, and the sequence

$$0 \longrightarrow \overline{U} \longrightarrow \overline{V} \longrightarrow \overline{W} \longrightarrow 0$$

is exact and possesses a continuous linear splitting.

Consequently:

Lemma 7.16. *A finite direct sum is regular if and only if each summand is. If V possesses a filtration of finite length by Harish-Chandra modules such that the associated graded modules are regular, then V is regular.*

8. Extensions associated to matrix coefficients. This section is devoted to several results due almost entirely to Wallach, whose methods I shall more or less follow. I hope to shorten and clarify his proofs by some slight reorganization.

Generally, in this section

$$\begin{aligned} (\pi, V_*) &= \text{ a Harish-Chandra representation of } G \\ V &= \text{ the subspace of } K\text{-finite vectors in } V_* \end{aligned}$$

Lemma 8.1. [**38**, 5.4]. *There exists $n > 0$, a norm ρ on V_*, and for each $\tilde{v} \in \tilde{V}$ a constant $C = C_{\tilde{v}}$ with*

$$|\langle \pi(g)v, \tilde{v}\rangle| \leqq C\|g\|^n \|v\|_\rho$$

for all $v \in V_*$.

Proof. By Corollary 5.10 there exist elements $\tilde{v}_1, \ldots, \tilde{v}_n$ of $\tilde{V}$ such that any $\tilde{v}$ in $\tilde{V}$ may be represented as $\sum \tilde{\pi}(f_i)\tilde{v}_i$ with the f_i in $C_c^\infty(G)$. Since the $\tilde{v}_i$ are continuous, one can certainly find a semi-norm ν such that for each of the generators $\tilde{v}$ there exists C with $|\langle v, \tilde{v}\rangle| \leqq C\|v\|_\nu$ for all v in V_*. Since

$$\langle v, \tilde{\pi}(f)\tilde{v}\rangle = \int f(h)\langle \pi(h^{-1})v, \tilde{v}\rangle dh,$$

we have the inequality

$$|\langle v, \tilde{\pi}(f)\tilde{v}\rangle| \leqq C \sup_{h\in \mathrm{Supp}(f)} \|\pi(h^{-1})v\|_\nu \left(\int |f(h)|dh\right).$$

Hence for any $\tilde{v} \in \tilde{V}$ there exists C with $|\langle v, \tilde{v}\rangle| \leqq C\|v\|_\nu$. But since π is of moderate growth, there exist constants C and n and a norm ρ such that

$$\|\pi(g)v\|_\nu \leqq C\|g\|^n\|v\|_\rho$$

for all v in V_*.

For the rest of this section, fix the semi-norm ρ according to Lemma 8.1. This semi-norm is actually a norm, which I shall write simply as $\|v\|$. Fix also a norm on $\tilde{V}$ by the formula

$$\|\tilde{v}\| := \sup_{g\in G, v\in V} \frac{|\langle \pi(g)v, \tilde{v}\rangle|}{\|g\|^n\|v\|}.$$

This definition is of course somewhat arbitrary since it depends on the choice of ρ, but then it will serve only a rather technical purpose. One of its convenient features is that if ρ is K-invariant on V, as we may assume, then this norm on $\tilde{V}$ will be K-invariant also.

The norm on $\tilde{V}$ is defined precisely so that the following is just a reformulation of Lemma above:

PROPOSITION 8.2. *There exists $n > 0$ such that*

$$|\langle \pi(g)v, \tilde{v}\rangle| \leqq \|v\|\,\|\tilde{v}\|\,\|g\|^n$$

for any $v \in V, \tilde{v} \in \tilde{V}, g \in G$.

Of course as it stands, this n may depend on the particular extension V_*. The next few steps will show that it may be chosen to depend only on the underlying Harish-Chandra module.

Fix now for the rest of the section a minimal parabolic subgroup R. Suppose temporarily that P is a maximal proper parabolic subgroup of G containing R

and Q a parabolic subgroup opposite to P. Let Δ be a basis for the roots of the adjoint action of A_R on $\mathfrak{n}_R$. Let α be the unique element of Δ such that

$$A_P = \bigcap_{\beta \neq \alpha} \mathrm{Ker}(\beta).$$

Use α to identify A_P with $\mathbf{R}^{\mathrm{pos}}$, so that A_P^{--} corresponds to (0, 1). If ϵ is a positive character of A_R such that $\epsilon(a) = O(\|a\|^n)$ on A_R^{--}, then from Proposition 8.2 it follows immediately that

(D) *There exists* $C > 0$ *such that*

$$|\langle \pi(a)v, \tilde{v}|\rangle \leqq C\|v\|\,\|\tilde{v}\|\epsilon(a)$$

for all v *in* $V_*, \tilde{v}$ *in* $\tilde{V}, a$ *in* A_R^{--}.

Call a character ϵ *dominant* for (π, V_*) if Condition (D) holds and dominant for (π, V) if it holds with V_* replaced by V.

LEMMA 8.3. *If* ϵ *is dominant for* (π, V_*) *then for each* Y *in* $\mathfrak{n}_Q^k$ *there exists a constant* C *such that*

$$|\langle \pi(a)v, \tilde{\pi}(Y)\tilde{v}\rangle| \leqq C\|\tilde{v}\|\,\|v\|\epsilon(a)\alpha(a)^k$$

for all $a \in A_R^{--}, v \in V, \tilde{v} \in \tilde{V}$.

Proof. If Y is an eigenvector in $\mathfrak{n}_Q^k$ with respect to the character γ^{-1}, then

$$\langle \pi(a)v, \tilde{\pi}(Y)\tilde{v}\rangle = \gamma(a)\langle \pi(a)\pi(Y^{\iota})v, \tilde{v}\rangle.$$

The result follows since $\gamma(a) \leqq \alpha(a)^k$ for a in A_R^{--}, and of course the map $\pi(Y^{\iota})$ is continuous on V_*.

LEMMA 8.4. [**38**, 5.8]. *There exist a finite set* S *in* $\mathbf{C}$, m *in* $\mathbf{N}$, *and formal series*

$$f_{k,s} = \sum_{n \geqq 0} f_{k,s,n}(v, \tilde{v})\alpha(a)^n \quad (s \in S, k \leqq m)$$

whose coefficients $f_{k,s,n}$ *are continuous bilinear functions on* $V_* \otimes \tilde{V}$, *such that for all* $v \in V_*, \tilde{v} \in \tilde{V}$, *the matrix coefficient* $\langle \pi(a)v, \tilde{v}\rangle$ *is as a function of* $a \in A_Q$ *asymptotic to*

$$\sum_{s \in S, k \leqq m} f_{k,s}\alpha(a)^s \log^k(\alpha(a))$$

in the sense that for every N

$$R_N = \langle \pi(a)v, \tilde{v}\rangle - \sum_{\substack{Re(n+s) \leqq N \\ k \leqq m}} f_{k,s,n}\alpha(a)^{s+n} \log^k(\alpha(a)) = o(\alpha(a)^N)$$

as $\alpha(a)$ goes to 0.

If ϵ is a dominant character of (π, V_*) and

$$F = f_{k,s,n}(v, \tilde{v}) \quad or \quad F = R_N(v, \tilde{v})$$

then there exists a constant C such that

$$|F(\pi(a)v, \tilde{v})| \leqq C\epsilon(a)\|v\| \, \|\tilde{v}\|$$

for all v in V_*, $\tilde{v}$ in $\tilde{V}$, and a in A_R^{--}.

Proof. Let X be a generator of $\mathfrak{a}_P$, let $\tilde{v}_1, \ldots \tilde{v}_l$ be a basis for $\tilde{V}/\mathfrak{n}_Q^k\tilde{V}$, with k assumed large. Define the function Φ with values on $[0, \infty)$ with values in $\mathbf{C}^l$: $\Phi(t) = (\langle\pi(\exp(tX))v, \tilde{v}_i\rangle)$. Suppose that $\pi(X)\tilde{v}_i = \sum c_{i,j}\tilde{v}_j$ defines the action of $\mathfrak{a}_P$ on $\tilde{V}/\mathfrak{n}_Q^k\tilde{V}$. Then according to Lemma 8.3

$$X\Phi - C\Phi = O(\epsilon(a)\alpha^k\|v\| \, \|\tilde{v}\|).$$

The lemma then follows from Proposition A.1 and Equations (A.5)–(A.7) of the Appendix to this section, if we let k get larger and larger.

LEMMA 8.5. *Any character of A_R dominant for (π, V) is also dominant for (π, V_*).*

Proof. Apply the previous lemma to successive maximal proper parabolic subgroups, keeping in mind that the coefficients of the asymptotic expansions are continuous.

This amounts to the earlier remark that the order of growth of matrix coefficients of extensions of moderate growth depends on the underlying Harish-Chandra module. More precisely:

LEMMA 8.6. *There exists a single n depending only on V and for each Harish-Chandra extension V_* of V norms $\|v\|$ and $\|\tilde{v}\|$ on V_* and $\tilde{V}$ such that*

$$|\langle\pi(g)v, \tilde{v}\rangle| \leqq \|g\|^n\|v\| \, \|\tilde{v}\|$$

for all g in G, v in $\tilde{V}$, and $\tilde{v}$ in $\tilde{V}$.

LEMMA 8.7. [**38**, 6.9]. *If V is essentially square-integrable then so is V_*.*

Proof. This follows immediately from Proposition 3.4 and 8.5.

Suppose that $\tilde{v}_1, \ldots, \tilde{v}_m$ are generators for $\tilde{V}$. Then the map taking v in V to the function $(\langle\pi(g)v, \tilde{v}_i\rangle)$ on G with values in $\mathbf{C}^m$ is a $(\mathfrak{g}, K)$-embedding of π into $C^\omega(G)^m$. According to Lemma 8.6, if V_* is any Harish-Chandra extension of V then this embedding of V extends to a (unique) continuous embedding of V_* into $\mathbf{B}^m$, where $\mathbf{B}$ is (just for the moment) the Banach space of continuous functions on G bounded by a multiple of $\|g\|^n$. Define $\overline{\overline{V}}$ to be the closure of V

in the Fréchet space of smooth vectors in $\mathbf{B}^m$. Thus $\overline{\overline{V}}$ is therefore in a precise way the largest smooth extension of V of moderate growth. In particular the definition of $\overline{\overline{V}}$ doesn't depend on the initial choice of generators of $\tilde{V}$.

PROPOSITION 8.8. [**38**, 5.10] *The canonical injection of $\overline{V}$ into $\overline{\overline{V}}$ is an isomorphism.*

Proof. It is required to construct a continuous map from $\overline{\overline{V}}$ to $\overline{V}$ extending the canonical embedding of V. Following [**38**] I do this by induction on the semi-simple rank of G. Of course for G compact there is nothing to be done. Continue to let P be a maximal proper parabolic subgroup of G. Let ϵ be dominant for V. Since the matrix coefficients of K-finite vectors have converging asymptotic expansions, we may choose k so large that every non-trivial $(\mathfrak{g}, K)$-stable subspace of V has terms of order no more than $\epsilon\alpha^k$ in expansion of its matrix coefficients. If U is the canonical P-representation on V_* modulo the closure of $\mathfrak{n}^k V_*$, then the canonical G-map from V_* into $\mathrm{Ind}(U|P, G)$ is an embedding.

In view of this result, from now I will identify $\overline{V}$ with $\overline{\overline{V}}$.
Define

$$A(G) := \text{ the strong dual of } \mathcal{S}(G).$$

It contains all functions of moderate growth on G, i.e., those f such that for every $X \in U(\mathfrak{g})$

$$\|f\|_{X,m} = \sup_{g \in G} \frac{|R_X f(g)|}{\|g\|^m} < \infty$$

for some $m > 0$, and this in turn contains the subspace

$$A_{\mathrm{umg}}(G) := \{f \in C_c^\infty(G) | \text{there exists a single } m \text{ such that } \|f\|_{X,m} < \infty \\ \text{for all } X \text{ in } (\mathfrak{g})\}.$$

Thus $A_{\mathrm{umg}}(G)$ is the space of smooth functions of *uniform moderate growth*. For a fixed m the subspace $A_{\mathrm{umg},m}$ becomes a Fréchet space with the semi-norms $\|f\|_{X,m}$. Assign to $A_{\mathrm{umg}}(G)$ itself the inductive limit topology as in [**36**: Exercise 13.6 and Section 50]. The right- and left-regular representation of $G \times G$ on each of these spaces is continuous and the one on each $A_{\mathrm{umg},m}(G)$ is of moderate growth.

It is a special case of [**9**, Theorem 1.16] that the space A_{umg} is the Gårding subspace of the right (or left) regular representation of G on $A(G)$. A dual result will be needed here:

LEMMA 8.9. *The Schwartz space $\mathcal{S}(G)$ is in turn the Gårding subspace of the dual of $A_{\mathrm{umg}}(G)$.*

Proof. Since the space $\mathcal{S}(G)$ is Fréchet and nuclear its dual $A(G)$ is nuclear according to [**35**, Proposition 50.6]. Since $C_c^\infty(G)$ is also nuclear so is the topological tensor product $C_c(G)\hat{\otimes}A(G)$ [**36**, Proposition 50.1]. Let $B(G)$ be for the moment the dual of $A_{\mathrm{ung}}(G)$. Then the coupling $\langle F, \hat{f} * D\rangle$ induces a continuous functional on the topological tensor product of the three spaces $B(G), C_c^\infty(G))$, and $A(G)$. If D lies in A_{umg} then this is the same as $\langle f * F, D\rangle$. It is straightforward to conclude from this that if F lies in $B(G)$ and f in $C_c^\infty(G)$ then $f * F$ may be identified with an element of the dual of $A(G)$, which is of course $\mathcal{S}(G)$. Hence the Gårding subspace of $B(G)$ is contained in $\mathcal{S}(G)$. The implication in the other direction follows from [**15**, Théorème 3.3].

Define $\underline{\underline{V}}$ to be the subspace of $\overline{\overline{V}}$ spanned by the $\pi(f)v$ with f in $\mathcal{S}(G)$ and v in $\overline{V}$. This contains V itself according to Corollary 5.10 and may also be expressed as the canonical image of the Fréchet space $\mathcal{S}(G) \otimes U$ if U is a finite-dimensional subspace of V generating V as a $U(\mathfrak{g})$-module. Assigned the quotient topology it becomes also an extension of V of moderate growth, which clearly possesses a canonical and continuous injection into any other smooth extension of moderate growth. It is therefore the smallest smooth extension of moderate growth, again in a precise sense.

Corollary 8.10 [**38**, 6.5]. *The canonical injection of $\underline{\underline{V}}$ into $\underline{V}$ is an isomorphism.*

Proof. Let U be the contragredient of V. The definition of $\underline{V}$ implies immediately, in view of [**15**], that it is the Gårding subspace of the topological dual of $\overline{U}$, while $\underline{\underline{V}}$ is because of Lemma 8.9 the Gårding subspace of the topological dual of $\overline{\overline{U}}$.

From now on I identify $\underline{V}$ with $\underline{\underline{V}}$.

Proposition 8.11. *Let (π, V) be an irreducible square-integrable Harish-Chandra module, and V_* the space of smooth vectors in the associated unitary representation of G. The canonical injections of $\underline{V}$ into V_* and of V_* into $\overline{V}$ are isomorphisms.*

Proof. From Lemma 8.7 it follows that $\overline{V}$ is contained in V_*. Hence since $\overline{V}$ is the largest possible Harish-Chandra extension, V_* and $\overline{V}$ must be the same. By duality, keeping in mind that V_* is the Gårding subspace of its own dual, V_* must agree with $\underline{V}$.

Corollary 8.12. *Every essentially square-integrable Harish-Chandra module is regular.*

Appendix to section 8. Suppose A to be an m by m matrix with complex coefficients and $S(x)$ a continuous function on $[0, \infty)$ with values in $\mathbf{C}^m$ such that for some constant C_S and real number s

$$\|S(x)\| \leqq C_S e^{sx}. \tag{A.1}$$

This appendix is concerned with the following elementary result and variations:

PROPOSITION A.1. *Any solution $y(x)$ of the system*

(A.2) $\quad dy/dx = Ay + S(x)$

possesses a unique decomposition

$$y(x) = y_1(x) + y_2(x)$$

where

(a) y_1 *is a solution of the homogeneous form of* (A.2), *i.e., with* $S(x) = 0$, *which can be expressed as*

$$\sum P_c(x)e^{cx},$$

each $P_c(x)$ an m-vector with polynomial coefficients and $\mathrm{Re}(c) > s$ *for each c, and*

(b) y_2 *satisfies*

$$\|y_2(x)\| \leqq C(1 + x^d)e^{sx}$$

for some constants $d, C > 0$.

Proof. Choose Q so that $B = Q^{-1}AQ$ is a sum of blocks B_+ and B_- such that the eigenvalues b of B_+ have real parts larger than s and those of B_- have real parts at most s. Then $y(x)$ is a solution of (A.2) if and only if $z(x) = Q^{-1}y(x)$ satisfies

(A.3) $\quad dz/dx = Bz + T(x)$

where $T(x) = Q^{-1}S(x)$. Furthermore, this equation splits up, so that in effect the Proposition must be proven in the two separate cases when (a) all the eigenvalues λ of A satisfy $\mathrm{Re}(\lambda) > s$ or (b) they all satisfy $\mathrm{Re}(\lambda) \leqq s$. In case (a) apply the formula

$$\text{(A.4a)} \quad \begin{aligned} y(x) &= e^{Ax}\left(y(0) + \int_0^\infty e^{-At}S(t)dt\right) - e^{Ax}\int_x^\infty e^{-At}S(t)dt \\ &= y_1(x) + y_2(x) \end{aligned}$$

where

$$\begin{aligned} y_1(x) &= e^{Ax}\left(y(0) + \int_0^\infty e^{-At}S(t)dt\right), \\ y_2(x) &= -e^{Ax}\int_x^\infty e^{-At}S(t)dt. \end{aligned}$$

In case (b) apply the formula

$$\text{(A.4b)} \quad y(x) = e^{Ax}y(0) + e^{Ax}\int_0^x e^{-At}S(t)dt,$$

or in other words let y_2 be all of y.

The proof tells much more about the decomposition. The functions $y_1(x)$ and $y_2(x)$ are linear functions of $y(0)$ and $S(x)$, and one can give good estimates on their magnitudes in terms of these data. First of all we have in case (a) above the estimate

$$\begin{aligned}\left\|\int_0^\infty e^{-At}S(t)dt\right\| &\leqq \int_0^\infty \|e^{-At}\|\,\|S(t)\|dt \\ &\leqq C_S C_A \int_0^\infty (1 + t^{(d-1)})e^{-(a-s)t}dt \\ &= C_S C_A\left(\frac{1}{a-s} + \frac{\Gamma(d)}{(a-s)^d}\right)\end{aligned}$$

if C_A is chosen suitably, where d is the size of the largest blocks in the Jordan decomposition of A, and a is the smallest eigenvalue of A. If the (+)-eigenspace of A denotes the sum of λ-primary components for $\mathrm{Re}(\lambda) > s$, then without any assumption relating s and the eigenvalues of A, this translates to the equation

$$\text{(A.5)} \quad y_1(x) = e^{Ax}(y + (0) + y_S)$$

where $y_+(0)$ is the projection of $y(0)$ onto the (+)-eigenspace of A and y_S, also in the (+)-eigenspace, depends only on S and A (not on y) and satisfies the inequality

$$\text{(A.6)} \quad \|y_S(x)\| \leqq C_A C_S\left(\frac{1}{a-s} + \frac{\Gamma(d)}{(a-s)^d}\right)$$

where C_A again depends only on A and a is now the least real part of an eigenvalue of A greater than s.

Second, an estimate of the integrals

$$\int_x^\infty e^{A(x-t)}S(t)dt, \qquad \int_0^x e^{A(x-t)}S(t)dt$$

leads to the estimate

$$\text{(A.7)} \quad \|y_2(x)\| \leqq D_A(1 + x^d)e^{sx}(\|y - (0)\| + C_S)$$

where C_S and d are the same as before, $y_-(0)$ is the projection of $y(0)$ onto the ($-$)-eigenspace of A (the complement of the (+)-eigenspace), and D_A depends only on A and the difference $a - s$.

9. Analytic families of Harish-Chandra modules. Let X be a complex analytic manifold, V the space of a representation of K which is simply the algebraic direct sum of irreducible finite-dimensional representations, each with finite multiplicity. Suppose for convenience that X is connected. There are two equivalent ways to describe an analytic family of Harish-Chandra modules based on the pair (X, V). In the first more naïve way it is a family of Harish-Chandra modules V_x indexed by X and varying holomorphically with x. Each element of $\mathfrak{g}$ thus corresponds to an infinite-dimensional matrix whose coefficients are holomorphic functions on X, with the property that all but a finite number of entries in each row and column vanish identically. For the second, define $\mathcal{V}$ to be the sheaf of germs analytic functions on X with values in V, which is simply the direct limit of the sheaves of functions with values in finite-dimensional subspaces. Then the analytic family becomes a representation of $(\mathfrak{g}, K)$ on $\mathcal{V}$ commuting with O (the structure sheaf on X), such that the associated representation on each fibre is a Harish-Chandra module, whose restriction to K is the given one.

The fibre V_x at a point x is the $(\mathfrak{g}, K)$-module $\mathcal{V}_x/m_x\mathcal{V}_z$, where m_x is the maximal ideal of O_x and $\mathcal{V}_x$ is the stalk of the sheaf $\mathcal{V}$ at x. If I is any ideal of O_x of finite codimension, then since $\mathcal{V}$ is a locally free sheaf we have this identification of O_z-modules:

$$(9.1) \qquad (O_x/I) \otimes V \cong \mathcal{V}_x/I\mathcal{V}_x.$$

But as a $(\mathfrak{g}, K)$-module the structure of $U = \mathcal{V}_x/I\mathcal{V}_x$ is a bit more complicated. Note that it is a module over both $(\mathfrak{g}, K)$ and O_x/I, which commute with each other. If I and J are ideals of O_x then the product map at least amounts to a $(\mathfrak{g}, K)$-morphism

$$(9.2) \qquad (J/I) \otimes \mathcal{V}_x \longrightarrow J\mathcal{V}_x/I\mathcal{V}_x.$$

It is straightforward to see:

LEMMA 9.1. *If $I \subseteq J$ are ideals of finite codimension in O_x with $m_xJ \subseteq I$ then the map induced from the product map* (9.2)

$$(J/I) \otimes V_x \longrightarrow J\mathcal{V}_x/I\mathcal{V}_x$$

is an isomorphism of $(\mathfrak{g}, K)$*-modules.*

As a consequence:

PROPOSITION 9.2. *If I is an ideal of finite codimension in O_x then the* $(\mathfrak{g}, K)$*-module $\mathcal{V}_x/I\mathcal{V}_x$, filtered by the images of the* $(\mathfrak{g}, K)$*-stable modules $m_x^n\mathcal{V}_x$, has as associated graded module a direct sum of copies of the fibre representation V_x.*

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It seems difficult to form an intuitive sense about these local representations. The simplest example of an analytic family is in some sense the most important one, and understanding it may help. Let A be the topologically connected component of a split torus over $\mathbf{R}$, and let X be $X(A)$, the space of ($\mathbf{C}^{\times}$-valued) characters of A. The canonical family of A-representations over this base space is on the direct product $A \times \mathbf{C}$, and associates to each χ in X simply the character χ. The universal enveloping algebra $U(\mathfrak{a})$ maps canonically onto the ring of polynomials on X: to α in $\mathfrak{a}$ corresponds the function taking χ to $d\chi(\alpha)$, and the representation of α on the analytic family amounts simply to multiplication by this polynomial. If I is $\mathfrak{m}_x$-primary (hence of finite codimension) then the representation of $U(\mathfrak{a})$ on $\mathcal{O}_x/I$ is that on $U(\mathfrak{a})/I$. Duality might improve one's intuition about the nature of the local representations: $U(\mathfrak{a})$ acts by differentiation on the space of functions on A annihilated by the differential operators in I, and the pairing $\langle D, F\rangle = DF(1)$ identifies this with the dual of $U(\mathfrak{a})/I$. The dual of the projective limit of the $U(\mathfrak{a})/\mathfrak{a}^n$, where $\mathfrak{a}^n$ is the n-th power of the augmentation ideal, is thus the space of all polynomials in the logarithms of (multiplicative) coordinates on A. If $A = \mathbf{R}^{\text{pos}}$, say, then what this amounts to is that the polynomials in $\log(a)$ of degree n form the subspace of functions on A annihilated by the differential operator ad/da. More generally the functions $a^s \log^n(a)$ span the space of polynomials annihilated by some power of $ad/da - s$, and form the dual of the projective limit of the $U(\mathfrak{a})/\mathfrak{m}_s$, where $\mathfrak{m}_s$ is generated by $ad/da - s$.

Suppose for the moment that P is a parabolic subgroup of G, and that $(\sigma_x, \mathcal{U})$ is an analytic family of Harish-Chandra modules of P, over the space X and associated to the $K(P)$-representation space U. Then one can define in the obvious way a family $\mathcal{I}$ with $I_x = \operatorname{Ind}(\sigma_x|P, G)$, based on the K-space $\operatorname{Ind}(\sigma|K(P), K)$. It is useful to observe that one has the canonical isomorphism (from Frobenius reciprocity)

$$\mathcal{I}_x/\mathfrak{m}_x^k \mathcal{I}_x \cong \operatorname{Ind}(\mathcal{V}_x/\mathfrak{m}_x^k \mathcal{V}_x|P, G).$$

If V_x is a regular Harish-Chandra module in the sense of the previous section and I is an ideal of finite codimension, then according to Lemma 7.16 and Proposition 9.2 the quotient $\mathcal{V}_x/I\mathcal{V}_x$ is regular as well, and the identification (9.1) extends to a topological isomorphism:

$$(9.3)\qquad (\mathcal{O}_x/I) \otimes \overline{V} \cong \overline{\mathcal{V}_x/I\mathcal{V}_x}.$$

This means, for example, that if J is an ideal containing I then the image of $I\overline{\mathcal{V}}_x$ in $\overline{\mathcal{V}_x/I\mathcal{V}_x}$ is closed.

If we have two analytic families of Harish-Chandra modules $\mathcal{U}$ and $\mathcal{V}$ based on X then an analytic $(\mathfrak{g}, K)$-map $\mathcal{T}$ from the first to the second is simply a $(\mathfrak{g}, K)$-morphism of $\mathcal{O}$-sheaves from $\mathcal{U}$ to $\mathcal{V}$, acting say as $\mathcal{T}_x$ on the stalk $\mathcal{U}_x$ and with fibre map T_x. It is said to be *meromorphically invertible* when there

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exists an analytic map $\mathcal{S}$ from $\mathcal{V}$ to $\mathcal{U}$ and a scalar analytic function $c(x)$ such that the composition of $\mathcal{S}$ and $\mathcal{T}$ as well as that in the other direction amount to multiplication by $c(x)$.

Assume now that X has dimension one. Fix a point x in X, and let z be a parameter for $\mathfrak{m} = \mathfrak{m}_x$. To make notation bearable, drop the subscript x when not confusing. Assume given two families $\mathcal{U}$ and $\mathcal{V}$ with the fibres U_x and V_x regular. For each positive integer m define U_m to be the quotient $\mathcal{U}/\mathfrak{m}_x^m\mathcal{U}$, and similarly for V_m. Assume given an analytic map $\mathcal{T}$ from $\mathcal{U}$ to $\mathcal{V}$, $\mathcal{S}$ a meromorphic inverse such that the composition of $\mathcal{S}$ and $\mathcal{T}$ amounts to multiplication with the analytic function c, and let n be the order of c at x. Then the composition of $\mathcal{T}$ and $\mathcal{S}$ also amounts to multiplication by c. By functoriality, each of the maps induced by $\mathcal{T}$ from U_m to V_m extens to a continuous G-map $\overline{T}_m : \overline{U}_m \longrightarrow \overline{V}_m$. The following elementary result is the lynch-pin of the entire paper:

PROPOSITION 9.3. *If $m \geqq 2n$, the image of $\overline{T}_m$ is closed.*

Proof. Fix m, and just refer to S, T, U, V, etc. Let C_V, C_U be multiplication by c on U, V. The beginning of the proof is the remark (which follows from (9.3)) that the image of the map $\overline{C}_U$ is closed, and in fact the kernel of multiplication by z^{m-n}. I now claim that $\mathrm{Im}(\overline{T})$ is the same as $\overline{S}^{-1}(\mathrm{Im}(\overline{C}_U))$, which will therefore prove the proposition. To verify this claim, the only difficulty is seeing that if v lies in $\overline{V}$ with $\overline{S}(v) = \overline{C}(u)$ for some u in $\overline{U}$, then v lies in $\mathrm{Im}(\overline{T})$. But under this assumption $\overline{T}(u) - v$ lies in $\mathrm{Ker}(\overline{S})$ on the one hand, and on the other

$$\begin{aligned}
\mathrm{Ker}(\overline{S}) &\subseteq \mathrm{Ker}(\overline{C}_V) && (\text{since } TS = C_V)\\
&\subseteq \mathrm{Im}(\overline{C}_V) && (\text{since } m \geqq 2n)\\
&\subseteq \mathrm{Im}(\overline{T}) && (\text{again since } TS = C_V).
\end{aligned}$$

This concludes the proof.

10. The main theorem. The next few results could have just as easily been proven earlier.

LEMMA 10.1. *If σ is a regular Harish-Chandra module over $(\mathfrak{p}, K(P))$ then* $\mathrm{Ind}(\sigma|P, G)$ *is a regular Harish-Chandra module over* $(\mathfrak{g}, K)$.

Proof. Let R be a minimal parabolic subgroup contained in P. From regularity, $\overline{\sigma}$ fits into the diagram of $(\mathfrak{m}, K(M))$-modules

$$I^{\mathrm{sm}}(\chi|R, P) \longrightarrow \overline{\sigma} \hookrightarrow I^{\mathrm{sm}}(\tau|R, P),$$

where the first map is surjective and the second has closed image. According to Lemma 7.15 there are continuous linear splittings of these maps, so that inducing in turn to G we have a corresponding diagram

$$\mathrm{Ind}^{\mathrm{sm}}(\chi|R, G) \longrightarrow \mathrm{Ind}^{\mathrm{sm}}(\overline{\sigma}|P, G) \longrightarrow \mathrm{Ind}^{\mathrm{sm}}(\tau|R, G),$$

where again the first map is surjective and the second has closed image.

From Lemma 10.1, Corollary 8.12, the Proposition 4.5 it follows immediately that:

LEMMA 10.2. *Every essentially tempered Harish-Chandra module is regular.*

Another technical consequence of earlier results:

LEMMA 10.3. *If U is regular and $\overline{U} \longrightarrow \overline{V}$ is surjective, then V is regular.*

Finally we come to the main result of this paper:

THEOREM 10.4. *Every Harish-Chandra module is regular.*

According to Lemma 7.15 it suffices to prove this for irreducible Harish-Chandra modules. The proof will proceed by a kind of induction (used by [**5**] and [**40**] in different context), based on the classification (recalled in Section 4) of Langlands, together with the algebraic interpretation in Section 9 of the meromorphicity of intertwining operators.

Define (temporarily) the *rank* of an irreducible Harish-Chandra module (π, V) in the following recursive manner: If π is of the form $\mathrm{Ind}(\sigma|P, G)$ with σ essentially tempered and $\langle\sigma\rangle$ in $X_{\mathbf{R}}^{--}(P)$ (i.e., as opposed to being a proper quotient) then the rank of π is defined to be 0. In general, if π is a proper quotient $\pi(P, \sigma)$ then it is one more than the maximal rank of any constituent of $\mathrm{Ind}(\sigma|P, G)$ other than π itself. It is Proposition 4.9 that assures that this definition is not circular.

The proof of Theorem 10.4 now goes by induction on the rank of the representation (π, V). If it has rank 0 then it is regular by Lemmas 10.1 and 10.2. Suppose now that it is a proper quotient $\pi(P, \omega_0)$. Recall that π is then the image of the intertwining operator

$$T_{\omega_0} : \mathrm{Ind}(\omega_0|P, G) \longrightarrow \mathrm{Ind}(\omega_0|P^{\mathrm{opp}}, G),$$

which by Proposition 4.6 is analytic and meromorphically invertible in the neighborhood of ω_0. Let S_ω be a meromorphic inverse; i.e., a map running back the other way such that the composition ST amounts to multiplication by a holomorphic scalar function c_ω. Let Ω be a small one-dimensional line segment through the representation ω_0, situated regularly so that the restriction of c_ω to X is not identically 0. Let n be the order of the zero of c_ω at ω_0. Introduce notation to conform to Section 9: Let $\mathcal{U}$ be the family $\mathrm{Ind}(\omega|P, G)$, $\mathcal{V}$ the family $\mathrm{Ind}(\omega_0|P^{\mathrm{opp}}, G)$, $\mathcal{S}$ and $\mathcal{T}$ the maps S_ω and T_ω, x the point ω, etc. Let $m = 2n$, C the endomorphism of either $\mathcal{U}$ or $\mathcal{V}$ induced by c_ω. Let

$$X = \text{the image of } \mathcal{U}/C^2\mathcal{U} = U_m\text{rm in } \mathcal{V}/C^2\mathcal{V} = V_m$$
$$Y = \text{the image of } \mathfrak{m}_x\mathcal{V} \text{ in } V_m$$
$$Z = \text{the sum of the two subspaces } X \text{ and } Y \text{ in } V_m.$$

Thus π is the quotient of Z/Y. Let $\overline{X}, \overline{Y}, \overline{Z}$ be their closures in $\overline{V}_m$ (this notation is consistent by Corollary 7.9). By Lemma 9.3 the subspace $\overline{X}$ is the image of $\overline{U}_m$ in $\overline{V}_m$ with respect to $\overline{\mathcal{T}}_m$, and by Lemma 10.3 the representation X is regular. In order to show that π is regular, it suffices by Lemma 10.3 to show that the subspace $\overline{Z}$ in $\overline{V}_m$ is the sum of the closed subspaces $\overline{X}$ and $\overline{Y}$. Since the representations X and Y are regular, $\overline{X} = \underline{X}$ and $\overline{Y} = \underline{Y}$, and since $\underline{Z}$ is the linear sum of $\underline{X}$ and $\underline{Y}$ (by Corollary 7.14), it must in effect be shown that Z is regular.

First of all, the Harish-Chandra module Z has finite length by Corollary 5.8 and since every occurrence of π is already in X, every component of Z/X has rank smaller than that of π. So by the induction assumption and Lemma 7.15 the quotient Z/X is itself regular. I have already remarked that X is regular. Therefore Z itself is regular.

The consequences of this main result I now summarize:

COROLLARY 10.5 *Let (σ, U) and (π, V) be smooth Fréchet representations of G of moderate growth such that the underlying K-finite subspace V^K is a Harish-Chandra module. Any linear $(\mathfrak{g}, K)$-map from U^K to V^K extends uniquely to a continuous G-map from U to V, and the image of the extension has closed range.*

This is because the quotient of U by the closure of the kernel of the original map is a Harish-Chandra representation.

COROLLARY 10.6. *If (π, V) is a smooth Fréchet representation of G whose underlying Harish-Chandra module is generated by the finite-dimensional subspace U, then every element in V is a finite linear combination $\sum f_i u_i$ with the u_i in U and the f_i in $\mathcal{S}(G)$.*

11. The asymptotic behaviour of matrix coefficients. Let (π, V) be a Harish-Chandra module. In Section 8 the main arguments described in relatively crude terms the asymptotic behaviour of matrix coefficients

$$\langle \pi(g)v, \tilde{v} \rangle$$

where v lies in $\overline{V}$, the space of the Harish-Chandra representation extending π, and $\tilde{v}$ is a K-finite vector in the contragredient of π. This crude description has been refined somewhat in Section 7 of [**38**], but with the same assumption on $\tilde{v}$. In this section a much more precise description of asymptotic behaviour of matrix coefficients will be given, and under the much weaker assumption that $\tilde{v}$ lies in the space of the contragredient Harish-Chandra representation. This description appears to be new.

The real point is to define local semi-norms on sections of the sheaves $\mathcal{A}_{S,m}$ on the Oshima completion $\overline{G}$. I do this first in great generality. For a while, let $\overline{Y}$ be a cube $|y_i| \leqq \epsilon$ in $\mathbf{R}^q$ and Z a cube $0 \leqq z_i \leqq \epsilon$ in $\mathbf{R}^n$, and let X be

$Y \times Z$. I shall label the y_i the *transverse* and the z_i the *corner* coordinates. Then for a finitely generated $\mathbf{N}$-stable subset S of $\mathbf{C}^n$ and m in $\mathbf{N}^n$ define the space $A_{S,m} = A_{S,m}(X)$ to be that of all functions on the subspace $\{z_i > 0\}$ of X which may be expressed as a finite sum

$$(11.1) \qquad \sum_{s\in S, k\leqq m} f_{s,k} z^s \log^k(z)$$

where each $f_{s,k}$ is a smooth function on the closed cube $Z \times Y$, and it is assumed that all but a finite number of the $f_{s,k}$ are null. I want to make $A_{S,m}$ into a Fréchet space.

To make the description a bit easier to follow let me first deal with the one-dimensional corner $X = [0, 1]$; i.e., take $q = 0, n = 1$. Defining a topology on $A_{S,m}$ even in this simple case proceeds in several steps.

Step (1). Let f in $A_{S,m}$ be expressed as in (11.1). The functions $f_{s,k}$ are hardly unique, since they may be mutually and easily modified in (0, 1]. However I claim that, roughly speaking, the Taylor series of the $f_{s,k}$ at 0 is uniquely determined by f, so that the image of each $f_{s,k}$ in the ring of formal series $\mathbf{C}[[z]]$ is essentialy well defined. The roughness in the statement is because two of the possible indices s may differ by an integer. However, after combining terms it may be seen that every f in $A_{S,m}$ determines a unique formal power series

$$\sum_{s\in S, k\leqq m} c_{s,k} z^s \log^k(z)$$

which is in fact an asymptotic series for f as $z \longrightarrow 0$. It is even a *strong asymptotic series* for f in the sense that its derivatives give the asymptotic expansions of the derivatives of f. The uniqueness of asymptotic expansions is well known but I recall the argument here: Let D be for the moment the differential operator zd/dz. Thus

$$Dz^s \log^k(z) = sz^s \log^k(z) + kz^s \log^{k-1}(z).$$

Suppose that σ is the maximum value of $\mathrm{Re}(s)$ as s ranges over S. Then for any f in $A_{S,m}$ and extremal s in S

$$\lim_{z\rightarrow 0} \frac{1}{z^s \log^m(z)} \frac{\left(\prod_t (D-t)\right) f}{\prod_t (s-t)} = c_{s,m}$$

where each product is over all t in S with $\mathrm{Re}(t) = \sigma, t \neq s$. An inductive argument shows then that in the strong symptotic expansion

$$f \sim \sum_{s\in S, k\leqq m} c_{s,k} z^s \log^k(z)$$

each coefficient $c_{s,k}$ is uniquely determined by the asymptotic behaviour of f and its derivatives as $z \longrightarrow 0$.

Step (2). Define $C_0^\infty[0, 1]$ to be the subspace of functions in $C^\infty[0, 1]$ vanishing of infinite order at 0. It is the kernel of the map taking f to its formal Taylor series $\hat{f}$ at 0, hence closed in $C^\infty[0, 1]$. Furthermore, according to the theorem of E. Borel [**36**, Theorem 38.1] every formal series in z is the Taylor series of some smooth function, so that all in all this sequence is an exact sequence of Fréchet spaces:

$$(11.2) \quad 0 \longrightarrow C_0^\infty[0, 1] \longrightarrow C^\infty[0, 1] \longrightarrow \mathbf{C}[[z]] \longrightarrow 0.$$

The remark that the Taylor series of the $f_{s,k}$ are determined, coupled with this remark, means that we also have an exact sequence of Fréchet spaces

$$(11.3) \quad 0 \longrightarrow C_0^\infty[0, 1] \longrightarrow A_{S,m} \longrightarrow \hat{A}_{S,m} \longrightarrow 0,$$

where $\hat{A}_{S,m}$ is the space of formal series of terms $c_{s,k} z^s k \log^k(z)$ ($s\ hbox \mathrm{in} S, k \leqq m$).

Step (3). The space $\hat{A}_{S,m}$ may be identified with the projective limit of the finite-dimensional quotients

$$(11.4) \quad A_{S,m}/\mathfrak{m}^l A_{S,m},$$

where $\mathfrak{m}$ is the maximal ideal in $C^\infty[0, 1]$ comprising functions vanishing at 0. It is a Fréchet space with semi-norms $|c_{s,k}|$. The space $C_0^\infty[0, 1]$ is also a Fréchet space with the semi-norms

$$\sup_z \frac{|Df|}{|z|^\sigma}$$

where D is some power of $zd/dz, \sigma$ a positive integer. I claim now that there exists a unique topology on the space $A_{S,m}$ making the sequence (11.3) into a continuous exact sequence of Fréchet spaces.

If the sequence (11.3) were to possess a continuous splitting, then there would be no problem, but of course it does not. The projections onto the quotients (11.4), however, do split, and in a rather trivial way. In other words, if f lies in $A_{S,m}$ and σ is any real number then

$$f - \sum_{\mathrm{Re}(s) \leqq \sigma} c_{s,k} z^s \log^k(z) = o(z^\sigma).$$

The semi-norms

$$|c_{s,k}| (s \in S, k \leqq m)$$

$$\sup_z \frac{|(zd/dz)^l(f - \sum_{Re(s)\leqq\sigma} c_{s,k} z^s \log^k(z))|}{|z^\sigma|} \quad (l, \sigma \in \mathbf{N})$$

define the Fréchet topology required.

The case where the transverse space Y has arbitrary dimension is almost as simple, with the functions $f_{s,k}$ taken in $C^\infty([0,1] \times Y)$ or, equivalently, the coefficients $c_{s,k}$ taken as elements of $C^\infty(Y)$.

For the case of higher dimensional corners $X = Y \times Z$, the notation seems to be necessarily cumbersome. Let Δ be the set $\{1, \ldots, n\}$ and $\Theta \subseteq \Delta$. Define X_Θ to be the part of the boundary of X where $z_i = 0$ for i not in Θ. Thus the z_i for i in Θ are parameters on X_Θ, and X_Δ is just X itself. For any z in $\mathbf{C}^\Delta$ let z_Θ be its canonical image in $\mathbf{C}^\Theta$. Finally, let I_Θ be the ideal in $C^\infty(X)$ of functions vanishing along $X(\Delta - \Theta)$, generated by the z_i with i in Θ.

To every function in $C^\infty(X)$ may be associated its formal power series along one of the boundary pieces $X(\Theta)$. Hence every section of $A_{S,m}(X_\Delta)$ determines as well a formal series expansion

$$\sum_{s\in\alpha(S), k\leqq\alpha(m)} c_{s,k} z_\Theta^s \log^k(z_\Theta)$$

where each of the coefficients $c_{s,k}$ is now a section of $A_{\beta(S),\beta(m)}(X(\Theta))$. Define $\hat{A}^{[\Theta]}_{S,m}$ to be the space of such formal series. By a simple generalization of the arguments given above in the one-dimensional case, and in particular by a mild extension of the theorem of E. Borel already mentioned (and the simplest case of Whitney's extension theorem), the map $\mathcal{P}_\Theta$ taking f in $A_{S,m}(X_\Delta)$ to its image in $\hat{A}^{[\Theta]}_{S,m}$ is surjective. The kernel is the subspace which I call momentarily $I^\infty_\Theta A_{S,m}$ of functions in $A_{S,m}(X_\Delta)$ with all the $f_{s,k}$ vanishing of infinite order along X_Θ. Hence we have an exact sequence

$$(11.5) \quad 0 \longrightarrow I^\infty_\Theta A_{S,m} \longrightarrow A_{S,m} \longrightarrow \hat{A}^{[\Theta]}_{S,m} \longrightarrow 0.$$

Arguing by recursion on Θ one can prove as in the one-dimensional case:

PROPOSITION 11.1. *There exists on the space $A_{S,m}$ a unique structure as Fréchet space such that all the exact sequences* (11.5), *as Θ ranges over all subsets of Δ, are exact sequences of Fréchet spaces.*

In effect the semi-norms are defined by recursion on the dimension of the corner. There is a slight different way to phrase this result. The intersection of the kernels of all the $\mathcal{P}_\Theta$may be identified with the subspace $C_0^\infty(X)$ of functions in $C^\infty(X)$ vanishing of infinite order along the corners. This is a Fréchet space. The topology on $A_{S,m}(X)$ is determined in fact by the conditions that (1) the topology induced on the subspace $C_0^\infty(X)$ be the usual one and (2) the projections $\mathcal{P}_\Theta$ for Θ the complement of a single element in Δ are continuous.

This construction can be applied to define local semi-norms on the sections of the sheaf $\mathcal{A}_{S,m}$ over $\overline{G}$. Unless the centre Z_G of G is compact, the space of global sections of this sheaf will not be a Fréchet space, but generally only an LF space, since $\overline{G}$ is only a partial compactification. However, if I is an ideal of finite codimension in $Z(\mathfrak{g})$ the sections of this sheaf annihilated by I are completely determined by their values in a compact subset of $\overline{G}$. More explicitly, any Z_G-finite function on G is determined by its values in an infinitesimal neighborhood of G_*, the intersection of the kernels of the $\mathbf{R}$-rational characters of G, and the closure of G_* in $\overline{G}$ is compact. Hence for any ideal I of finite codimension in $Z(\mathfrak{g})$ the semi-norms defined locally on this sheaf makes the subspace $A_{S,m}(I)$ of such sections into a Fréchet space. It is straightforward to verify that this subspace becomes therefore the space of a Harish-Chandra representation of the product $G \times G$.

In the next result, suppose (π, V) to be a Harish-Chandra module over $(\mathfrak{g}, K)$ annihilated by the ideal I of finite codimension in $Z(\mathfrak{g})$. Let U be the contragradient $\tilde{V}$. Matrix coefficients give a canonical embedding of the tensor product $V \otimes U$ into the space of global sections of $\mathcal{A}_{S,m}$ over $\overline{G}$, which extends by Theorem 10.4 and functoriality to a continuous map from $\overline{V}\hat{\otimes}\overline{U}$ into the same space.

THEOREM 11.2. *The image of $\overline{V}\hat{\otimes}\overline{U}$ coincides with the closure of the image of $V \otimes U$ in $A_{S,m}(I)$.*

This says not only that the matrix coefficients

$$\langle \pi(g)v, \tilde{v} \rangle \quad (v \in \overline{V}, \tilde{v} \in \overline{U})$$

possess a certain asymptotic behaviour, but yields as well the curious bonus that any function with such asymptotic behaviour is a limit in some sense of such matrix coefficients.

The sheaves $\mathcal{A}_{S,m}$ occur frequently in harmonic analysis, in particular in the theory of Whittaker models.

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University of British Columbia,
Vancouver, British Columbia