A NOTE ON CLIFFORD ALGEBRAS AND CENTRAL DIVISION ALGEBRAS WITH INVOLUTION

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In this note we consider the question as to which central division algebras occur as the Clifford algebra of a quadratic form over a field. Non-commutative ones other than quaternion division algebras can occur and it is also the case that there are certain central division algebras D which, while not themselves occurring as a Clifford algebra, are such that some matrix ring over D does occur as a Clifford algebra. We also consider the further question as to which involutions on the division algebra can occur as one of two natural involutions on the Clifford algebra.

We use the standard notation and terminology of quadratic forms and Clifford algebras as in [**8**] for example. We write $C(q)$ for the Clifford algebra of a non-singular quadratic form q over a field F, char $F \neq 2$, and we write $\langle a_1, a_2, \dots, a_n \rangle$ for a diagonalization of q, where $a_i \in F$, $i = 1, 2, \dots, n$. We write $((a_1, a_2)/F)$ for the quaternion algebra over F generated by elements i, j, with $i^2 = a$, $j^2 = b$, $ij = -ji$. Note that $((a_1, a_2)/F)$ is either a division algebra or else isomorphic to M_2F, the ring of all 2×2 matrices over F.

All our quadratic forms will be assumed to be non-singular, i.e. represented by a non-singular matrix.

PROPOSITION 1. *Let $A = Q_1 \otimes Q_2 \otimes \cdots \otimes Q_s$ be a tensor product of s quaternion algebras over F, s being a positive integer. (The Q_i need not necessarily be division algebras.) Then there exists a quadratic form q over F such that $C(q)$ is isomorphic to A.*

Proof. We proceed by induction on s.

If $s = 1$ we may take $A = ((a_1, b_1)/F)$ and if $q = \langle a_1, b_1 \rangle$ then $C(q)$ is isomorphic to A.

Now assume the result holds for $s-1$ and let $A = Q_1 \otimes Q_2 \otimes \cdots \otimes Q_s$. Write $Q_i = ((a_i, b_i)/F)$ for each i, for suitable a_i, b_i in F, $i = 1, 2, \dots, s$. A fundamental property of Clifford algebras, see [**7**] or [**8**], is that if q_1 is two dimensional then $C(q_1 \perp q_2)$ is isomorphic to $C(q_1) \otimes_F C(-\delta q_2)$, q_2 being any non-singular form over F, δ being the determinant of q_1 (modulo squares in F) and $\perp$ denoting the orthogonal sum of forms. Let $q_1 = \langle a_1, b_1 \rangle$ so that $C(q_1)$ is isomorphic to Q_1. By the inductive assumption there is a quadratic form, q_2 say, such that $C(q_2)$ is isomorphic to $Q_2 \otimes Q_3 \otimes \cdots \otimes Q_s$. Then taking q to be $q_1 \perp (-a_1 b_1 q_2)$ yields that $C(q)$ is isomorphic to A.

Note that we could in fact have written down q directly as follows:

$$q = \Big\langle a_1, b_1, -a_1b_1a_2, -a_1b_1b_2, a_1b_1a_2b_2a_3, a_1b_1a_2b_2b_3, \dots, \\ (-1)^{s-1}\left(\prod_{i=1}^{s-1} a_ib_i\right)a_s, (-1)^{s-1}\left(\prod_{i=1}^{s-1} a_ib_i\right)b_s\Big\rangle.$$

COROLLARY. *Let D be any central division algebra over F admitting an involution of the*

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first kind (i.e. one which keeps fixed all the elements of F). Then for some integer $r \geq 1$, the matrix ring $M_r D$ occurs as the Clifford algebra $C(q)$ of some quadratic form q over F.

Proof. The algebra D represents an element of $B_2(F)$, the subgroup of elements of order two in the Brauer group. (This is because the involution gives an isomorphism between D and D^{op}, the opposite algebra of D, and the class of D^{op} is the inverse to that of D in the Brauer group.) Note also that D must have degree equal to a power of two by a result of Albert [**1**]. (The degree of D is the square root of its dimension as an F-vector space.) The recent theorem of Merkurjev [**10**] implies that B_2F is generated by the quaternion algebras. Thus, for some integer r, the matrix ring $M_r D$ must be a tensor product of quaternion algebras over F and hence is $C(q)$ for some quadratic form q over F.

COMMENT 1. Our proposition gives a specific way of constructing the form q such that $C(q)$ is isomorphic to the algebra A provided that A has been decomposed into a tensor product of quaternion algebras. Of course given an algebra A it is not necessarily easy to see whether or how such a decomposition can be effected. However in many cases such a decomposition can be done and so in these cases we have a way of constructing examples of forms with specified Clifford algebras.

We should mention that Merkurjev's theorem in [**10**] gives an isomorphism of a quotient of the algebraic K-group K_2F with the group B_2F. It can be reformulated as in [**4**], in terms of quadratic forms without any K-theory, and our corollary is also an immediate consequence of this version of Merkurjev's theorem.

COMMENT 2. A partial converse to Proposition 1 is the standard result, see [**7**] or [**8**], that the Clifford algebra of any non-singular even-dimensional quadratic form over F is always expressible as a tensor product of quaternion algebras over F. For an odd-dimensional form q, $C(q)$ is expressible as a tensor product of quaternion algebras over F together with the centre of the algebra, the centre being either a quadratic extension field of F or a sum of two copies of F. If we confine ourselves to looking at central division algebras over F with an involution of the first kind then these can only occur as the Clifford algebras of even-dimensional forms if at all.

COMMENT 3. Given a central division algebra D over F with an involution of the first kind, we may ask what is the least integer r such that $M_r D$ is the Clifford algebra of a quadratic form over F. Denote this integer by r_D. We could also define, for the field F, $r_F = \max r_D$, maximum over all central division algebras D over F with an involution of the first kind. (Possibly for some fields r_F could be infinite.)

The values of r_F and r_D depend very much on the particular field F and algebra D. The determination of r_F and r_D in general does not seem at all easy although there are a lot of results in some special cases which we will summarize later on in this article.

COMMENT 4. A natural question to ask is the following:

When is a tensor product of s quaternion algebras a division algebra?

When $s = 1$ it is so if and only if the norm form of the quaternion algebra is anisotropic [**8**]. When $s=2$ it is so if and only if a certain six-dimensional quadratic form is anisotropic [**2**]. Specifically if $Q_i=((a_i,b_i)/F)$, $i=1,2$, then this form is $\langle a_1,b_1,-a_1b_1,-a_2,-b_2,-a_2b_2\rangle$.

We may ask whether or not there is a quadratic form criterion for general s. It is easy to check that for $s=1$ an equivalent criterion to the norm form one is that $\langle a_1,b_1,-a_1b_1\rangle$ is anisotropic. This suggests the following conjecture which we are unable to prove.

CONJECTURE. $\prod_{i=1}^{s} Q_i$ is a division algebra if and only if the quadratic form

$$\sum_{i=1}^{s}(-1)^{i+1}\langle a_i,b_i,-a_ib_i\rangle,$$

of dimension $3s$, is anisotropic.

We now summarize some of the known facts about r_F and r_D.

(i) For any field F, if D has degree two or four then $r_D=1$. The degree two case follows from the fact that any such D must be a quaternion division algebra and the degree four case from the fact that D must be isomorphic to a tensor product of two quaternion division algebras. Both these results are essentially due to Albert [**1**], [**2**]. See also [**11**], [**13**]. Hence $r_F=1$ for any field in which each division algebra with involution of the first kind has degree at most four. This includes a lot of familiar fields e.g. local fields and global fields, where in fact the degree is at most two.

(ii) For any field F, if D has degree eight we have $r_D\leqslant 2$. This follows from a result of Tignol [**14**]. He proves that any central simple F-algebra A of degree eight and with an involution of the first kind must be similar to a tensor product of quaternion algebras provided that A is split by a Galois extension of rank eight and exponent two (i.e. a triquadratic extension). When A is a division algebra D it has been shown by Rowen [**12**] that this condition about the triquadratic extension is automatically satisfied. Thus M_2D is a tensor product of four quaternion algebras over F and hence a Clifford algebra over F.

(iii) There exist division algebras D for which $r_D=2$. In [**3**] an example is produced of a central division algebra D with an involution of the first kind, D being of degree eight, for which D is not a tensor product of quaternion algebras. The field F for this example is $F=\mathbb{Q}(\lambda)$, the rational functions in an indeterminate λ, $\mathbb{Q}$ being the rationals. Also in [**5**] this example is generalized to quite a few other fields of characteristic zero.

(iv) The techniques of [**3**], using abelian crossed products, leads to the construction of division algebras with involution of the first kind of degree 2^n, $n\geqslant 3$, which are not tensor products of quaternion algebras. The centre of these division algebras is a purely transcendental extension of $\mathbb{Q}$ whose degree must get larger as the value of n increases. It seems likely that as n increases the value of r_D increases. So it is to be expected that there are fields F and division algebras D with r_D arbitrarily large.

We have the following general result.

PROPOSITION 2. *Let D be a division algebra with an involution of the first kind over any field F with* char $F\neq 2$. *Let D have degree 2^k. Then* (i) *r_D is a power of two,* (ii) *if D is cyclic then $r_D\leqslant 2^{2^{k-1}-k}$.*

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Proof. $M_rD \cong \prod_{i=1}^{s} Q_i$ so that $r^2 2^{2k} = 4^s$. Hence $r = 2^{s-k}$, i.e. r is a power of two. For (ii) we appeal to a recent result of Tignol [**15**] who shows that if F contains a primitive nth root of unity then a cyclic algebra of exponent m and degree mn is similar to a tensor product of at most m symbols of degree n. In our case $n=2$, a symbol of degree 2 is a quaternion algebra, and $m=2^{k-1}$ so that $s \leqslant 2^{k-1}$ yielding the result.

COMMENT 1. For $k=1, 2$ this gives $r_D = 1$, and for $k=3$ it gives $r_D \leqslant 2$. These results are valid even without D being cyclic as we have seen earlier. (Non-cyclic division algebras exist of course. Albert [**2**] gave the first example.) For $k=4$ we obtain $r_D \leqslant 16$. We would suspect that there is a lower bound than 16 in this case.

COMMENT 2. A natural question to ask is whether r_F is related to any of the usual field invariants such as the level s_F [**8**]. While both r_F and s_F must each be a power of two, there seems no obvious relation between them except that it appears to be the case that $r_F \leqslant s_F$ for all examples we have examined. However, see note added in proof.

Finally we consider which involutions can occur. By an involution we mean an anti-automorphism of period two. On a Clifford algebra $C(q)$ of a quadratic form $q: V \to R$ we have two natural involutions J_ε, $\varepsilon = \pm 1$, induced by $J_\varepsilon(x) = \varepsilon x$ for all x in V.

Recall first the *type* of an involution as in [**6**] for example. We are interested here in simple algebras A and the type of an involution is either $+1$ or -1 according as our involution is the adjoint involution of a symmetric or skew-symmetric bilinear form. If A has dimension n^2 over F then the dimension of the subspace of A fixed by the involution is either $n(n+1)/2$ or $n(n-1)/2$ according as to whether it is of type $+1$ or -1.

PROPOSITION 3. *Let $C(q)$ be the Clifford algebra of a quadratic form of dimension n, n even. Let J_ε be the involution of $C(q)$, $\varepsilon = \pm 1$.*

If $n \equiv 0 \pmod 8$, J_ε has type $+1$ for $\varepsilon = \pm 1$.
If $n \equiv 2 \pmod 8$, J_ε has type ε.
If $n \equiv 4 \pmod 8$, J_ε has type -1 for $\varepsilon = \pm 1$.
If $n \equiv 6 \pmod 8$, J_ε has type $-\varepsilon$.

Proof. Choosing a basis $\{e_1, e_2, \ldots, e_n\}$ for the underlying space of q, we have that $C(q)$ is spanned by the 2^n possible monomials of the form $e_1^{\alpha_1} e_2^{\alpha_2} \cdots e_n^{\alpha_n}$, $\alpha_n = 0$ or 1. It is easy to see that the dimension of the subspace fixed by J_1 must equal a sum of binomial coefficients

$$S = \binom{n}{0} + \binom{n}{1} + \binom{n}{4} + \binom{n}{5} + \binom{n}{8} + \binom{n}{9} + \ldots$$

and for that fixed by J_{-1} it will be

$$S' = \binom{n}{0} + \binom{n}{3} + \binom{n}{4} + \binom{n}{7} + \binom{n}{8} + \ldots.$$

For $n \equiv 0 \pmod 8$ and $n \equiv 4 \pmod 8$, the final term in each sum is $\binom{n}{n}$ and it is easy to see that $S=S'$ from the fact that $\binom{n}{r}=\binom{n}{n-r}$, i.e. J_1 and J_{-1} are of the same type for these cases. Also, for $n\equiv 2$ or 6 (mod 8), it is easy to see that $S+S'=2^n$ and so J_1 and J_{-1} are of opposite types in these cases. To see precisely which types occur, we observe that $S=\frac{1}{2}\{2^n+(1+i)^n\}$ for $n\equiv 0$ or 4 (mod 8), and that $S=\frac{1}{2}\{2^n+\mathrm{Im}(1+i)^n\}$ for $n\equiv 2$ or 6 (mod 8).

To see this write out the binomial expansion of $(1+i)^n$ and compare with S, noting that $(1+i)^2=2i$ so that $(1+i)^n$ is entirely real for $n\equiv 0$ or 4 (mod 8) and entirely imaginary for $n\equiv 2$ or 6 (mod 8). For $n\equiv 0$ (mod 8), $(1+i)^n>0$ so that $S-2^{n-1}>0$ implying that J_1 has type $+1$. For $n\equiv 4$ (mod 8), $(1+i)^n<0$ so that $S-2^{n-1}<0$ implying that J_1 has type -1. For $n\equiv 2$ (mod 8), $\mathrm{Im}(1+i)^n>0$ so that J_1 has type $+1$ and for $n\equiv 6$ (mod 8) $\mathrm{Im}(1+i)^n<0$ so that J_1 has type -1. This completes the proof.

COMMENT. This result gives us information about the division algebras with involution (A, J) which occur as $(C(q), J_\varepsilon)$ for some q and $\varepsilon=-1$. For example if D is a division algebra of degree four which is isomorphic to a tensor product $Q_1\otimes Q_2$ of quaternion algebras and J is an involution of type $+1$, (e.g. that induced by taking the standard involution on each Q_i), then (D, J) cannot occur as $(C(q), J_\varepsilon)$ for any q since from our proposition J_ε always has type -1.

We finish off with a couple of final remarks.

REMARK 1. We have talked about division algebras with involutions of the first kind and their appearance as Clifford algebras with involution. A division algebra D over F with involution J_ε of the second kind may well occur as $(C(q), J)$ for an odd-dimensional quadratic form q over F_0 where F is a quadratic extension of F_0.

REMARK 2. Our initial interest in this question stemmed from [**9**] where we constructed exact octagons of Witt groups of hermitian forms over Clifford algebras with involution.

ADDED IN PROOF. D. Shapiro has pointed out to me that for $F=\mathbb{C}(x, y)$, $S_F=1$ while $r_F\geqslant 2$ by results in [**5**, §5].

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