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SOLUTION OF A PROBLEM OF L. FUCHS CONCERNING FINITE INTERSECTIONS OF PURE SUBGROUPS

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1. Introduction. L. Fuchs states in his book "Infinite Abelian Groups" [**6**, Vol. I, p. 134] the following

Problem 13. Find conditions on a subgroup of A to be the intersection of a finite number of pure (p-pure) subgroups of A.

The answer to this problem will be given as a special case of our theorem below. In order to find a better setting of this problem recall that a subgroup $S \subseteq E$ is p-pure if $p^nE \cap S = p^nS$ for all natural numbers. Then S is pure in E if S is p-pure for all primes p. This generalizes to p^σ-isotype, a definition due to L. J. Kulikov, cf. [**6**, Vol. II, p. 75] and [**11**, pp. 61, 62]. If σ is an ordinal, then S is p^σ-isotype if

$$p^\nu E \cap S = p^\nu S \text{ for all } \nu \leqq \sigma.$$

Obviously p^ω-isotype is purity and p^1-isotype is neatness. This concept extends to valuated abelian groups. Recall that (E, v) is a valuated abelian group if E is an abelian group and $v = \{v_p, p \text{ prime}\}$ a set of p-valuations v_p, i.e., $v_p : E \to \mathbf{O} \cup \{\infty\}$ is a map from E into the ordinals $\mathbf{O}$ and $\{\infty\}$ such that the following holds:

(1) $v_p(x + y) \geqq \min\{v_p(x), v_p(y)\}$

(2) $v_p(px) > v_p(x)$ (assume $\infty < \infty$, $\alpha < \infty$ if $\alpha \in \mathbf{Q}$),

(3) $v_p(nx) = v_p(x)$ if n is not divisible by p, c.f. [**10**].

If $h_p : E \to \mathbf{Q} \cup \{\infty\}$ is the p-height-function then

$$(E, h = \{h_p, p \text{ prime}\})$$

is a valuated group. Let

$$E(p^\nu) = \{e \in E, v_p(e) \geqq \nu\} \quad \text{and}$$

$$E[p] = \{e \in E, pe = 0\}$$

be the p-socle of E. Observe that

$$p^\nu E = E(p^\nu) \quad \text{if } v_p = h_p.$$

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We will use the following notation throughout this paper. If $S \subseteq E$ and (E, v) is a valuated group and $\nu \in \mathbf{O}$, we denote

$$E[p](p^\nu/S) := (E(p^\nu)[p] + S)/S$$

and

$$E(p^\nu/S)[p] := ((E(p^\nu) + S)/S)[p].$$

If π is a set of primes, we say that S is v_π^σ-isotype in (E, v) if

$$E[p](p^\nu/S) = E(p^\nu/S)[p]$$

for all $\nu < \sigma$ and all $p \in \pi$.

If $\pi = \{p\}$ then S is v_p^σ-isotype in E and if π is the set of all primes P, then S is v^σ-isotype in E. This notation coincides with Kulikov's definition if $v = h$, cf. [7, p. 527, Proposition 3.2]. The main result of [7, 526, Theorem] is a characterization of arbitrary intersections of v^σ-isotype groups. We recall from [7] the following

PROPOSITION. *For a subgroup $S \subseteq (E, v)$ of a valuated group and an ordinal $\sigma > 0$ the following are equivalent*:

(1) *S is an intersection of v^σ-isotype subgroups of E.*

(2) *$E(p^\nu/S)[p] \neq 0$ implies $E[p](p^\nu/S) \neq 0$ for all $p \in \mathbf{P}$ and all $\nu < \sigma$.*

If $\sigma = \omega$ and $v_p = h_p$ we derive a characterization of intersections of pure subgroups. Then (2) becomes

(2*) If $x \in E - S$ and $p^n x \in S$, $p^{n-1}x \notin S$, then there exists $y \in E$ such that $p^n y = 0$ and $p^{n-1}y \notin S$.

This case is due to D. Boyer and K. M. Rangaswamy [**3**]. It answers a question in [**5**]. Another special case of our proposition was derived independently by J. Becvár [**1**]. Other corollaries from the proposition are obtained in [**4**, **8**] and papers mentioned in [**7**]. In the case of finite intersections condition (2) must obviously be sharpened. In fact we will prove the following

THEOREM. *For a subgroup S of a valuated group (E, v) and ordinal $\sigma > 0$, a set $\pi \neq \emptyset$ of primes the following are equivalent*:

(1) *S is a finite intersection of v_π^σ-isotype subgroups of E.*

(2) *Either*

$$\dim \mathrm{E}(p^\nu/S)[p] = \dim_{\mathbf{Z}_p} E[p](p^\nu/S) \geqq \aleph_0$$

or else $\dim \mathrm{E}(p^\nu/S)[p]$ *is finite and in this case*

$$E(p^\nu/S)[p] \neq 0$$

if and only if $E[p](p^\nu/S) \neq 0$ for all primes $p \in \pi$ and ordinals $\nu < \sigma$.

If in (2) $\dim \mathrm{E}(p^\nu/S)[p]$ *is infinite or* 0 *for all $p \in \pi$ and $\nu < \sigma$, then S is an intersection of two v_π^σ-isotype subgroups of E.*

The case $\sigma = \omega$ and $v_p = h_p$ is the characterization of finite intersections of pure (p-pure) subgroups which solves L. Fuchs' problem. If $\sigma = 1$ and $v_p = h_p$ the theorem characterizes finite intersections of neat subgroups. Moreover we have a stronger

COROLLARY. *For a subgroup S of a valuated group (E, v), a set $\pi \neq \emptyset$ of primes and an integer $n \geqq 2$ the following are equivalent*:

(1) *S is an intersection of n v_π^1-isotype subgroups of (E, v).*

(2) $\dim(E/S)[p] \leqq n \cdot \dim(E[p] + S)/S$.

In the case $v_p = h_p$ the v_π^1-isotype subgroups of E are called π-neat subgroups. Therefore the corollary characterizes intersections of n neat subgroups. This was recently shown by K. Benabdallah and S. Robert [**2**].

The strategy of the proof consists of two parts. First, in Section 4 we transform the major burden of the problem into linear algebra and will solve a problem on double-filtered vector spaces (Section 3).

In Section 5 we put all pieces together and prove the theorem mentioned above. Finally we will construct a subgroup $S \subseteq E$ which is an intersection of pure subgroups but not of finitely many pure subgroups.

2. Definitions. Let $\mathbf{O}$ be the class of all ordinals with the natural well-ordering. If σ is an ordinal, we will identify σ with the set of all ordinals $\alpha < \sigma$. In particular $\alpha < \sigma$ if and only if $\alpha \in \sigma$. A cardinal κ is identified with the ordinal

$$\inf\{\alpha \in \mathbf{O}, |\alpha| = \kappa\}.$$

If α is an ordinal then

$$\operatorname{cof} \alpha = \inf\{ |X|, X \subseteq \alpha, \sup X = \alpha\}.$$

For ordinals $\alpha \leqq \beta$ we will consider the open and closed intervals

$$(\alpha, \beta) = \{\gamma \in \mathbf{O}, \alpha < \gamma < \beta\}$$

respectively

$$[\alpha, \beta] = \{\gamma \in \mathbf{O}, \alpha \leq \gamma \leq \beta\}$$

and the intervals

$$[\alpha, \beta] = \{\gamma \in \mathbf{O}, \alpha \leqq \gamma < \beta\}$$

and

$$(\alpha, \beta] = \{\gamma \in \mathbf{O}, \alpha < \gamma \leqq \beta\}.$$

If V is a vector space over a field F, then the dimension $\dim_F V = \dim V$ of V is a cardinal.

Thus $\langle X \rangle \subseteq V$ denotes the subspace of V generated by $X \subseteq V$. The

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same notation will be used for groups. The letter p always denotes a prime and $\mathbf{Z}(p) = \mathbf{Z}/p\mathbf{Z}$ the cyclic group (respectively the field) of p elements. All other notations are also standard and can be found in [**6**].

3. Some results on linear algebra. Let V be a vector space over the field F and $\mu < \sigma$ two ordinals. A family

$$\mathscr{F}_1 = \{V_\alpha, \alpha \in [\mu, \sigma)\}$$

of subspaces $V_\alpha \subseteq V$ with $V_\beta \subseteq V_\alpha$ for all $\alpha \leqq \beta$ is a $[\mu, \sigma)$-filtration on V and the pair $(V, \mathscr{F}_1)$ will be a filtered vector space on $[\mu, \sigma)$.

If $\mathscr{F}^2 = \{V^\alpha, \alpha \in [\mu, \sigma)\}$ is another $[\mu, \sigma)$-filtration on V, then $\mathscr{F}_1 \subseteq \mathscr{F}^2$ if $V_\alpha \subseteq V^\alpha$ for all $\alpha \in [\mu, \sigma)$.

A vector space V with two comparable filtrations $\mathscr{F}_1 \subseteq \mathscr{F}^2$ will be called a *double-filtered* vector space on $[\mu, \sigma)$. We denote this space by $(V, \mathscr{F}_1, \mathscr{F}^2)$ and we will always use the notation $V_\alpha \in \mathscr{F}_1$ and $V^\alpha \in \mathscr{F}^2$ as above. If $\mu = 0$ we replace $[\mu, \sigma)$ by σ. Motivated by abelian p-groups, we will use the following

Definition 3.1. Let $(V, \mathscr{F}_1, \mathscr{F}^2)$ be a double filtered vector space on $[\mu, \sigma)$, U a subspace of V and $\mu \leqq \alpha \leqq \beta \leqq \sigma$. Then U will be called *dense* on $[\alpha, \beta)$ if

$$V^\alpha \subseteq V_\delta + U$$

for all $\delta \in [\alpha, \beta)$.

The subspace U will be called *piece-wise dense* on $[\alpha, \beta)$ if there is a finite chain $\alpha = \alpha_0 \leqq \alpha_1 \leqq \ldots \leqq \alpha_{n+1} = \beta$ such that U is dense on (α_i, α_{i+1}) for all $i \in [0, n]$.

The main result of this section will be a theorem on double-filtered vector spaces and piece-wise dense subspaces:

THEOREM 3.2. *Let $(V, \mathscr{F}_1, \mathscr{F}^2)$ be a double-filtered vector space on σ such that*

(a) $\dim V_\alpha \geqq \aleph_0$ *implies* $\dim V^\alpha = \dim V_\alpha$ *for all* $\alpha \in [0, \sigma)$

(b) $0 \neq \dim V^\alpha < \aleph_0$ *implies* $V_\alpha \neq 0$.

Then we can find finitely many subspaces U^j $(j \in [1, n])$ such that

(1) *U^j is piece-wise dense on* $[0, \sigma]$

(2) $\bigcap_{j \in [1,n]} U^j = 0$.

If $\dim V_\alpha \geqq \aleph_0$ *or* $= 0$ *for all* $\alpha \in \sigma$ *then we can choose* $n = 2$.

This will follow from a sequence of Lemmata. We begin with a trivial observation which is used several times.

Observation 3.3. *Let $(V, \mathscr{F}_1, \mathscr{F}^2)$ be a double-filtered vector space on σ and $\mu \in [0, \sigma)$ such that*

$$\dim V^{\mu} < \dim V_{\alpha} = \dim V^{\alpha} \geqq \aleph_0 \quad \textit{for all } \alpha \in [0, \mu).$$

Then

$$\dim V^{\alpha}/V^{\mu} = \dim(V_{\alpha} + V^{\mu})/V^{\mu} \quad \textit{for all } \alpha \in [0, \mu).$$

Proof. If $\mu = 0$, then $[0, \mu) = \emptyset$ and (3.3) holds trivially. Therefore let $\mu > 0$ and $\alpha \in [0, \mu)$. Since $\dim V^{\mu} < \dim V_{\alpha}$ and $\dim V_{\alpha}$ is infinite, we obtain from a simple cardinal argument that

$$\dim(V_{\alpha} + V^{\mu})/V^{\mu} = \dim V_{\alpha} = \dim V^{\alpha} = \dim(V^{\alpha}/V^{\mu}).$$

LEMMA 3.4. *Let U be a subspace of the vector space V such that* $\dim V \leqq n \cdot \dim U$ *for some integer $n \geqq 2$. Then we can find n complements U^j of U in V for $j \in [1, n]$ such that*

$$\bigcap_{j \in [1,n]} U^j = 0.$$

Proof. Case $n = 2$. Let $V = U^1 \oplus U$ by any decomposition and B any basis of U^1. We may assume $U^1 \neq 0$ and hence $B \neq \emptyset$. Since $\dim V \leqq 2 \dim U$, also

$$|B| = \dim U^1 \leqq \dim U.$$

Therefore we can find a linearly independent subset $\{b^* | b \in B\}$ of U. Now we choose

$$U^2 = \bigoplus_{b \in B} \langle b + b^* \rangle.$$

Obviously

$$U^2 \oplus U = U^1 \oplus U = V \quad \text{and} \quad U^1 \cap U^2 = 0.$$

We now proceed by induction on $n \geqq 3$ and assume that (3.4) holds for all $n' < n$.

If $U \subseteq V$ such that $\dim V \leqq n \cdot \dim U$, choose any decomposition $V = C \oplus U$. Therefore

$$\dim C \leqq (n - 1) \dim U.$$

If $\dim C \leqq \dim U$ we have

$$\dim V \leqq 2 \cdot \dim U.$$

Therefore (3.4) follows from the case $n = 2$ above. Thus we assume $\dim U \leqq \dim C$ and decompose $C = D \oplus E$ such that $\dim U = \dim D$. We conclude

$$\begin{aligned} \dim(U \oplus E) &= \dim U + \dim E = \dim D + \dim E \\ &= \dim C \leqq (n - 1) \dim U. \end{aligned}$$

Therefore

$$\dim(U \oplus E) \leqq (n - 1) \dim U$$

and we can apply induction for $U \subseteq U + E$. There are decompositions

$$U \oplus E = V^j \oplus U \quad \text{for } j \in [1, n - 1]$$

such that

$$\bigcap_{j \in [1,n-1]} V^j = 0.$$

Let

$$D = \bigoplus_{b \in B} \langle b \rangle$$

and from

$$|B| = \dim D = \dim U$$

we have a linearly independent subset $\{b^* | b \in B\}$ of U. Choose

$$U^n = \bigoplus_{b \in B} \langle b + b^* \rangle \oplus E \quad \text{and}$$

$$U^j = D \oplus V^j \quad \text{for } j \in [1, n - 1].$$

Therefore $U \oplus U^j = V$ for all $j \in [1, n]$ and

$$\bigcap_{j \in [1,n-1]} U^j = D$$

implies

$$\bigcap_{j \in [1,n]} U^j = D \cap U^n = 0.$$

LEMMA 3.5. *Let $(V, \mathscr{F}_1, \mathscr{F}^2)$ be a double-filtered vector space on $[\mu, \sigma)$ such that*
(a) $0 \neq \dim V^\mu < \aleph_0$
(b) $V^\alpha \neq 0$ *implies* $V_\alpha \neq 0$ *for all* $\alpha \in [\mu, \sigma)$.
Then we can find finitely many subspaces $U^j \subseteq V^\mu$ which are dense on $[\mu, \sigma)$ with

$$\bigcap_j U^j = 0.$$

Proof. Let $\mu \leqq \rho \leqq \nu \leqq \sigma$ such that:
(i) $V_\alpha = 0$ for all $\alpha \in [\nu, \sigma)$ and ν minimal
(ii) $\dim V_\alpha = \dim V_\beta$ for all $\alpha, \beta \in [\rho, \nu)$.
If $y \in V_\rho$ and $\dim V^\mu = n$, we apply (3.4) and find n subspaces U^j such that

$$\langle y \rangle \oplus U^j = V^\mu \quad \text{and} \quad \bigcap_{j \in [1,n]} U^j = 0.$$

From $\alpha \in [\mu, \nu)$ we have $y \in V_\alpha$ and therefore

$$V^{\mu} = \langle y\rangle \oplus U^{j} = V_{\alpha} + U^{j}.$$

If $\alpha \in [\nu, \sigma)$ ($\neq \emptyset$) then $V_{\alpha} = 0$ by (i) and $V^{\alpha} = 0$ from (b). Therefore $V^{\alpha} \subseteq V_{\alpha} + U^{j}$ holds trivially in this case and U^{j} is dense on $[\mu, \sigma)$ for all $j \in [1, n]$.

LEMMA 3.6. *If $\aleph_0 \leqq \rho \leqq \kappa$ are cardinals then there is a decomposition $\{\kappa_{\beta}, \beta \in \kappa\}$ of κ into κ many subsets κ_{β} of cardinality ρ such that the following holds*:

(*) *If E is a finite subset of κ and*

$$E \subseteq \bigcup_{\beta\in E} \kappa_{\beta},$$

then $E = \emptyset$.

Proof. Let $f:\kappa \to \kappa$ be a bijection such that f has only infinite cycles. Then κ decomposes into κ cycles $\mathbf{Z}_i (i \in \kappa)$ such that f acts on $\mathbf{Z}_i = \{z_i, z \in \mathbf{Z}\}$ as

$$f(z_i) = (z + 1)_i \quad \text{for all } z \in \mathbf{Z}.$$

Since $|\kappa \times \rho| = \kappa$, there is a bijection

$$\nu:\kappa \times \rho \to \kappa$$

and the canonical projection

$$\pi:\kappa \times \rho \to \kappa(\,(k, r) \to k)$$

defines a trivial fibration on $\kappa \times \rho$ with fibres isomorphic to ρ. This is used to define an induced fibration on κ with κ many fibres κ_{β} ($\beta \in \kappa$) each of cardinality ρ. Let

$$\bar{\gamma} = \{\beta \in \kappa | \pi\nu^{-1}\beta = \gamma\} \quad \text{and}$$

$$\bar{\kappa}_{\gamma} = \nu(f^{-1}\gamma \times \rho) \quad \text{for all } \gamma \in \kappa.$$

Then

$$\kappa = \dot{\bigcup_{\gamma\in\kappa}} \bar{\gamma} = \dot{\bigcup_{\gamma\in\kappa}} \bar{\kappa}_{\gamma} \quad \text{and} \quad |\bar{\gamma}| = |\bar{\kappa}_{\gamma}| = \rho.$$

Therefore we can decompose

$$\bar{\kappa}_{\gamma} = \dot{\bigcup_{\beta\in\bar{\gamma}}} \kappa_{\beta}$$

such that $|\kappa_{\beta}| = \rho$ for all $\beta \in \bar{\gamma}$ and $\gamma \in \kappa$. Hence we have

$$\kappa = \dot{\bigcup_{\beta\in\kappa}} \kappa_{\beta}$$

such that

$$\kappa_{\beta} \subseteq \nu(\,(f^{-1}\pi\nu^{-1}\beta) \times \rho) \quad \text{and}$$

$$|\kappa_\beta| = \rho \quad \text{for all } \beta \in \kappa.$$

In order to show (*) consider $E \subseteq \kappa$ such that

$$E \subseteq \bigcup_{\beta \in E} \kappa_\beta.$$

By definition of κ_β we have

$$E \subseteq \bigcup_{\beta \in E} \nu((f^{-1}\pi\nu^{-1}\beta) \times \rho).$$

Application of ν^{-1} and $\alpha = \nu^{-1}\beta$ leads to

$$\nu^{-1}E \subseteq \bigcup_{\beta \in E} (f^{-1}\pi\nu^{-1}\beta) \times \rho = \bigcup_{\alpha \in \nu^{-1}E} f^{-1}\pi\alpha \times \rho.$$

If $F = \nu^{-1}E$ and $\pi\alpha = \gamma$, then

$$F \subseteq \bigcup_{\alpha \in F} f^{-1}\pi\alpha \times \rho = \bigcup_{\gamma \in \pi F} f^{-1}\gamma \times \rho.$$

Now suppose that E is a finite non-empty set. Then $F \neq \emptyset$ is finite and also πF is a non-empty finite subset of κ. Since

$$\kappa = \bigcup_{i \in \kappa} \mathbf{Z}_i$$

we find a largest integer $z \in \mathbf{Z}$ such that $z_i \in \pi F$ for some $i \in \kappa$. In particular there is an $r \in \rho$ such that $(z_i, r) \in F$. We want to show that

$$(z_i, r) \notin \bigcup_{\gamma \in \pi F} f^{-1}\gamma \times \rho$$

which contradicts

$$F \subseteq \bigcup_{\gamma \in \pi F} f^{-1}\gamma \times \rho$$

and (*) is shown. If

$$(z_i, r) \in f^{-1}\gamma \times \rho \quad \text{for some } \gamma \in \pi F,$$

then $\gamma = w_j \in \mathbf{Z}_j$ for some $j \in \kappa$ and $w \leqq z$ by the maximality of z. However

$$z_i = f^{-1}\gamma = (w - 1)_j$$

leads to the contradiction $z = w - 1 < z$.

The proof of the following lemma is similar to [**9**, Lemma 15, pp. 318, 319].

LEMMA 3.7. *Let $(V, \mathcal{F}_1)$ be a σ-filtered vector space and $\nu < \lambda \leqq \sigma$ with λ a limit ordinal such that* $\dim V_\beta = \kappa \geqq \aleph_0$ *for all $\beta \in [\nu, \lambda)$ and*

$$\dim \bigcap_{\beta \in [\nu,\lambda)} V_\beta < \kappa.$$

Then we can find a disjoint family $\{X_\alpha, \alpha \in \kappa\}$ *of subsets of* V *with*

(i) $|X_\alpha| = \mathrm{cf}(\lambda)$ *for all* $\alpha \in \kappa$

(ii) $|X_\alpha \cap V_\beta| = \mathrm{cf}(\lambda)$ *for all* $\alpha \in \kappa$ *and* $\beta \in [\nu, \lambda)$

(iii) $\dot{\bigcup}_{\alpha \in \kappa} X_\alpha$ *is linearly independent.*

Proof. If

$$\dim \bigcap_{\beta \in [\nu,\lambda)} V_\beta < \kappa \quad \text{and} \quad \dim V_\beta = \kappa$$

then $\mathrm{cf}(\lambda) \leqq \kappa$ and we can find a sequence

$$X = \{x_\alpha | \alpha \in \mathrm{cf}(\lambda)\}$$

with

(a) $|X \cap V_\beta| = \mathrm{cf}(\lambda)$ for all $\beta \in [\nu, \lambda)$

(b) X is linearly independent and $|X| = \mathrm{cf}(\lambda)$.

Since $\mathrm{cf}(\lambda) \leqq \kappa$ either $\kappa = \mathrm{cf}(\lambda)$ or $\mathrm{cf}(\lambda) < \kappa$. In the first case we can choose κ disjoint subsequences of X with (a) and (b). Hence (3.7) is shown in the first case.

Now assume $\mathrm{cf}(\lambda) < \kappa$ and consider the sets

$$M = \{X \subseteq V, X \text{ satisfies (a) and (b)}\}$$

such that different $X, X' \in M$ are disjoint and

$$\bigcup_{X \in M} X$$

is linearly independent.

Let $\mathfrak{M}$ be the collection of all these sets M. Then $\{X\} \in \mathfrak{M}$ as shown above and $\mathfrak{M}$ is obviously an inductive set. From the maximum principle of set theory we obtain a maximal element $M \in \mathfrak{M}$. If $|M| = \kappa$ the lemma is shown. Therefore we assume $|M| < \kappa$ for contradiction. Since

$$\dim U < \kappa \quad \text{for } U = \langle \bigcup_{X \in M} X \rangle,$$

we derive a λ-filtered vector space

$$(V/U, \{\bar{V}_\alpha = (V_\alpha + U)/U, \alpha \in \lambda\})$$

with

$$\dim \bar{V}_\alpha = \kappa \quad \text{for all } \alpha \in [\nu, \lambda].$$

Therefore we can choose a sequence

$$\bar{X} = \{0 \neq x_\alpha + U, \alpha \in \mathrm{cf}(\lambda)\}$$

of V/U with the properties

(a′) $|\bar{X} \cap \overline{V_\beta}| = \mathrm{cf}(\lambda)$ for all $\beta \in [\nu, \lambda)$
(b′) $\bar{X}$ is linearly independent and $|\bar{X}| = \mathrm{cf}(\lambda)$.
From (a′) we obtain a subset $\{\xi_\alpha, \alpha \in \mathrm{cf}(\lambda)\}$ of $[\nu, \lambda)$ with

$$\sup\{\xi_\alpha, \alpha \in \mathrm{cf}(\lambda)\} = \lambda \quad \text{and} \quad x_\alpha + U \in \overline{V_{\xi\alpha}}.$$

Let $x_\alpha + U = x'_\alpha + U$ with $x'_\alpha \in V_{\xi\alpha}$, then

$$M \,\dot\cup\, \{x'_\alpha, \alpha \in \mathrm{cf}(\lambda)\}$$

contradicts the maximality of M.

LEMMA 3.8. *Let* $(V, \mathscr{F}_1, \mathscr{F}^2)$ *be a double-filtered vector space on* σ *such that* $\dim V_\alpha = \dim V^\alpha$ *is* 0 *or infinite for all* $\alpha \in [0, \sigma)$. *Let* $\mu \in \sigma$ *be the smallest ordinal with* $\dim V_\alpha = \kappa$ *for all* $\alpha \in [\mu, \sigma)$. *Then*

(i) $\dim(V_\alpha + V^\mu)/V^\mu = \dim V^\alpha/V^\mu$ *for all* $\alpha \in [0, \mu)$
(ii) *There are two subspaces* X^1, X^2 *of* V^μ *such that*
(a) X^1 *and* X^2 are dense on $[\mu, \sigma)$
(b) $X^1 \cap X^2 = 0$.

Proof. (i). Since $\dim V^\mu < \dim V_\alpha = \dim V^\alpha \geqq \aleph_0$, we apply (3.3) and derive (i).
(ii). Let

$$D = \bigcap_{\beta \in [\mu,\sigma)} V_\beta$$

and consider first
Case 1. $\dim D = \kappa$.
Since $\dim D = \kappa = \dim V_\mu = \dim V^\mu$ and $D \subseteq V^\mu$ there are two complements X^1 and X^2 of D in V^μ such that $X^1 \cap X^2 = 0$. Therefore (ii) (b) holds. If $i \in \{1, 2\}$, then

$$V^\mu = X^i \oplus D \quad \text{and} \quad V_\delta \supseteq \bigcap_{\beta \in [\mu,\sigma)} V_\beta = D$$

for all $\delta \in [\mu, \sigma)$ implies

$$V^\mu = X^i + V_\delta.$$

Hence X^1 and X^2 are dense on $[\mu, \sigma)$ and (ii) (a) is shown; compare (3.1).
Case 2. $\dim D < \kappa$.
If $\sigma = \delta + 1$, then

$$\dim D = \dim \bigcap_{\beta \in [\mu,\delta+1)} V_\beta = \dim V_\delta = \kappa,$$

which is excluded. Hence σ is a limit ordinal.
From (3.6) we obtain a disjoint family

$$\{X_\alpha \subseteq V, \alpha \in \kappa\}$$

such that

(*) $|X_\alpha| = \mathrm{cf}(\sigma)$ for all $\alpha \in \kappa$.

(**) $|X_\alpha \cap V_\beta| = \mathrm{cf}(\sigma)$ for all $\alpha \in \kappa$ and $\beta \in [\mu, \sigma)$

(***) $\dot\cup X_\alpha$ is linearly independent.

Now we decompose

$$V^\mu = \bigoplus_{\alpha \in \kappa} \langle X_\alpha \rangle \oplus C$$

where

$$C = \bigoplus_{\alpha \in \bar\kappa} \langle c_\alpha \rangle$$

is an arbitrary complement. Since

$$\dim V^\mu = \dim V_\mu = \kappa,$$

also $\dim C = \bar\kappa \leqq \kappa$. Consider the family

$$Y_\alpha = \begin{cases} X_\alpha \text{ for } \alpha \in \kappa \setminus \bar\kappa \\ X_\alpha \cup \{c_\alpha\} \text{ for } \alpha \in \bar\kappa. \end{cases}$$

Then the following conditions are obviously satisfied:

$(+)$ $|Y_\alpha \cap V_\beta| = \mathrm{cf}(\sigma)$ for all $\alpha \in \kappa$ and $\beta \in [\mu, \sigma)$

$(++)$ $Y_\alpha \cap Y_{\alpha'} = \emptyset$ for different $\alpha, \alpha' \in \kappa$ and $|Y_\alpha| = \mathrm{cf}(\lambda)$ for all $\alpha \in \kappa$.

$(+++)$ $\dot\cup Y_\alpha$ is linearly independent and $V^\mu = \langle Y_\alpha, \alpha \in \kappa \rangle$.

Therefore we can decompose $Y_\alpha = Y_\alpha^1 \cup Y_\alpha^2$ such that

$$|Y_\alpha^1| = |Y_\alpha^2| = \mathrm{cf}(\sigma) \quad \text{and}$$

$$|Y_\alpha^i \cap V_\beta| = \mathrm{cf}(\sigma) \quad \text{for all } \alpha \in \kappa \text{ and } \beta \in [\mu, \sigma).$$

Let

$$Y_\alpha^i = \{ (i, \alpha, \eta), \eta \in \mathrm{cf}(\sigma) \}$$

be an enumeration of Y_α^i for all $\alpha \in \kappa$. Also decompose

$$\kappa = \dot\bigcup_{\alpha \in \kappa} \kappa_\alpha$$

with $|\kappa_\alpha| = \mathrm{cf}(\sigma)$ and enumerate

$$\kappa_\alpha = \{\alpha_\eta, \eta \in \mathrm{cf}(\sigma) \}$$

with the help of (3.3). Then the following holds:

(****) If E is a finite subset of κ and

$$E \subseteq \bigcup_{\alpha \in E} \kappa_\alpha$$

then $E = \emptyset$. Finally we let

$$I = \kappa \times \mathrm{cf}(\sigma), \quad I^+ = \{ (\beta, \delta) \in I | \delta \neq 0\}$$

and define

$$X^1 = \bigoplus_{(\alpha,\eta)\in I} \langle (1, \alpha, \eta) + (2, \alpha_\eta, 0) \rangle \oplus \bigoplus_{(\beta,\delta)\in I^+} \langle (2, \beta, \delta) - (2, \beta, 0) \rangle$$

and

$$X^2 = \bigoplus_{(\alpha,\eta)\in I} \langle (2, \alpha, \eta) + (1, \alpha_\eta, 0) \rangle \oplus \bigoplus_{(\beta,\delta)\in I^+} \langle (1, \beta, \delta) - (1, \beta, 0) \rangle.$$

We want to show that X^1 and X^2 satisfy (ii) (b). Therefore assume $X^1 \cap X^2 \neq 0$ for contradiction. Then we can find subsets $E_1, E_1' \subseteq I$ and $E_2, E_2' \subseteq I^+$ such that the following holds.

$$\sum_{(\alpha,\eta)\in E_1} (\alpha, \eta)^*((1, \alpha, \eta) + (2, \alpha_\eta, 0)) + \sum_{(\beta,\delta)\in E_2} (\beta, \delta)^{**}((2, \beta, \delta) - (2, \beta, 0))$$
$$= \sum_{(\alpha,\eta)\in E_1'} (\alpha, \eta)'((2, \alpha, \eta) + (1, \alpha_\eta, 0)) + \sum_{(\beta,\delta)\in E_2'} (\beta, \delta)''((1, \beta, \delta) - (1, \beta, 0))$$

with elements $(\alpha, \eta)^*$, $(\alpha, \eta)'$, $(\beta, \delta)^{**}$ and $(\beta, \delta)''$ in the field F and different from 0 and $E_1 \cup E_2 \neq \emptyset$.

Since

$$\langle Y^1_\alpha, \alpha \in \kappa\rangle \cap \langle Y^2_\alpha, \alpha \in \kappa\rangle = 0,$$

we derive two equations

$$0 = \sum_{(\alpha,\eta)\in E_1} (\alpha, \eta)^*(1, \alpha, \eta) - \sum_{(\beta,\delta)\in E_2'} (\beta, \delta)''(1, \beta, \delta) + \sum_{(\beta,\delta)\in E_2'} (\beta, \delta)''(1, \beta, 0) - \sum_{(\alpha,\eta)\in E_1} (\alpha, \eta)'(1, \alpha_\eta, 0)$$

and

$$0 = \sum_{(\alpha,\eta)\in E_1'} (\alpha, \eta)'(2, \alpha, \eta) - \sum_{(\beta,\delta)\in E_2} (\beta, \delta)^{**}(2, \beta, \delta) + \sum_{(\beta,\delta)\in E_2} (\beta, \delta)^{**}(2, \beta, 0) - \sum_{(\alpha,\eta)\in E_1} (\alpha, \eta)^*(2, \alpha_\eta, 0).$$

Let

$$G_1 = E_1 \cap (\kappa \times 0) \quad \text{and} \quad E_1 = G_1 \,\dot{\cup}\, H_1$$

and similarly

$$G'_1 = E'_1 \cap (\kappa \times 0) \quad \text{and} \quad E'_1 = G'_1 \,\dot{\cup}\, H'_1.$$

From the last two equations we derive

$$\sum_{H_1} (\alpha, \eta)^*(1, \alpha, \eta) - \sum_{E'_2} (\beta, \delta)''(1, \beta, \delta)$$

$$= \sum_{E_1} (\alpha, \eta)'(1, \alpha_\eta, 0) - \sum_{G_1} (\alpha, 0)^*(2, \alpha, 0) - \sum_{E'_2} (\beta, \delta)''(1, \beta, 0)$$

and

$$\sum_{H'_1} (\alpha, \eta)'(2, \alpha, \eta) - \sum_{E_2} (\beta, \delta)^{**}(2, \beta, \delta)$$

$$= \sum_{E_1} (\alpha, \eta)^*(2, \alpha_\eta, 0) - \sum_{G'_1} (\alpha, 0)'(2, \alpha, 0) - \sum_{E_2} (\beta, \delta)^{**}(2, \beta, 0)$$

with the obvious summation parameters.

Since

$$\langle (i, \alpha, \eta), (\alpha, \eta) \in I^+ \rangle \cap \langle (i, \alpha, 0), \alpha \in \kappa \rangle = 0 \quad \text{for } i = 1, 2,$$

we conclude $H_1 = E'_2$ and $H'_1 = E_2$. Let

$$S = \{\alpha_\eta, (\alpha, \eta) \in E_1\},$$

$$S' = \{\alpha_\eta, (\alpha, \eta) \in E'_1\} \quad \text{and}$$

$$\iota : I \rightarrow \kappa(\,(\alpha, \eta) \rightarrow \alpha)$$

the canonical projection. Then

$$S' \subseteq G_1^\iota \cup E_2'^\iota \quad \text{and} \quad S \subseteq G_1'^\iota \cup E_2^\iota$$

follows also from the last two equations. Hence we derive

$$S' \subseteq G_1^\iota \cup E_2'^\iota = G_1^\iota \cup H_1^\iota = (G_1 \cup H_1)^\iota = E_1^\iota \quad \text{and}$$

$$S \subseteq G_1'^\iota \cup E_2^\iota = G_1'^\iota \cup H_1'^\iota = (G'_1 \cup H'_1)^\iota = E_1'^\iota.$$

Therefore

$$|E'_1| = |S'| \leqq |E_1^\iota| \leqq |E_1| = |S| \leqq |E_1'^\iota| \leqq |E'_1|$$

implies $S' = E_1$ and $S = E'_1$. Finally we consider

$$S \cup S' = E_1^\iota \cup E_1'^\iota = (E_1 \cup E'_1)^\iota \subseteq \bigcup_{\alpha \in (E_1 \cup E'_1)^\iota} \kappa_\alpha$$

and derive from (***) that $E_1 \cup E'_1 = \emptyset$. Therefore also $E_2 = E'_2 = \emptyset$ and $E_1 \cup E_2 = \emptyset$ contradicts our choice of these sets.

Finally we show (ii) (a) and let $\gamma \in [\mu, \sigma)$; compare (3.1). We will restrict our consideration to X^1. A similar argument holds for X^2. We want to show that

$$V^\mu = V_\gamma + X^1.$$

From $(+++)$ we see that it suffices to show that

$$(i, \alpha, \beta) \in V_\gamma + X^1 \text{ for all } i \in \{1, 2\}, (\alpha, \beta) \in I.$$

From

$$|Y^i_\alpha \cap V_\beta| = \text{cf}(\sigma)$$

we obtain that $Y^2_\alpha \cap V_\gamma \neq \emptyset$ and we find some $(2, \alpha, \delta) \in Y^2_\alpha \cap V_\gamma$. We want to show that

$$(2, \alpha, 0) \in V_\gamma + X^1.$$

Therefore we have finished if $\delta = 0$ and assume $\delta \neq 0$. Hence

$$(2, \alpha, \delta) - (2, \alpha, 0) \in X^1 \quad \text{and}$$

$$(2, \alpha, 0) = (2, \alpha, \delta) - ((2, \alpha, \delta) - (2, \alpha, 0)) \in V_\gamma + X_1.$$

If $\beta \neq 0$ also

$$(2, \alpha, \beta) - (2, \alpha, 0) \in X^1 \quad \text{and}$$

$$(2, \alpha, \beta) = (2, \alpha, 0) - ((2, \alpha, \beta) - (2, \alpha, 0)) \in V_\gamma + X^1$$

and $(1, \alpha, \beta) \in V_\gamma + X^1$ follows from

$$(1, \alpha, \beta) + (2, \alpha_\beta, 0) \in X^1.$$

Consequently X^1 is dense on $[\mu, \sigma)$ and (3.8) is shown.

Proof of Theorem 3.2. We use transfinite induction on σ. If $\sigma = 1$, then (3.4) implies (3.2). Therefore let σ be an ordinal > 1 such that (3.2) holds for all vector spaces on μ with $\mu < \sigma$. Now let μ be the smallest ordinal such that one of the following three conditions holds:

(i) If $\dim V_\alpha \geqq \aleph_0$, then $\dim V_\alpha = \text{const.}$ for all $\alpha \in [\mu, \sigma)$.

(ii) If there is $\mu' \in [0, \sigma)$ such that $V_{\mu'} = 0$, then $V_\alpha = 0$ for all $\alpha \in [\mu, \sigma)$.

(iii) If there is $\mu' \in [0, \sigma)$ such that

$$0 \neq \dim V_{\mu'} < \aleph_0$$

then $\dim V_\alpha < \aleph_0$ for all $\alpha \in [\mu, \sigma)$.

Now we apply (3.5) and (3.8). If $\dim V_\mu \geqq \aleph_0$ or 0 we find $n = 2$ and in general a finite number n of subspaces $X^j (j \in [1, n])$ which are dense on $[\mu, \sigma)$ and

$$\bigcap_{j \in [1,n]} X^j = 0$$

as shown in (3.8) and (3.5) respectively.

Now we consider factor spaces $\bar{Y} = (Y + V^{\mu})/V^{\mu}$ for $Y \subseteq V$. In particular

$$(\bar{V}, \{\bar{V}_{\alpha}, \alpha \in [0, \mu)\}, \{\bar{V}^{\alpha}, \alpha \in [0, \mu)\})$$

is a (new) double-filtered vector space. By our choice of μ and (3.3) we derive

$$\dim \bar{V}_{\alpha} = \dim \bar{V}^{\alpha} \geqq \aleph_0 \quad \text{for all } \alpha \in [0, \mu).$$

From the induction hypothesis we find two subspaces $W^j \subseteq V$ such that $W^j \cap V^{\mu}, = 0$, $\bar{W}^j$ is piece-wise dense on $[0, \mu)$ for $j = 1, 2$ and $\bar{W}^1 \cap \bar{W}^2 = 0$. If $V = W \oplus V_{\mu}$, then also $\bar{W}$ is dense on $[0, \mu)$ and we choose $U^j = X^j + W^j$ for $j \in [1, n]$ where $W^j = W$ for $j \in [3, n]$ and $n \geqq 3$.

Next we will show (3.2) (1) and consider a subspace U^j. Since X^j and $\bar{W}^j$ are piece-wise dense on $[\mu, \sigma)$ respectively on $[0, \mu)$, we find an increasing finite sequence $\{\nu_i, i \in [0, s]\}$ of ordinals such that $\nu_0 = 0$, $\nu_m = \mu$ for some $m \in [0, s]$ and $\nu_s = \sigma$ from Definition 3.1. We want to show that U^j is dense on all intervals $[\nu_i, \nu_{i+1})$. This is trivial for $i \geqq m$. Therefore let $i < m$, $\alpha \in [\nu_i, \nu_{i+1})$ and $a \in V^{\nu i}$. We have finished if $a \in U^j + V_{\alpha}$ [compare (3.1)]. Since

$$a + V^{\mu} \in V^{\nu i} \subseteq \bar{W}^j + \bar{V}_{\alpha},$$

we find $W^j \in w^j$, $v_{\alpha} \in V_{\alpha}$ and $v^{\mu} \in V^{\mu}$ such that

$$a = w^j + v_{\alpha} + v^{\mu}.$$

From $\alpha < \mu$ we have

$$V^{\mu} \subseteq X^j + V_{\alpha}$$

and therefore

$$a \in W^j + V_{\alpha} + V^{\mu} \subseteq W^j + X^j + V_{\alpha} = U^j + V_{\alpha}.$$

Finally we want to show (3.2) (2) that is

$$\bigcap_{j \in [1,n]} U^j = 0.$$

If

$$D = \bigcap_{j \in [1,n]} U^j \quad \text{and} \quad a \in D,$$

then

$$a + V^{\mu} \in \bigcap_{j \in [1,n]} \overline{(X^j + W^j)} = \bigcap_{j \in [1,n]} \overline{X^j} = 0$$

and therefore $a \in V^{\mu}$. Since $a \in D$, we find $x^j \in X^j$ and $w^j \in W^j$

such that $a = x^j + w^j$ for all $j \in [1, n]$. Because $a \in V^\mu$, $X^j \subseteq V^\mu$ and $W^j \cap V^\mu = 0$, we obtain

$$a - x^j = w^j \in V^\mu \cap W^j = 0 \quad \text{and} \quad a = x^j.$$

Therefore also

$$a \in \bigcap_{j \in [1,n]} X^j = 0$$

by our choice of X^j. Consequently $a = 0$ and $D = 0$.

4. The link between valuated abelian groups and double-filtered vector spaces. In this section we will restrict ourselves to vector spaces over the field $F = \mathbf{Z}(p)$ and these are related to abelian groups as follows.

Let (E, v) always denote a valuated abelian group, $S \subseteq E$ a fixed subgroup of E and $\sigma > 0$ some ordinal. The $\mathbf{Z}(p)$-vector space $(E/S)[p]$ has two natural comparable σ-filtrations

$$\mathscr{F}_1 = \{V_\alpha = E[p](p^\alpha/S) \,|\alpha < \sigma\}$$

and

$$\mathscr{F}_2 = \{V^\alpha = E(p^\alpha/S)[p] \,|\alpha < \sigma\}$$

compare Sections 2 and 3.

We will call $((E/S)[p], \mathscr{F}_1, \mathscr{F}_2)$ the canonically double-filtered vector space and $\mathscr{F}_1 \subseteq \mathscr{F}_2$ the σ-filtrations induced from the valuation map v_p. A subspace $U/S \subseteq (E/S)[p]$ is called v_p^σ-dense on a certain interval of ordinals $< \sigma$ (respectively piece-wise dense) if U/S is dense (respectively piece-wise dense) with respect to $((E/S)[p], \mathscr{F}_1, \mathscr{F}_2)$; compare Definition 3.1.

The the following holds

LEMMA 4.1. *Let (E, v) be a valuated group and let $U/S \subseteq (E/S)[p]$ be piece-wise v_p^σ-dense. Then we can find $M \subseteq E$ such that*

(i) $(M/S)[p] = U/S$

(ii) $M/S \subseteq (E/S)_p$

(iii) *M is v_p^σ-isotype in (E, v).*

Proof. By our choice of U/S we can find a finite chain $0 = 0^* < \ldots < n^* = \sigma$ of ordinals j^* for $j \in [0, n]$ such that U/S is v_p^σ-dense on $[(j - 1)^*, j^*)$ for all $j \in [1, n]$. Next we construct by induction a chain $M_n \subseteq M_{n-1} \subseteq \ldots \subseteq M_0$ of subgroups with the following properties for $j \in [0, n]$:

$(1j)$ $S \subseteq M_j \subseteq E(p^{j^*}) + S$

$(2j)$ M_j/S is a p-group

$(3j$ $(M_j/S)[p] = (U/S) \cap E(p^{j^*}/S)[p]$

$(4j)$ $E(p^\delta/M_j)[p] = E[p](p^\delta/M_j)$ for all $\delta \in [j^*, \sigma)$.

Let $M_n \subseteq E$ such that

$$S \subseteq M_n \quad \text{and} \quad M_n/S = U/S \cap E(p^{n^*}/S)[p].$$

Obviously ($1n$), ($2n$) and ($3n$) are satisfied and since $n^* = \sigma$ implies $[n^*, \sigma) = \emptyset$, condition ($4n$) holds trivially.

We assume that $M_n \subseteq \ldots \subseteq M_i$ with ($1j$) to ($4j$) for $j \in [i, n]$ are constructed for some $i \in [0, n]$. In order to define M_{i-1} let

$$\mathfrak{M}_{i-1} := \{M \subseteq E(p^{(i-1)^*}) + S, M_i \subseteq M, M/S \text{ } p\text{-group},$$
$$(M/S)[p] = U/S \cap E(p^{(i-1)^*}/S)[p]\}.$$

First we observe that

$$\mathfrak{M}_{i-1} \neq \emptyset.$$

To see this let

$$M/S = \{U/S \cap E(p^{(i-1)^*}/S)[p]\} + M_i/S.$$

Therefore $M_i \subseteq M \subseteq E(p^{(i-1)^*}) + S$ follows from

$$M_i \subseteq E(p^{i^*}) + S \subseteq E(p^{(i-1)^*}) + S.$$

Because M_i/S and $U/S \cap E(p^{(i-1)^*}/S)[p]$ are p-groups also M/S is a p-group. Since

$$U/S \cap E(p^{(i-1)^*}/S[p] \subseteq (M/S)[p]$$

it remains to show for $M \in \mathfrak{M}_{i-1}$ that

$$(M/S)[p] \subseteq U/S \cap E(p^{(i-1)^*}/S)[p].$$

Hence we choose $m + S \in (M/S)[p]$. We can find

$$x + S \in U/S \cap E(p^{(i-1)^*}/S)[p] \quad \text{and} \quad y + S \in (M_i/S)$$

such that $m + S = (x + S) + (y + S)$. Since

$$px + S = 0 \quad \text{and} \quad 0 = pm + S = (px + S) + (py + S)$$

also

$$y + S \in (M_i/S)[p] = U/S \cap E(p^{i^*}/S)[p]$$
$$\subseteq U/S \cap E(p^{(i-1)^*}/S[p]$$

by induction hypothesis ($3i$). Therefore

$$m + S = x + y + S \in U/S \cap E(p^{(i-1)^*}/S)[p]$$

and $M \in \mathfrak{M}_{i-1}$, i.e., $\mathfrak{M}_{i-1} \neq \emptyset$.

The set $\mathfrak{M}_{i-1}$ is obviously inductive, since unions of chains in $\mathfrak{M}_{i-1}$ are in $\mathfrak{M}_{i-1}$. From the maximum principle we obtain a maximal element $M_{i-1} \in \mathfrak{M}_{i-1}$. By our choice of $\mathfrak{M}_{i-1}$ the group M_{i-1} satisfies the condition $(1(i-1))$, $(2(i-1))$ and $(3(i-1))$.

In order to show $(4(i-1))$ let

$$\delta \in [(i-1)^*, \sigma) \quad \text{and} \quad 0 \neq x + M_{i-1} \in E(p^\delta / M_{i-1})[p]$$

such that $x \in E(p^\delta)$. In the first case we assume $\delta \in (i-1)^*, i^*)$ and let $M = \langle x, M_{i-1} \rangle$. Obviously M satisfies the first three conditions of the set $\mathfrak{M}_{i-1}$ and since M_{i-1} is maximal in $\mathfrak{M}_{i-1}$ however $M_{i-1} \subsetneq M$ we conclude $M \notin \mathfrak{M}_{i-1}$ hence

$$(M/S)[p] \neq U/S \cap E(p^{(i-1)^*}/S)[p].$$

Therefore we can find

$$m + S \in (M/S)[p] \backslash (U/S \cap E(p^{(i-1)^*}/S)[p]).$$

Since $M \subseteq E(p^{(i-1)^*}) + S$ also

$$m + S \in E(p^{(i-1)^*}/S)[p].$$

By hypothesis U/S is v_p^σ-dense on $[(i-1)^*, i^*)$ and therefore

$$m + S \in E(p^{(i-1)^*}/S)[p] \subseteq U/S + E[p](p^\delta/S).$$

We find $u \in U$ and $e \in E(p^\delta)[p]$ such that

$$m + S = u + e + S.$$

Next we will show $u \in M_{i-1}$. From

$$u + S = m - e + S,$$

$$m + S \in E(p^{(i-1)^*}/S)[p] \quad \text{and}$$

$$e + S \in E[p](p^\delta/S) \subseteq E(p^\delta/S)[p] \subseteq E(p^{(i-1)^*}/S)[p]$$

it follows that

$$u + S \in E(p^{(i-1)^*}/S)[p].$$

Therefore

$$u + S \in U/S \cap E(p^{(i-1)^*}/S)[p] = (M_{i-1}/S)[p]$$

using $(3(i-1))$. In particular $u \in M_{i-1}$.

Since $m \in M = \langle x, M_{i-1} \rangle$ there are a natural number k and $y \in M_{i-1}$ such that $m = kx + y$. If p divides k we have $m \in M_{i-1}$ from $px \in M_{i-1}$. Therefore

$$m + S \in (M_{i-1}/S)[p] = U/S \cap E(p^{(i-1)^*}/S)[p]$$

by $(3(i-1))$. This contradicts our choice of m and hence

$$m + M_{i-1} = kx + M_{i-1}$$

where p does not divide k. Since

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$$m + S = kx + M_{i-1} = u + e + M_{i-1}$$
$$= e + M_{i-1} \in E[p](p^{\delta}/M_{i-1})$$

and p does not divide k, we derive

$$x + M_{i-1} \in E[p](p^{\delta}/M_{i-1}).$$

Therefore (4($i - 1$) is shown in this case.

Next we consider the remaining case $\delta \in [i^*, \sigma)$. We assume $px \notin M_i$ for contradiction. Since $px \in M_{i-1}$, $S \subseteq M_i$ and M_{i-1}/S is a p-group by (2($i - 1$)), we can find a natural number $m \geqq 1$ such that

$$0 \neq p^m x + M_i \in E(p^{i^*}/M_i)[p].$$

From (4i) we obtain $e \in E(p^{i^*})[p]$ such that

$$p^m x + M_i = e + M_i \neq 0.$$

Since $e \notin M_i$ also

$$e + S \notin (M_i/S)[p].$$

Since $px \in M_{i-1}$, $e - p^m x \in M_i \subseteq M_{i-1}$ and $m \geqq 1$ we derive $e \in M_{i-1}$. Therefore (3($i - 1$)) and $e \in E(p^{i^*})[p]$ imply

$$e + S \in (M_{i-1}/S)[p] = U/S \cap E(p^{(i-1)^*}/S)[p]$$

and a fortiori $e + S \in U/S$.

With the help of (3i) and

$$e + S \in E[p](p^{i^*}/S) \subseteq E(p^{i^*}/S)[p]$$

we conclude

$$e + S \in U/S \cap E(p^{i^*}/S)[p] = (M_i/S)[p]$$

contradicting

$$e + S \notin (M_i/S)[p].$$

Therefore $px \in M_i$ and

$$x + M_i \in E(p^{\delta}/M_i)[p] = E[p](p^{\delta}/M_i)$$

from (4i). In particular

$$x + M_{i-1} \in E[p](p^{\delta}/M_{i-1}).$$

Finally we choose $M = M_0$ from the constructed chain and properties (1o) to (4o) imply 4.1).

In order to mix different primes we derive

LEMMA 4.2. *Let* (E, v) *be a valuated group,* π *a set of primes,* $0 < \sigma \in \mathbf{O}$ *and* $S \subseteq M_p \subseteq E$ *for all* $p \in \pi$ *with:*

(i) *M_p/S is a p-group*
(ii) *M_p is v_p^σ-isotype in E.*
Then

$$M = \sum_{p \in \pi} M_p$$

is v_π^σ-isotype in E.

Proof. If $p \in \pi$ we want to show

$$E[p](p^\nu/M) = E(p^\nu/M)[p] \quad \text{for all } \nu < \sigma.$$

Since

$$E[p](p^\nu/M) \subseteq E(p^\nu/M)[p]$$

holds trivially let

$$0 \neq x + M \in E(p^\nu/M)[p] \quad \text{with } x \in E(p^\nu).$$

Since $px \in M$ and M/S is torsion, $px + S$ has finite order. Let

$$O(px + S) = k \cdot p^n$$

such that $(k, p) = 1$. Therefore

$$O(kx + S) = p^{n+1} \quad \text{and}$$

$$pkx + S \in M/S \cap (E/S)_p = M_p/S.$$

Since $(k, p) = 1$ we obtain from the definition of p-valuations that

$$v_p(x) = v_p(kx)$$

and therefore

$$kx + M_p \in E(p^\nu/M)[p].$$

From (ii) we have

$$E(p^\nu/M_p)[p] = E[p](p^\nu/M_p)$$

and there exists $e \in E(p^\nu)[p]$ such that

$$e + M_p = kx + M_p.$$

Also

$$kx + M = e + M \in E[p](p^\nu/M).$$

Since $(k, p) = 1$ and $px \in M$ Euclid's argument leads to

$$x + M \in E[p](p^\nu/M).$$

COROLLARY 4.3. *Let (E, v) be a valuated group, $S \subseteq E$, $0 < \sigma \in \mathbf{O}$, π a set of primes and $n > 1$ an integer. If $p \in \pi$ and $j \in [1, n]$, let*

$$U_p^j/S \subseteq (E/S)[p]$$

such that

(a) U_p^j/S *is piece-wise* v_p^σ*-dense in* $(E/S)[p]$

(b) $\bigcap_{j\in[1,n]} U_p^j = S.$

Then there exist subgroups $M^j \subseteq E$ *such that*

(1) $\bigcap_{j\in[1,n]} M^j = S$

(2) $M^j/S \subseteq \bigoplus_{p\in\pi} (E/S)_p$

(3) M^j *is* v_π^σ*-isotype in E.*

Proof. From (4.1) we obtain $M_p^j \subseteq E$ such that

(j) $(M_p^j/S)[p] = U_p^j/S$

(jj) $M_p^j/S \subseteq (E/S)_p$

(jjj) M_p^j is v_π^σ-isotype in (E, v)

for all $j \in [1, n]$ and $p \in \pi$. If

$$M^j = \sum_{p\in\pi} M_p^j,$$

then M^j is v_π^σ-isotype in (E, v) by (4.2) and obviously

$$M^j/S \subseteq \bigoplus_{p\in\pi} (E/S)p$$

from (jj). Therefore we only have to show (1).

Since

$$S \subseteq \bigcap_{j\in[1,n]} M^j$$

let $x \in E - S$ and it remains to show $x \in E - M^j$ for some $j \in [1, n]$. We consider different cases:

Case I. $O(x + S) = \infty$. Since M^j/S is torsion, also $x \notin M^j$.

Case II. Let $O(x + S)$ be finite and not divisible by p for all $p \in \pi$. Then

$$x + S \notin \bigoplus_{p\in\pi} (E/S)[p]$$

and in particular $x + S \notin M^j/S$ for all $j \in [1, n]$. Therefore $x \notin M^j$ in this case.

Case III. Let $p \in \pi$ such that $O(x + S) = p \cdot k$. Therefore

$$0 \neq kx + S \in (E/S)[p].$$

Since

$$\bigcap_{j\in[1,n]} U_p^j/S$$

we can find $l \in [1, n]$ such that

$$kx + S \notin U_p^l/S.$$

From

$$U_p^l/S = (M_p^l/S)[p]$$

we conclude $kx \notin M_p^l$ and also $kx \notin M^l$. Therefore $x \notin M^l$ and (1) is shown.

5. Proof of the theorem and the corollary. First we will show

PROPOSITION 5.1. *Let (E, v) be a valuated group, $\pi \neq \emptyset$ a set of primes $\sigma > 0$ an ordinal and $k \geqq 2$ an integer. If a subgroup $S \subseteq E$ is the intersection of k v_π^σ-isotype subgroups of (E, v), then the following condition holds*:

$$(*)\quad \dim E(p^\nu/S)[p] \leqq k \cdot \dim E[p](p^\nu/S) \quad \textit{for all } \nu < \sigma.$$

This is an immediate consequence of

Observation 5.2. Let $\varphi_i:V \to V_i$ be homomorphisms between vector spaces over F for $i \in [1, k]$ such that

$$\bigcap_{i\in[1,k]} \ker \varphi_i = 0.$$

Then

$$\dim V \leqq k \cdot \max\{\dim V_i, i \in [1, k]\}.$$

Proof. The homomorphism

$$\psi:V \to \bigoplus_{i\in[1,k]} V_i(v \to (v^{\varphi_i})_{i\in[1,k]})$$

is injective and therefore

$$\dim V \leqq \dim\left(\bigoplus_{i\in[1,k]} V_i\right) \leqq k \max\{\dim V_i, i \in [1, k]\}.$$

Proof of (5.1). Let $p \in \pi$ and $\nu < \sigma$ and consider the canonical homomorphisms

$$\varphi_i:E(p^\nu/S)[p] \to E(p^\nu/M_i)[p]\ (e + S \to e + M_i)$$

where M_i for $i \in [1, k]$ are the given v_π^σ-isotype subgroups of E with

$$S = \bigcap_{i\in[1,k]} M_i.$$

We conclude

$$\bigcap_{i \in [1,k]} \ker \varphi_i = 0$$

and let $j \in [1, k]$ such that

$$\dim E(p^\nu/M_j)[p] = \max\{\dim E(p^\nu/M_i)[p], \quad i \in [1, k]\}.$$

Since M_j is ν_π^σ-isotype in E and $p \in \pi$ we have

$$E(p^\nu/M_j)[p] = E[p](p^\nu/M_j) \quad \text{and}$$

$$\dim E(p^\nu/S)[p] \leqq k \dim E[p](p^\nu/M_j)$$

from (5.2). If

$$E[p](p^\nu/M_j) = \bigoplus_{i \in I} \langle x_i + M_j \rangle,$$

then from $S \subseteq M_j$ we derive that $\{x_i + S, i \in I\}$ is a linearly independent subset of $E[p](p^\nu/S)$. Therefore

$$\dim E[p](p^\nu/M_j) \leqq \dim E[p](p^\nu/S) \quad \text{and}$$

$$\dim E(p^\nu/S)[p] \leqq k \cdot \dim E[p](p^\nu/M_j) \leqq k \cdot \dim E[p](p^\nu/S).$$

Proof of the theorem in Section 1. (1) $\Rightarrow$ (2) follows from (5.1). (2) $\Rightarrow$ (1): If $p \in \pi$, we obtain from Theorem 3.2 subgroups $U_p^j/S \subseteq E/S[p]$ for $j \in [1, n]$ such that

(i) U_p^j/S is piece-wise dense on $[0, \sigma)$

(ii) $\bigcap_{j \in [1,n]} U_p^j/S = 0$

(iii) $n = 2$ if $\dim E(p^\nu/S)[p] = 0$ or infinite for all $\nu < \sigma$.

From Corollary 4.3 we find ν_π^σ-isotype subgroups M^j of E such that

$$\bigcap_{j \in [1,n]} M^j = S.$$

Proof of the corollary in Section 1. (1) $\Rightarrow$ (2) follows from (5.1). (2) $\Rightarrow$ (1): From (3.4) we find n complements U_p^j/S of $E[p](p^0/S)$ in $E(p^0/S)[p]$ such that

$$\bigcap_{j \in [1,n]} U_p^j/S = 0.$$

The subspaces U_p^j/S are dense on $[0, 1)$ by Definition 3.1. Hence we can apply (4.3) and derive (1).

Example 5.3. A subgroup $D \subseteq G$ which is an intersection of pure subgroups but not an intersection of finitely many pure subgroups. Let κ be an uncountable cardinal and

$$B_i = \bigoplus_{n \in \omega} Z(p^n) \quad \text{for all } i \in \kappa.$$

If

$$G = \bigoplus_{i \in \kappa} B_i \quad \text{and} \quad B' = \bigoplus_{\substack{i \in \kappa \\ i > 1}} B_i,$$

we want to show that $D = B'[p]$ is the desired example. Obviously

$$G = B_1 \oplus B' \quad \text{and}$$

$$p^\nu G[p] = p^\nu B_1[p] \oplus p^\nu B'[p] \quad \text{for all } \nu \in \omega.$$

Since G is a separable p-group and

$$G[p](p^\nu/D) \cong p^\nu B_1[p] \neq 0 \quad \text{for all } \nu \in \omega,$$

we derive from the proposition in Section 1 that D is an intersection of pure subgroups. On the other hand

$$\dim G(p^\nu/D)[p] = \kappa \geqq \aleph_1 \quad \text{and}$$

$$\dim G[p](p^\nu/D) = \aleph_0 \quad \text{for all } \nu \in \omega.$$

From our Theorem in Section 1 we derive that D cannot be a finite intersection of pure subgroups.

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