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CHIEF FACTORS COVERED BY PROJECTORS OF SOLUBLE LEIBNIZ ALGEBRAS

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Abstract

Let $\mathfrak{F}$ be a saturated formation of soluble Leibniz algebras. Let K be an $\mathfrak{F}$-projector and A/B a chief factor of the soluble Leibniz algebra L. It is well known that if A/B is $\mathfrak{F}$-central, then K covers A/B. I prove the converse: if K covers A/B, then A/B is $\mathfrak{F}$-central.

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The theory of saturated formations and projectors for finite-dimensional soluble Lie and Leibniz algebras has been developed analogously to that for finite soluble groups. In broad outline, the theories run parallel, but some results require very different proofs. Not all the results translate. One of the most troubling of these is the failure of the conjugacy theorem for projectors to translate to Lie and Leibniz algebras.

Definition 1. A saturated formation of soluble Leibniz algebras is a nonempty class $\mathfrak{F}$ of soluble Leibniz algebras which is closed with respect to quotients and subdirect sums and such that, if A is an ideal of L contained in the Frattini subalgebra of L with $L/A \in \mathfrak{F}$, then $L \in \mathfrak{F}$.

Definition 2. A subalgebra K of L is called an $\mathfrak{F}$-projector of L if, for every ideal A of L, $K + A/A$ is maximal in the set of those subalgebras of L/A which are in $\mathfrak{F}$.

Definition 3. Let V be an irreducible L-module. Denote by $C_L(V)$ the centraliser $\{x \in L \mid xV = Vx = 0\}$ of V in L. Then V is called $\mathfrak{F}$-central if the split extension of V by $L/C_L(V)$ is in $\mathfrak{F}$.

A well-known basic theorem in all three contexts is that if $\mathfrak{F}$ is a saturated formation, if A/B is an $\mathfrak{F}$-central chief factor of the group or algebra L, and if K is an $\mathfrak{F}$-projector of L, then K covers A/B, that is, $K + B \supseteq A$. In this note, I prove the converse for Lie and Leibniz algebras. For groups, if one $\mathfrak{F}$-projector covers the chief factor A/B then, trivially, every $\mathfrak{F}$-projector covers A/B because of the conjugacy theorem. Curiously, this result follows for Lie and Leibniz algebras as a consequence of the main theorem of this note, a result which has no group theory analogue.

All algebras considered in this note are soluble finite-dimensional Leibniz algebras over the field F and $\mathfrak{F}$ is a saturated formation. I denote the $\mathfrak{F}$-residual of the algebra L by $L_{\mathfrak{F}}$. If A is an ideal of L, then $C_L(A)$ denotes the centraliser of A in L.

A theorem of Loday and Pirashvili [5] (see also Barnes [1, Theorem 1.4]) states that if V is an irreducible module for the Leibniz algebra L, then $L/C_L(V)$ is a Lie algebra and that V is either symmetric ($vx = -xv$ for all $x \in L$ and $v \in V$) or antisymmetric ($VL = 0$). Irreducible modules come in pairs, one symmetric, one antisymmetric, with the same left action. By Barnes [2, Theorem 3.16], the symmetric member of the pair is $\mathfrak{F}$-central if and only if the antisymmetric member is $\mathfrak{F}$-central.

Theorem 4. *Let $\mathfrak{F}$ be a saturated formation of soluble Leibniz algebras. Let A/B be a chief factor and K an $\mathfrak{F}$-projector of the soluble Lie algebra L. Suppose that K covers A/B. Then A/B is $\mathfrak{F}$-central.*

Remark. In view of Barnes [2, Corollary 3.17], the corresponding result for saturated formations of soluble Lie algebras may be regarded as a special case, one to which, in the proof, I reduce the general result.

Proof. We can work in L/B, so we can assume that $B = 0$ and $K \supseteq A$. Put $L_1 = L/A$ and $K_1 = K/A$. Then K_1 is an $\mathfrak{F}$-projector of L_1 and A is an irreducible L_1-module which is $\mathfrak{F}$-hypercentral as K_1-module. The assertion is equivalent to the assertion that A is $\mathfrak{F}$-central as L_1-module. As we can work with $L_1/C_{L_1}(A)$, we can assume that $C_{L_1}(A) = 0$, and so, that $C_L(A) = A$. Thus L is primitive and A is the only minimal ideal of L. By the theorem of Loday and Pirashvili quoted above, L_1 is a Lie algebra and A is either symmetric or antisymmetric. By Barnes [2, Theorem 3.16], we need only consider the symmetric case and so may assume that L is a Lie algebra. Further, an irreducible symmetric L-module is $\mathfrak{F}$-central if and only if it is ${}^{\mathrm{Lie}}\mathfrak{F}$-central, where ${}^{\mathrm{Lie}}\mathfrak{F}$ is the saturated formation of soluble Lie algebras consisting of those Lie algebras which are in $\mathfrak{F}$.

If $L/A \in \mathfrak{F}$, then $K = L$ and the result holds. So we suppose that $L_{\mathfrak{F}}$ properly contains A. From this we shall derive a contradiction.

There exists an ideal C such that $L_{\mathfrak{F}}/C$ is a chief factor of L. Since $L/L_{\mathfrak{F}} \in \mathfrak{F}$ but $L/C \notin \mathfrak{F}$, the chief factor $L_{\mathfrak{F}}/C$ is an irreducible $\mathfrak{F}$-eccentric K-module. Now consider $V = L_{\mathfrak{F}}/A$ as a K-module. By Barnes [3, Theorem 4.4], we have a K-module direct decomposition $V = V^+ \oplus V^-$ where V^+ is $\mathfrak{F}$-hypercentral and V^- is $\mathfrak{F}$-hypereccentric. Since $L_{\mathfrak{F}}/C$ is $\mathfrak{F}$-eccentric, $V^- \neq 0$.

Consider the K-module homomorphism $\phi : V^- \otimes A \to A$ given by $\phi(\bar{v} \otimes a) = va$ for $\bar{v} = v + A \in V^-$ and $a \in A$. Put $W = \phi(V^- \otimes A)$. Since $C_L(A) = A$, $W \neq 0$. Since as K-modules, V^- is $\mathfrak{F}$-hypereccentric and A is $\mathfrak{F}$-hypercentral, $V^- \otimes A$ is $\mathfrak{F}$-hypereccentric by Barnes [4, Theorem 2.3]. Therefore W is $\mathfrak{F}$-hypereccentric. But W is a submodule of the $\mathfrak{F}$-hypercentral module A. This contradiction completes the proof. □

Corollary 5. *Suppose that some $\mathfrak{F}$-projector of L covers the chief factor A/B. Then every $\mathfrak{F}$-projector covers A/B.*

The group theory analogue of Theorem 4 is false.

Example 6. Let $\mathfrak{F}$ be the saturated formation of 2-groups. It is locally defined by the family of formations $f(2) = \{1\}$, $f(p) = \emptyset$ for $p \neq 2$. Then the $\mathfrak{F}$-projectors of a group G are its 2-Sylow subgroups. Let $G = S_4$ be the symmetric group on four symbols and let A be its normal subgroup of order four. Then A is covered by the 2-Sylow subgroups of G but is not $\mathfrak{F}$-central as the split extension of A by $G/C_G(A)$ is not in $\mathfrak{F}$.

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