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SAMELSON PRODUCTS OF $SO(3)$ AND APPLICATIONS†

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Abstract. Certain generalized Samelson products of $SO(3)$ are calculated and applications to the homotopy of gauge groups are given.

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1. Introduction and statement of results. Each space is assumed to have the homotopy type of a CW-complex. We often make no distinction between a continuous map and its homotopy class.

Let G be a topological group and let $\gamma : G \wedge G \to G$ denote the commutator of G. A *generalized Samelson product of maps* $\alpha : A \to G$ and $\beta : B \to G$ is defined as the homotopy class of the composition

$$A \wedge B \xrightarrow{\alpha\wedge\beta} G \wedge G \xrightarrow{\gamma} G$$

and denoted by $\langle\alpha, \beta\rangle$. We denote the adjoint map $\Sigma A \to BG$ of a map $\alpha : A \to G$ by $\mathrm{ad}(\alpha)$. Regarding the generalized Samelson product $\langle\alpha, \beta\rangle$, Arkowitz [**2**] showed that

$$\mathrm{ad}(\langle\alpha, \beta\rangle) = [\mathrm{ad}(\alpha), \mathrm{ad}(\beta)],$$

where [,] is the generalized Whitehead product.

The purpose of this paper is to calculate certain generalized Samelson products of $SO(3)$ and to give applications to the homotopy of gauge groups. Let ϵ_1 and ϵ_3 be generators of $\pi_1(SO(3)) \cong \mathbf{Z}/2$ and $\pi_3(SO(3)) \cong \mathbf{Z}$ respectively, and let $\hat{\epsilon}$ and ι be the

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natural inclusion $\mathbf{R}P^2 \hookrightarrow SO(3)$ $(= \mathbf{R}P^3)$ and the identity of $SO(3)$ respectively. Then we shall prove the following results.

THEOREM 1.1. *The order of the generalized Samelson product* $\langle \epsilon_3, \hat{\epsilon} \rangle$ *is* 4.

COROLLARY 1.1. *The order of the generalized Samelson product* $\langle \epsilon_3, \iota \rangle$ *is* 12.

Let G be a compact, connected Lie group and let P be a principal G-bundle over S^4. The *gauge group* $\mathcal{G}_P$ *of* P is the group of all G-equivariant automorphisms of P covering the identity of S^4. Atiyah and Bott [**4**] showed that

$$B\mathcal{G}_P \simeq \mathrm{Map}_P(S^4, BG),$$

where $\mathrm{Map}_P(S^4, BG)$ denotes the component of $\mathrm{Map}(S^4, BG)$ corresponding to the classifying map of P. We shall often identify $B\mathcal{G}_P$ with the $\mathrm{Map}_P(S^4, BG)$. For simplicity, when $G = SO(3)$ and P is classified by $k \in \mathbf{Z} \cong \pi_4(BSO(3))$, we replace $\mathcal{G}_P$ by $\mathcal{G}_k$. Let (n, m) be the GCD of n and m. As applications of the above results, we shall prove the following results.

PROPOSITION 1.1. $\mathcal{G}_k \simeq \mathcal{G}_l$ *if and only if* $(12, k) = (12, l)$.

PROPOSITION 1.2.

$$\pi_0(\mathcal{G}_k) \cong \begin{cases} \mathbf{Z}/2 & k \equiv 0\ (2) \\ 0 & k \equiv 1\ (2) \end{cases} \qquad \pi_1(\mathcal{G}_k) \cong \begin{cases} \mathbf{Z}/2 & k \equiv 1\ (2) \\ \mathbf{Z}/4 & k \equiv 2\ (4) \\ \mathbf{Z}/2 \oplus \mathbf{Z}/2 & k \equiv 0\ (4) \end{cases}$$

REMARK 1.1. Readers may refer to [**8**] for the relevant results of the homotopy of $\mathcal{G}_P$ when P is a principal $SU(2)$-bundle over S^4. Readers may also refer to [**7**] for an alternative calculation of $\pi_1(\mathcal{G}_k)$ in a different context.

REMARK 1.2. Regarding the homotopy of the classifying space $B\mathcal{G}_k$, we have the following result. Let P be a principal $SU(2)$-bundle over S^4 corresponding to $k \in \mathbf{Z} \cong \pi_4(BSU(2))$. Since the natural projection $\mathcal{G}_P \to \mathcal{G}_k$ is a double covering, the universal covering group of the identity components of $\mathcal{G}_P$ and $\mathcal{G}_k$ are isomorphic. Then it follows from Theorem 1.5 of [**10**] that $B\mathcal{G}_k \simeq B\mathcal{G}_l$ if and only if $k = \pm l$.

REMARK 1.3. Let P be as in Remark 1.2. Then it is straightforward to check that $\pi_2(\mathcal{G}_k) \cong \pi_2(\mathcal{G}_P)$. Hence, by a result of [**8**], one finds $\pi_2(\mathcal{G}_k) \cong \mathbf{Z}/(12, k)$.

2. Proofs of Theorem 1.1 and Corollary 1.1. Before starting the proofs, let us recall a result of Bott [**5**]. Denote a generator of $\pi_i(U(2))$ by $\tilde{\epsilon}_i$ for $i = 1, 3$. Then Bott [**5**] showed that the order of the Samelson product $\langle \tilde{\epsilon}_3, \tilde{\epsilon}_1 \rangle$ is 2 and hence $\langle \tilde{\epsilon}_3, \tilde{\epsilon}_1 \rangle$ is a generator of $\pi_4(U(2)) \cong \mathbf{Z}/2$.

Proof of Theorem 1.1. Let $\pi : U(2) \to SO(3)$ be the natural projection. It is obvious that $\pi_*(\tilde{\epsilon}_i) = \epsilon_i$ for $i = 1, 3$. Then one has

$$\pi_*(\langle \tilde{\epsilon}_3, \tilde{\epsilon}_1 \rangle) = \langle \epsilon_3, \epsilon_1 \rangle \in \pi_4(SO(3)).$$

Since $\pi_* : \pi_4(U(2)) \to \pi_4(SO(3))$ is an isomorphism, the order of $\langle \epsilon_3, \epsilon_1 \rangle$ is 2 and hence $\langle \epsilon_3, \epsilon_1 \rangle$ is a generator of $\pi_4(SO(3)) \cong \mathbf{Z}/2$. Let $i : S^1 \hookrightarrow \mathbf{R}P^2$ be the inclusion of the

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1-skeleton. Then $i^*(\hat{\epsilon}) = \epsilon_1$ and, by the above observation, one can see that

$$\langle \epsilon_3, \hat{\epsilon} \rangle \neq 2\gamma \tag{2.1}$$

for any $\gamma \in [S^3 \wedge \mathbf{R}P^2, SO(3)]$.

Since $S^3 \wedge \mathbf{R}P^2$ is 3-connected we have a group isomorphism

$$[S^3 \wedge \mathbf{R}P^2, SO(3)] \cong [S^3 \wedge \mathbf{R}P^2, Sp(1)].$$

By applying $[S^3 \wedge \mathbf{R}P^2, \;]$ to the fiber sequence

$$\Omega(Sp(\infty)/Sp(1)) \to Sp(1) \to Sp(\infty) \to Sp(\infty)/Sp(1),$$

we can derive an exact sequence

$$\begin{aligned}[S^3 \wedge \mathbf{R}P^2, \Omega(Sp(\infty)/Sp(1))] &\to [S^3 \wedge \mathbf{R}P^2, Sp(\infty)] \\ &\to [S^3 \wedge \mathbf{R}P^2, Sp(1)] \to [S^3 \wedge \mathbf{R}P^2, Sp(\infty)/Sp(1)].\end{aligned}$$

Since $S^3 \wedge \mathbf{R}P^2$ is 5-dimensional and $Sp(\infty)/Sp(1)$ is 6-connected, we obtain a group isomorphism

$$[S^3 \wedge \mathbf{R}P^2, Sp(1)] \cong [S^3 \wedge \mathbf{R}P^2, Sp(\infty)].$$

On the other hand, one has a sequence of group isomorphisms

$$[S^3 \wedge \mathbf{R}P^2, Sp(\infty)] \cong [S^4 \wedge \mathbf{R}P^2, BSp(\infty)] \cong \widetilde{KO}^0(\mathbf{R}P^2) \cong \mathbf{Z}/4,$$

where the second and the last isomorphisms are due to Bott periodicity and a result of Adams [**1**], respectively. Therefore we obtain

$$[S^3 \wedge \mathbf{R}P^2, SO(3)] \cong \mathbf{Z}/4$$

and, by (2.1), the proof is completed. □

Proof of Corollary 1.1. Since $SO(3)$ is parallelizable, the result of Atiyah [**3**] yields that it is stably homotopy equivalent to $\mathbf{R}P^2 \vee S^3$. Then it follows from the Freudenthal suspension theorem that the cofibration $S^3 \wedge \mathbf{R}P^2 \xrightarrow{1 \wedge \hat{\epsilon}} S^3 \wedge SO(3) \to S^6$ splits as $S^3 \wedge SO(3) \simeq (S^3 \wedge \mathbf{R}P^2) \vee S^6$. Hence the Samelson product $\langle \epsilon_3, \iota \rangle$ is factored as

$$S^3 \wedge SO(3) \simeq (S^3 \wedge \mathbf{R}P^2) \vee S^6 \xrightarrow{\langle \epsilon_3, \hat{\epsilon} \rangle \vee \alpha} SO(3)$$

by a map $\alpha : S^6 \to SO(3)$. One can see that $\alpha = \pi_*(\langle \tilde{\epsilon}_3, \tilde{\epsilon}_3 \rangle)$, when localized at any primes but 2. It is well known that the Samelson product $\langle \tilde{\epsilon}_3, \tilde{\epsilon}_3 \rangle$ is a generator of $\pi_6(U(2)) \cong \mathbf{Z}/12$. Then we obtain that the order of α is a divisor of 12 and it is divisible by 3, since $\pi_* : \pi_6(U(2)) \to \pi_6(SO(3))$ is an isomorphism. Hence, by Theorem 1.1, the order of $\langle \epsilon_3, \iota \rangle = \langle \epsilon_3, \hat{\epsilon} \rangle \vee \alpha$ is found to be 12. □

3. Proofs of Proposition 1.1 and Proposition 1.2.

Proof of Proposition 1.1. The idea of the proof is due to [**8**]. Let $e : B\mathcal{G}_k \simeq \mathrm{Map}_k(S^4, BSO(3)) \to BSO(3)$ denote the evaluation at the basepoint of $BSO(3)$. By the fibration

$$\mathcal{G}_k \simeq \Omega B\mathcal{G}_k \xrightarrow{\Omega e} SO(3) \xrightarrow{\Gamma_k} \Omega_0^3 SO(3),$$

$\mathcal{G}_k$ can be considered as a homotopy fiber of the above map Γ_k. Then we shall analyze the map Γ_k.

By Lang [**9**], it is shown that the homotopy class of Γ_k is $\mathrm{ad}^3(\langle k\epsilon_3, \iota\rangle)$. Since Samelson products are bilinear, we have $\Gamma_k \simeq k\Gamma_1$. By Corollary 1.1, the order of Γ_1 is 12. Since $\pi_*(\Omega_0^3 SO(3))$ is finite for all $*$, it follows from Lemma 3.2 of [**6**] that $\mathcal{G}_k \simeq \mathcal{G}_l$ if and only if $(12, k) = (12, l)$. Thus Proposition 1.1 is proved. □

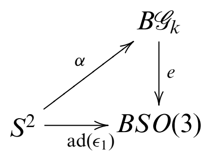
Proof of Proposition 1.2. Consider the homotopy sequence of the evaluation fibration $\Omega_0^3 SO(3) \to B\mathcal{G}_k \xrightarrow{e} BSO(3)$. Then we have an exact sequence

$$\begin{aligned} 0 = \pi_3(BSO(3)) &\to \pi_2(\Omega_0^3 SO(3)) \cong \mathbf{Z}/2 \to \pi_2(B\mathcal{G}_k) \\ &\xrightarrow{e_*} \pi_2(BSO(3)) \cong \mathbf{Z}/2 \xrightarrow{\delta} \pi_1(\Omega_0^3 SO(3)) \cong \mathbf{Z}/2 \to \pi_1(B\mathcal{G}_k) \to \pi_1(BSO(3)) = 0. \end{aligned} \tag{3.1}$$

Let $\Gamma_k : SO(3) \to \Omega_0^3 SO(3)$ be as in Proposition 1.1. Then $\Gamma_k = \mathrm{ad}^3(\langle k\epsilon_3, \iota\rangle)$ and the connecting homomorphism δ in (3.1) is the canonical isomorphism $\pi_2(BSO(3)) \cong \pi_1(SO(3))$ followed by $(\Gamma_k)_* : \pi_1(SO(3)) \to \pi_1(\Omega_0^3 SO(3))$. Hence we have

$$\delta(\mathrm{ad}(\epsilon_1)) = \mathrm{ad}^3(\langle k\epsilon_3, \epsilon_1\rangle).$$

Since the order of the Samelson product $\langle \epsilon_3, \epsilon_1\rangle$ is 2, the order of its 3-fold adjoint $\mathrm{ad}^3(\langle \epsilon_3, \epsilon_1\rangle)$ is 2 as well. Then there exists a map $\alpha : S^2 \to B\mathcal{G}_k$ satisfying the homotopy commutative diagram

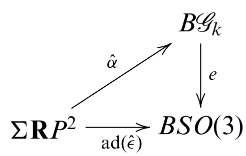


if and only if $k \equiv 0\ (2)$. Since $\mathrm{ad}(\epsilon_1)$ is the inclusion of the 2-skeleton of $BSO(3)$, $\pi_0(\mathcal{G}_k) \cong \pi_1(B\mathcal{G}_k)$ is obtained, as in the statement, by the exact sequence (3.1).

By the above argument, we have obtained $\pi_1(\mathcal{G}_k) \cong \pi_2(B\mathcal{G}_k) \cong \mathbf{Z}/2$ if $k \equiv 1\ (2)$. Then we shall consider the case that $k \equiv 0\ (2)$ and have an exact sequence

$$0 \to \mathbf{Z}/2 \to \pi_2(B\mathcal{G}_k) \xrightarrow{e_*} \pi_2(BSO(3)) \cong \mathbf{Z}/2 \to 0. \tag{3.2}$$

By Theorem 1.1, we see that the order of $\mathrm{ad}^3(\langle \epsilon_3, \hat{\epsilon}\rangle)$ is 4. Then, quite similarly to the above, there exists a map $\hat{\alpha} : \Sigma\mathbf{R}P^2 \to B\mathcal{G}_k$ satisfying the homotopy commutative diagram



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if and only if $k \equiv 0$ (4). Since $\mathrm{ad}(\hat{\epsilon})$ is the inclusion of the 3-skeleton of $BSO(3)$, we obtain, by (3.2), that $\pi_1(\mathcal{G}_k) \cong \pi_2(B\mathcal{G}_k) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2$ when $k \equiv 0$ (4).

In the case that $k \equiv 2$ (4), we suppose that $\pi_2(B\mathcal{G}_k) \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2$. Since $k \equiv 0$ (2), we have the above lift $\alpha : S^2 \to B\mathcal{G}_k$ of $\mathrm{ad}(\epsilon_1)$ and it is of order 2, by hypothesis. Note that $\Sigma\mathbf{R}P^2$ is the Moore space $S^2 \cup_2 e^3$ and the restriction of $\mathrm{ad}(\hat{\epsilon})$ to S^2 is $\mathrm{ad}(\epsilon_1)$. Then there exists the above lift $\hat{\alpha} : \Sigma\mathbf{R}P^2 \to B\mathcal{G}_k$. Hence $k \equiv 0$ (4) and this is a contradiction. Therefore we have obtained that $\pi_1(\mathcal{G}_k) \cong \pi_2(B\mathcal{G}_k) \cong \mathbf{Z}/4$ when $k \equiv 2$ (4). □

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