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AN OPERATOR SUMMABILITY OF SEQUENCES IN BANACH SPACES

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Abstract. Let $1 \leq p < \infty$. A sequence $\langle x_n \rangle$ in a Banach space X is defined to be p-operator summable if for each $\langle f_n \rangle \in l_p^{w^*}(X^*)$ we have $\langle \langle f_n(x_k) \rangle_k \rangle_n \in l_p^s(l_p)$. Every norm p-summable sequence in a Banach space is operator p-summable whereas in its turn every operator p-summable sequence is weakly p-summable. An operator $T \in B(X, Y)$ is said to be p-limited if for every $\langle x_n \rangle \in l_p^w(X)$, $\langle T x_n \rangle$ is operator p-summable. The set of all p-limited operators forms a normed operator ideal. It is shown that every weakly p-summable sequence in X is operator p-summable if and only if every operator $T \in B(X, l_p)$ is p-absolutely summing. On the other hand, every operator p-summable sequence in X is norm p-summable if and only if every p-limited operator in $B(l_{p'}, X)$ is absolutely p-summing. Moreover, this is the case if and only if X is a subspace of $L_p(\mu)$ for some Borel measure μ.

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1. Introduction. Let X be a Banach space, $\langle x_n \rangle$ a sequence in X and $1 \leq p < \infty$. We say that $\langle x_n \rangle$ is (norm) p-summable in X if $\sum_{n=1}^{\infty} \|x_n\|^p < \infty$. If $\sum_{n=1}^{\infty} |f(x_n)|^p < \infty$, for all $f \in X^*$, then we say that $\langle x_n \rangle$ is weakly p-summable in X. It is easy to note that a norm p-summable sequence is always a weakly p-summable whereas the converse, in general, is not true. In fact, in a Banach space X every weakly p-summable sequence is norm p-summable if and only if X is finite dimensional. These two types of summability were used by Grothendieck [**13**] to introduce the operator ideal of absolutely summing operators (for $p = 1$), further generalized by Piestch [**18**], who defined the operator ideal of absolutely p-summing operators for all $1 \leq p < \infty$. These operator ideals have been studied extensively in the literature.

Let $l_p^s(X)$ denote the set of all norm p-summable sequences and $l_p^w(X)$ denote that of all weakly p-summable sequences in X. Then these two sets become Banach spaces under suitable norms. More precisely, $l_p^s(X)$ can be identified as the 'countable p-direct sum' of X; similarly, $l_p^w(X)$ can be shown to be isometrically isomorphic to the space $B(l_{p'}, X)$ of operators if $p > 1$ (here p' is the harmonic conjugate of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$) and $l_{p'}$ is replaced by c_0 when $p = 1$.

In this paper we introduce a new kind of summability of sequences in Banach spaces using the notion of p-summing operators and call it the operator p-summability (definition below). This notion crops up naturally while extending the idea of limited sets to a p-level. In general, this type of summability of sequences is different from both weak and norm summability. In this paper we investigate Banach spaces for which this type of summability coincides either with weak or with norm summability. For the first type of Banach spaces in question, we encounter a p-level of the Dunford–Pettis property whereas for the other we are encouraged to introduce the notion of p-level of the Gelfand–Phillips property. The later type of Banach space ultimately reduces to subspaces of $L_p(\mu)$ for some Borel measure μ.

Example of a Banach space can be constructed, for which the operator p-summability is different from both norm and weak p-summabilities.

2. An operator summability. A non-empty subset S of a Banach space X is said to be limited if for every weak*-null sequence $\langle f_n \rangle$ in X^* (i.e. $lim_{n\to\infty} f_n(x) = 0$, for all $x \in X$), $f_n \to 0$ uniformly on S. Alternatively, given a weak*-null sequence $\langle f_n \rangle$ in X^*, there is an $\langle \alpha_n \rangle \in c_0$ such that $|f_n(x)| \leq \alpha_n$ for all $x \in S$ and all $n \in \mathbf{N}$. We can extend this idea to the 'p-sense' in the following way. We define a subset S of X to be p-limited in X $(1 \leq p < \infty)$ if for every weak*-p-summable sequence $\langle f_n \rangle$ in X^* (i.e. $\sum_{n=1}^{\infty} |f_n(x)|^p < \infty$ for all $x \in X$) there is an $\langle \alpha_n \rangle \in l_p$ such that $|f_n(x)| \leq \alpha_n$ for all $x \in S$ and $n \in \mathbf{N}$.

The history of limited sets originated from the following error of Gelfand [**12**]: A set S in Banach space X is compact if and only if every weak*-null sequence in X^* is uniformly null on S. Clearly, every compact set has this property. However, Phillips [**17**] came out with an example of a non-compact set with the above property, i.e. of a limited non-compact set. The authors [**22**] (followed by Delgado et al.[**7**, **8**], Pineiro and Delgado [**19**] and Choi and Kim [**6**]) recently studied the concept of p-compact sets for $1 \leq p < \infty$. It is interesting to note that the above-mentioned analogy carries over to p-level too. Firstly, we show that p-compact sets are p-limited.

We begin with some definitions. For $x = \langle x_n \rangle \in l_p^w(X)$, we define an operator $E_x : l_{p'} \to X$ given by $E_x(\alpha) = \sum_{n=1}^{\infty} \alpha_n x_n$, $\alpha = \langle \alpha_n \rangle \in l_{p'}$. Then $E_x \in B(l_{p'}, X)$. Moreover, in this identification $l_p^w(X)$ is isometrically isomorphic to $B(l_{p'}, X)$. For $p = 1$, $l_{p'}$ is replaced by c_0. We say that $K \subset X$ is relatively p-compact if there is an $x = \langle x_n \rangle \in l_p^s(X)$ such that $K \subset E_x(Ball(l_{p'}))$. Similarly, $K \subset X$ is said to be (relatively) weakly p-compact if there is an $x = \langle x_n \rangle \in l_p^w(X)$ such that $K \subset E_x(Ball(l_{p'}))$.

Before we come to operator summability, we prove some elementary facts about p-limited sets.

PROPOSITION 2.1. *Let $1 \leq p < \infty$ and X be a Banach space. Then every p-compact subset of X is p-limited.*

Proof. Let $K \subset X$ be a p-compact and $\langle f_n \rangle \in l_p^{w^*}(X^*)$. There is an $x = \langle x_k \rangle \in l_p^s(X)$ such that $K \subset E_x(Ball(l_{p'}))$. Then

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |f_n(x_k)|^p \leq (\| \langle f_n \rangle \|_p^{w^*})^p \sum_{k=1}^{\infty} \|x_k\|^p$$

so that

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}|f_n(x_k)|^p = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}|f_n(x_k)|^p \le \left(\|\langle f_n\rangle\|_p^{w^*}\|x\|_p^s\right)^p.$$

Set $(\sum_{k=1}^{\infty}|f_n(x_k)|^p)^{\frac{1}{p}} = \alpha_n$ for all n so that $\langle\alpha_n\rangle \in l_p$. Now if $z \in K$, then $z = \sum_{k=1}^{\infty}\beta_k x_k$ for some $\langle\beta_k\rangle \in Ball(l_{p'})$, and for each n we have

$$|f_n(z)| = \left|\sum_{k=1}^{\infty}\beta_k f_n(x_k)\right| \le (\sum_{k=1}^{\infty}|\beta_k|^{p'})^{\frac{1}{p'}}\left(\sum_{k=1}^{\infty}|f_n(x_k)|^p\right)^{\frac{1}{p}} \le \alpha_n.$$

Hence, K is p-limited. □

PROPOSITION 2.2. *Let A and B be two subsets of a Banach space X.*
(a) *If B is p-limited and $A \subset B$, then A is also p-limited.*
(b) *If A is p-limited, then $\overline{A}$ is p-limited.*
(c) *If A and B are p-limited sets, so are $A \cup B$, $A + B$ and $A \cap B$.*
(d) *If A is p-limited and $T \in B(X, Y)$, then $T(A)$ is p-limited in Y.*

Proof. Suppose A is p-limited. We prove (b). Let $\langle f_n\rangle \in l_p^{w*}(X^*)$. Then there is an $\langle\alpha_n\rangle \in l_p$ such that $|f_n(x)| \le \alpha_n$ for all $x \in A$ and $n \in \mathbf{N}$. Let $x \in \overline{A}$. Then there is an $\langle x_k\rangle$ in A such that $x_k \to x$. Thus, for each $n, f_n(x_k) \to f_n(x)$. Fixing n we have $|f_n(x_k)| \le \alpha_n$ for all k. It follows that $|f_n(x)| \le \alpha_n$ for all n so that $\overline{A}$ is p-limited. Thus, (b) follows. The proofs of (a), (c) and (d) are immediate. □

The following result describes a new class amongst weakly p-summable sequences in a Banach space.

LEMMA 2.3. *Let $\langle x_n\rangle \in l_p^w(X)$. Then $E_x(Ball(l_{p'}))$ is p-limited if and only if for every $\langle f_n\rangle \in l_p^{w^*}(X^*)$ we have $\left\langle\langle f_n(x_k)\rangle_k\right\rangle_n \in l_p^s(l_p)$.*

Proof. Consider $x = \langle x_n\rangle \in l_p^w(X)$ such that $S = E_x(Ball(l_{p'}))$ is p-limited. Then given $\langle f_n\rangle \in l_p^{w^*}(X^*)$, there is an $\langle\alpha_n\rangle \in l_p$ such that for each $\beta = \langle\beta_k\rangle \in Ball(l_{p'})$ we have

$$|f_n(E_x(\beta))| \le \alpha_n, \text{ for all } n,$$

i.e.

$$|\langle\beta, \langle f_n(x_k)\rangle\rangle_{k=1}^{\infty}| \le \alpha_n, \text{ for all } n.$$

It follows that $\|\langle f_n(x_k)\rangle_{k=1}^{\infty}\|_p \le \alpha_n$ for all n. Thus, $\left\langle\langle f_n(x_k)\rangle_k\right\rangle_n \in l_p^s(l_p)$ for all $\langle f_n\rangle \in l_p^{w^*}(X^*)$.

Now tracing back the proof, we can also prove the converse. □

Since $l_p^{w^*}(X^*)$ can be identified with $B(X, l_p)$, where to each $f = \langle f_n\rangle \in l_p^{w^*}(X^*)$ we get $(E_f)_* \in B(X, l_p)$ given by $(E_f)_*(x) = \langle f_n(x)\rangle$ with $\|f\|_p^{w^*} = \|(E_f)_*\|$, and since $l_p^{w^*}(X^*) = l_p^w(X^*)$, the above lemma can be reorganized as follows.

PROPOSITION 2.4. *Let $x = \langle x_n\rangle$ be a sequence in X. The following are equivalent.*
(a) *$x \in l_p^w(X)$ and $E_x(Ball(l_{p'}))$ is a p-limited set in X.*
(b) *$\langle Tx_n\rangle \in l_p^s(l_p)$ for all $T \in B(X, l_p)$.*

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Let us rename this phenomena as follows.

DEFINITION 2.5. A sequence $\langle x_n \rangle$ in X is said to be operator p-summable in X if it satisfies one (and hence all) of the conditions of Proposition 2.4.

Note that every norm p-summable sequence in X is operator p-summable. To see this, let $\langle x_n \rangle \in l_p^s(X)$ and $T \in B(X, l_p)$. Then $\|Tx_n\| \leq \|T\|\|x_n\|$ for all n so that $\langle Tx_n \rangle \in l_p^s(l_p)$. Thus, $\langle x_n \rangle$ is operator p-summable. We have already seen that an operator p-summable sequence is weakly p-summable.

3. Towards weak summability. In this section we characterize Banach spaces with the property that every weakly p-summable sequence is operator p-summable and give some examples of such spaces. We shall call a Banach space with this property a *weak p-space*. A simple characterization of such spaces in terms of operator ideals is given below. Let X and Y be Banach spaces and let $1 \leq p < \infty$. Then an operator $T \in B(X, Y)$ is called absolutely p-summing if for every $\langle x_n \rangle \in l_p^w(X)$, $\langle Tx_n \rangle \in l_p^s(Y)$. The set of all absolutely p-summing operators in $B(X, Y)$ is denoted by $\Pi_p(X, Y)$.

PROPOSITION 3.1. *Let X be a Banach space and $1 \leq p < \infty$. Then X is a weak p-space if and only if $\Pi_p(X, l_p) = B(X, l_p)$.*

Proof. Let $T \in B(X, l_p)$ and $x = \langle x_n \rangle \in l_p^w(X)$. Suppose X is a weak p-space. Then $\langle x_n \rangle$ is operator p-summable so that $\langle Tx_n \rangle \in l_p^s(l_p)$. Thus, $T \in \Pi_p(X, l_p)$. Tracing back, we can prove the converse. □

Before we give some examples of weak p-spaces we shall further explore Banach spaces that satisfy an operator ideal equation of the above type. Given Banach spaces X and Y, let $W(X, Y)$ and $\nu(X, Y)$ denote the sets of weakly compact and completely continuous operators from X to Y respectively. Recall that a Banach space X is said to have the Dunford–Pettis property (DPP, for short) if for any Banach space Y, $W(X, Y) \subset \nu(X, Y)$.

Dunford and Pettis [**11**] in 1940 proved that every weakly compact operator defined on a $L_1(\mu)$ space takes weakly compact sets to norm compact sets. Grothendieck [**14**] in 1953 defined a Banach space X to have the Dunford–Pettis property if weakly compact operators defined on X are completely continuous and proved that $C(K)$ spaces also have this property. This result was also obtained independently in 1955 by Bartle et al. [**1**]. Brace [**3**] and Grothendieck [**14**] gave some nice characterizations of the Dunford–Pettis property. A detailed survey of the Dunford–Pettis property can be found in [**9**]. In this section we propose to extend this property to a p-setting to meet our above-mentioned end. For this purpose we recall the following characterization of the Dunford–Pettis property, essentially due to Grothendieck [**14**].

THEOREM 3.2. *Let X be a Banach space, then the following statements are equivalent*:

(a) *$W(X, Y) \subset \nu(X, Y)$ for all Banach spaces Y.*
(b) *$W(X, c_0) \subset \nu(X, c_0)$.*
(c) *For $\langle x_n \rangle \in c_0^w(X)$ and $\langle f_n \rangle \in c_0^w(X^*)$, $\left\langle \langle f_k(x_n) \rangle_k \right\rangle_n \in c_0^s(c_0)$.*
(d) *For $\langle x_n \rangle \in c_0^w(X)$ and $\langle f_n \rangle \in c_0^w(X^*)$, $\langle f_n(x_n) \rangle \in c_0$.*

Picking up (c) as an end, we now propose the following definition.

DEFINITION 3.3. Let $1 \leq p, q \leq \infty$. A Banach space X is said to have the (p, q)-Dunford–Pettis property ((p, q)-DPP, for short) if given $\langle x_n \rangle \in l_q^w(X)$ and $\langle f_n \rangle \in l_p^w(X^*)$,

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we have $\left\langle \langle f_k(x_n)\rangle_k \right\rangle_n \in l_q^s(l_p)$. For p(or q) $= \infty$, l_p (or l_q) is replaced by c_0. For all p, the (p, p)-Dunford–Pettis property shall be called the p- Dunford–Pettis property.

It is immediate from Theorem 3.2 that the ∞-DPP is the classical Dunford–Pettis property. In what follows, we shall extend the above characterization theorem to the (p, q)-setting. Towards this end the notion of weak p-compactness studied by the authors [**22**] (also see Castillo and Sanchez [**4**, **5**]) fits smugly in the scheme. Let X and Y be Banach spaces and let $1 \le p < \infty$. An operator $T \in B(X, Y)$ is said to be p-compact (weakly p-compact) if $T(Ball(X))$ is relatively p-compact (respectively, relatively weakly p-compact). Here $l_{p'}$ is replaced by c_0 if $p = 1$. Let $W_p(X, Y)$ denote the set of all weakly p-compact operators and $K_p(X, Y)$ that of all p-compact operators. The next result was obtained by the authors [**22**].

THEOREM 3.4. *Let X and Y be Banach spaces, $1 \le p < \infty$ and $T \in B(X, Y)$. Then the following statements are equivalent*:

(a) *T is weakly p-compact.*

(b) *There are $y \in l_p^w(Y)$ and $S_y \in B(R(y), X^*)$ such that $T^* = S_y \cdot E_y^*$, where $R(y) = \overline{Range(E_y^*)} \subset l_p$.*

The set $W_p(X, Y)$ of all weakly p-compact operators in $B(X, Y)$ is a Banach operator ideal with the factorization norm ω_p defined as follows:

$$\omega_p(T) = inf\{\|S_y\|\|E_y\| : T^* = S_y \cdot E_y^* \text{ as in Theorem 3.4(b)}\}.$$

Let (A, α) be an operator ideal. For Banach spaces X and Y we put

$$A^d(X, Y) = \{T \in B(X, Y) : T^* \in A(Y^*, X^*)\}.$$

For an operator $T \in A^d(X, Y)$, we put $\alpha^d(T) = \alpha(T^*)$. With these notations (A^d, α^d) is also a Banach operator ideal and is called the dual ideal of (A, α).

COROLLARY 3.5. *For Banach spaces X and Y, $T \in W_p^d(X, Y)$ if and only if there are $f = \langle f_n\rangle \in l_p^w(X^*)$ and $S_f \in B(R(f), Y)$ such that $T = S_f \cdot (E_f)_*$. Here $R(f) = \{\langle f_n(x)\rangle :$ $x \in X\} \subset l_p$ and $(E_f)_* = E_f^*|_X$.*

We can now extend the classical characterization theorem for the Dunford–Pettis property to the (p, q)-setting. Let X and Y be a pair of Banach spaces and $T \in B(X, Y^*)$. Then we have $T = i_Y^* \cdot T^{**} \cdot i_X$. Indeed, for $x \in X$ and $y \in Y$ we have $\left\langle i_Y^* \cdot T^{**} \cdot i_X(x), y\right\rangle = \langle Tx, y\rangle$. Here $i_X : X \hookrightarrow X^{**}$ is the canonical embedding.

THEOREM 3.6. *Let $1 \le p < \infty$, $1 \le q \le \infty$ and X a Banach space. Then the following statements are equivalent*:

(a) *X has the (p, q)-Dunford–Pettis property.*

(b) *$W_p(Y, X^*) \subset \Pi_q^d(Y, X^*)$ for every Banach space Y.*

(c) *$\Pi_q^d(l_{p'}, X^*) = B(l_{p'}, X^*)$.*

Proof. It only remains to show that (a) implies (b), for $W_p(l_{p'}, X^*) = B(l_{p'}, X^*)$. To this end, assume that X has the (p, q)-DPP and let $T \in W_p(Y, X^*)$. Then by Theorem 3.4, there are $\mathbf{f} = \langle f_n\rangle \in l_p^w(X^*)$ and $S_\mathbf{f} \in B(R(\mathbf{f}), Y^*)$ such that $T^* = S_\mathbf{f} \cdot E_\mathbf{f}^*$. Firstly we show that $T^*|_X \in \Pi_q(X, Y^*)$. To see this, let $\langle x_n\rangle \in l_q^w(X)$. Then as $\langle f_n\rangle \in l_p^w(X^*)$, we have $\langle (E_\mathbf{f})^* \cdot i_X(x_n)\rangle = \left\langle \langle f_k(x_n)\rangle_k \right\rangle_n \in l_q^s(l_p)$. Thus, $\langle T^* \cdot i_X(x_n)\rangle \in l_q^s(Y^*)$. In other

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words, $T^* \cdot i_X \in \Pi_q(X, Y^*)$. It follows from Proposition 2.19 in [**10**] that $i_X^* \cdot T^{**} \in \Pi_q^d(Y^{**}, X^*)$ so that $T = i_X^* \cdot T^{**} \cdot i_Y \in \Pi_q^d(Y, X^*)$. This completes the proof. □

Note. The authors in [**22**] have observed that absolutely p-summing operators may be regarded as p-completely continuous operators as they take weakly p-compact sets to p-compact sets. Thus, the classical Dunford–Pettis property may be traced back, provided we regard absolutely p-summing operators as p-completely continuous operators.

In view of Proposition 3.1 and Theorem 3.6, we have the following characterization for weak p-spaces.

THEOREM 3.7. *Let $1 \leq p < \infty$. Then for a Banach space X the following statements are equivalent*:

(a) *X is a weak p-space.*
(b) *X has the p-Dunford–Pettis property.*
(c) $W_p(Y, X^*) \subset \Pi_p^d(Y, X^*)$ *for every Banach space Y.*
(d) $\Pi_p^d(l_{p'}, X^*) = B(l_{p'}, X^*)$.
(c′) $W_p^d(X, Y^*) \subset \Pi_p(X, Y^*)$ *for every Banach space Y.*
(d′) $\Pi_p(X, l_p) = B(X, l_p)$.

Proof. Note that $W_p^d(X, l_p) = B(X, l_p)$. Thus, in the light of Proposition 3.1 and Theorem 3.6, it is enough to show that (c)⇔(c′) and that (d′)⇒(d). Firstly, assume that $W_p(Y, X^*) \subset \Pi_p^d(Y, X^*)$ and let $T \in W_p^d(X, Y^*)$. Then $T^* \cdot i_Y \in W_p(Y, X^*) \subset \Pi_p^d(Y, X^*)$. Thus, $i_Y^* \cdot T^{**} \in \Pi_p(X^{**}, Y^*)$ so that $T = i_Y^* \cdot T^{**} \cdot i_X \in \Pi_p(X, Y^*)$ [**10**, 2.4]. Therefore, $W_p^d(X, Y^*) \subset \Pi_p(X, Y^*)$. Next, let $W_p^d(X, Y^*) \subset \Pi_p(X, Y^*)$. If $T \in W_p(Y, X^*)$, then by Theorem 3.4, $T^* = S_f \cdot E_f^*$ for some $f \in l_p^w(X^*)$. Thus, by Corollary 3.5, $T^* \cdot i_X \in W_p^d(X, Y^*) \subset \Pi_p(X, Y^*)$. It follows from Proposition 2.19 in [**10**] that $T = i_X^* \cdot T^{**} \cdot i_Y \in \Pi_p^d(Y, X^*)$. Thus, $W_p(Y, X^*) \subset \Pi_p^d(Y, X^*)$ so that (c)⇔(c′). Now let $1 < p < \infty$ and assume that $\Pi_p(X, l_p) = B(X, l_p)$. Let $T \in B(l_{p'}, X^*)$. Then $T^* \cdot i_X \in B(X, l_p) = \Pi_p(X, l_p)$. Thus, $i_X^* \cdot T^{**} \in \Pi_p^d(l_{p'}, X^*)$. As $l_{p'}$ is reflexive, we have $T = i_X^* \cdot T^{**}$ so that $\Pi_p^d(l_{p'}, X^*) = B(l_{p'}, X^*)$. Finally, suppose that $\Pi_1(X, l_1) = B(X, l_1)$. Let $T \in B(c_0, X^*)$. Then $T^* \cdot i_X \in B(X, l_1) = \Pi_1(X, l_1)$. Thus, $i_X^* \cdot T^{**} \cdot i_{c_0} \in \Pi_1^d(c_0, X^*)$ so that $\Pi_1^d(c_0, X^*) = B(c_0, X^*)$. Therefore, (d′)⇒(d), which completes the proof. □

Some more consequences of Theorem 3.6 are in order.

COROLLARY 3.8. *If $1 \leq q \leq p < \infty$ and if X has the (p, q)-Dunford–Pettis property, then it has the p-Dunford–Pettis property. In particular, X is a weak p-space.*

COROLLARY 3.9. *If X^* has the p-Dunford–Pettis property, so does X. In other words, if X^* is a weak p-space, so is X.*

Remark. The $p = \infty$ case of the above corollary, i.e. if X^* has the classical Dunford–Pettis property, so does X, was proved by Grothendieck [**14**].

Examples (1) Let X be an $\mathcal{L}_\infty$-space. If $1 \leq p \leq 2$, then X has the $(p, 2)$-DPP and 2 is sharp [**13**, **16**]. If $2 < p < q < \infty$, then X has the (p, q)-DPP and q is sharp, that is to say (by the abuse of the language) that X has the almost p-DPP for every $p > 2$ [**15**, **20**].

(2) In view of Theorem 3.6, every $\mathcal{L}_1$-space has the above properties. In particular, c_0 and l_1 have the 2-DPP, the almost p-DPP if $p > 2$ and also the ∞-DPP (=Dunford–Pettis property).

It is interesting to note that these are the only $\mathcal{L}_p$-spaces with any (r, s)-DPP, $1 \le r, s \le \infty$.

THEOREM 3.10. *Let* $1 < p < \infty$. *Then* l_p *does not have the r-Dunford–Pettis property for any* $r > 1$. *In other words, for* $1 < p < \infty$, l_p *is not a weak r-space for any* $r > 1$.

Proof. We divide the proof in several parts.
Case 1. Let $r \ge max\{p, p'\}$. Let $\{e_n\}$ be the standard unit vector basis of l_p and $\{f_n\}$ that of $l_{p'}$. Then $\langle e_n\rangle \in l_r^w(l_p)$ and $\langle f_n\rangle \in l_r^w(l_{p'})$. Since $\langle\langle f_k(e_n)\rangle_k\rangle_n = \langle\langle \delta_k^n\rangle_k\rangle_n \notin l_r^s(l_r)$, where δ_k^n is the Kronecker delta, we conclude that l_p does not have the r-DPP if $r \ge max\{p, p'\}$.

Before we proceed to the other cases, we need to prove the following lemma.

LEMMA 3.11. *Let* $1 \le s \le p'$, *where* p' *is the harmonic conjugate of* p, $1 < p < \infty$. *Find* $t > s$ *such that* $\frac{1}{s} - \frac{1}{p'} = \frac{1}{t}$. *Then for any* $\langle \alpha_n\rangle \in l_t$, $\langle \alpha_n e_n\rangle \in l_s^w(l_p)$.

Proof of Lemma 3.11. If $\langle \beta_n\rangle \in l_{p'}$, then

$$\begin{aligned}\left(\sum_{n=1}^{\infty} |\langle \beta, \alpha_n e_n\rangle|^s\right)^{1/s} &= \left(\sum_{n=1}^{\infty} |\alpha_n \beta_n|^s\right)^{1/s}\\ &\le \left(\sum_{n=1}^{\infty} |\alpha_n|^t\right)^{1/t} \left(\sum_{n=1}^{\infty} |\beta_n|^{p'}\right)^{1/'}\\ &< \infty.\end{aligned}$$

Thus, $\langle \alpha_n e_n\rangle \in l_s^w(l_p)$. □

Proof. Now we consider the other cases of the theorem.

Case 2. Let $1 < r < min\{p, p'\}$. Find $t_1, t_2 > 1$ such that $\frac{1}{t_1} = \frac{1}{r} - \frac{1}{p'}$ and $\frac{1}{t_2} = \frac{1}{r} - \frac{1}{p}$. Then $\frac{1}{t_1} + \frac{1}{t_2} = \frac{2}{r} - 1 < \frac{1}{r}$. Thus, we can find $\langle \alpha_n\rangle \in l_{t_1}$ and $\langle \beta_n\rangle \in l_{t_2}$ such that $\langle \alpha_n\beta_n\rangle \notin l_r$. Now by the above lemma $\langle \alpha_n e_n\rangle \in l_r^w(l_p)$ and $\langle \beta_n f_n\rangle\ l_r^w(l_{p'})$. But

$$\langle\langle\langle \beta_k f_k, \alpha_n e_n\rangle\rangle_k\rangle_n \notin l_r^s(l_r).$$

Thus, l_p does not have the r-DPP if $1 \le r < min\{p, p'\}$.

Case 3. Let r lie between p and p'. Note that l_p has the r-DPP if and only if $l_{p'}$ has the r-DPP. Thus, without any loss of generality, we may assume that $p < r < p'$. Find $t > 1$ such that $\frac{1}{t} = \frac{1}{r} - \frac{1}{p'}$. Then $r < t$ so that we can find $\langle \alpha_n\rangle \in l_t$ with $\langle \alpha_n\rangle \notin l_r$. Then $\langle \alpha_n e_n\rangle \in l_r^w(l_p)$. Also, $\langle f_n\rangle \in l_r^w(l_{p'})$. But

$$\langle\langle\langle f_k, \alpha_n e_n\rangle\rangle_k\rangle_n \notin l_r^s(l_r).$$

Thus, l_p does not have the r-DPP if r lies between p and p'.

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Finally, since $\Pi_p(l_p) \neq B(l_p)$, we conclude that both l_p and $l_{p'}$ do not have the p- and p'-DPP. This completes the proof. □

COROLLARY 3.12. *For $1 < p < \infty$, l_p does not have*

(a) *The* (r, s)*-Dunford–Pettis property if* $1 < r, s < \infty$.
(b) *The* $(r, 1)$*-Dunford–Pettis property if* $1 < r < \infty$.
(c) *The* $(1, r)$*-Dunford–Pettis property if* $1 < r < \infty$.

Proof. (a) For $1 < s \leq r$, if l_p has the (r, s)-DPP then it also has the r-DPP. Thus, if $1 < s \leq r < \infty$, then l_p does not have the (r, s)-DPP.

Next, let $1 < r < s$. Find $\langle \alpha_n \rangle \in l_s^w(l_p)$ such that $\langle \alpha_n \rangle \notin l_r^w(l_p)$. Find $\beta \in l_{p'}$ such that $\sum_{n=1}^{\infty} | \langle \beta, \alpha_n \rangle |^r = \infty$. Putting $\beta_1 = \beta$ and $\beta_n = 0$ for $n \geq 2$, $\langle \beta_n \rangle \in l_r^w(l_{p'})$. However,

$$\langle \langle \langle \beta_k, \alpha_n \rangle \rangle_k \rangle_n \notin l_s^s(l_r).$$

Thus, l_p does not have the (r, s)-DPP for $1 < r, s < \infty$.

Now both (b) and (c) can be obtained on the lines of (a). □

Note. We have not been able to settle whether for $1 < p < \infty$, l_p has the 1-DPP.

4. Towards norm summability. In this section we shall examine a condition that forces every operator p-summable sequence to become norm-p-summable. Let X be a Banach space and $1 \leq p < \infty$. If $x \in l_p^w(X)$ is such that $E_x \in \Pi_p(l_{p'}, X)$, then it follows from Proposition 5.5(a) in [**22**] and by Proposition 2.4 that x is an operator p-summable sequence in X. In the light of this observation, we propose to study an operator version of the operator p-summable sequences.

DEFINITION 4.1. An operator $T \in \mathcal{B}(X, Y)$ is said to be p-limited if $T(Ball X)$ is p-limited in Y, and T is said to be sequentially p-limited if $\langle Tx_n \rangle$ is operator p-summable for all $\langle x_n \rangle \in l_p^w(X)$.

It follows from Proposition 2.4 that a sequence $\mathbf{x} = \langle x_n \rangle$ in X is operator p-summable if and only if $E_x \in \mathcal{B}(l_{p'}, X)$ is a p-limited operator if and only if $E_x \in \Pi_p^d(l_{p'}, X)$. Further, we have the following.

PROPOSITION 4.2. *Let $1 \leq p < \infty$. Every p-limited operator $T \in \mathcal{B}(X, Y)$ is sequentially p-limited.*

Proof. Let $\mathbf{x} = \langle x_n \rangle \in l_p^w(X)$. We may assume that $\| \langle x_n \rangle \|_p^w \leq 1$ so that $E_x(Ball(l_{p'})) \subset Ball(X)$. Since $T(Ball(X))$ is p-limited in Y, $T(E_x(Ball(l_{p'})))$ is also p-limited in Y. Now by Lemma 2.3, $\langle Tx_n \rangle$ is operator p-summable in Y. □

The following result will be used to characterize sequentially p-limited operator.

LEMMA 4.3. *Let $1 \leq p < \infty$ and let $\alpha \in l_p^w(l_p)$. Then $\alpha \in l_p^s(l_p)$ if and only if $E_\alpha \in \Pi_p(l_{p'}, l_p) = \Pi_p^d(l_{p'}, l_p)$. Here $l_{p'} = c_0$ when $p = 1$.*

Proof. When $1 < p < \infty$, this fact follows from Remark (v) of Proposition 5.5 in [**22**]. Thus, we may assume that $p = 1$. Again in this case, it follows from [**22**, Proposition 5.5(a)] that if $E_\alpha \in \Pi_1(c_0, l_1)$, then $\alpha \in l_1^s(l_1)$.

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Conversely, let $\alpha \in l_1^s(l_1)$, $\alpha = \langle \alpha_n \rangle = \langle \langle \alpha_n^k \rangle_k \rangle_n$ such that $\alpha_n = \langle \alpha_n^k \rangle_k \in l_1$ for all n. Put $\tilde{\alpha}_k = \langle \alpha_n^k \rangle_n$ for all k. Then

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |\alpha_n^k| = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\alpha_n^k| = \|\alpha\|_1^s$$

so that $\tilde{\alpha}_k \in l_1$ for all k with $\tilde{\alpha} = \langle \tilde{\alpha}_k \rangle \in l_1^s(l_1)$. If $\beta = \langle \beta_n \rangle = \langle \langle \beta_n^m \rangle_m \rangle_n \in l_1^w(c_0)$. Then

$$\begin{aligned} \langle E_\alpha(\beta_n) \rangle_n &= \left\langle \sum_{m=1}^{\infty} \beta_n^m \alpha_m \right\rangle_n \\ &= \left\langle \left\langle \sum_{m=1}^{\infty} \beta_n^m \alpha_m^k \right\rangle_k \right\rangle_n \\ &= \langle \langle (\beta_n, \tilde{\alpha}_k \rangle_k \rangle_n. \end{aligned}$$

Since

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |(\beta_n, \tilde{\alpha}_k)| \leq \|\beta\|_1^w \sum_{k=1}^{\infty} \|\tilde{\alpha}_k\| = \|\beta\|_1^w \|\alpha\|_1^s,$$

we get $\langle E_\alpha(\beta_n) \rangle_n \in l_1^s(l_1)$ with $\| \langle E_\alpha(\beta_n) \rangle_n \|_1^s \leq \|\beta\|_1^w \|\alpha\|_1^s$. Thus, $E_\alpha \in \Pi_1(c_0, l_1)$ with $\pi_1(E_\alpha) \leq \|\alpha\|_1^s$. Since $\|\alpha\|_1^s \leq \pi_1(E_\alpha)$, we conclude $\pi_1(E_\alpha) = \|\alpha\|_1^s$. □

THEOREM 4.4. *Let $T \in \mathcal{B}(X, Y)$. For $1 \leq p < \infty$, the following are equivalent*:

(1) *T is sequentially p-limited.*
(2) *$TU \in \Pi_p^d(l_{p'}, Y)$ for all $U \in \mathcal{B}(l_{p'}, X)$.*
(3) *$ST \in \Pi_p(X, l_p)$ for all $S \in \mathcal{B}(Y, l_p)$.*

Proof. That (1) is equivalent to (2) follows from Proposition 2.4.

Now let (1) hold. If $S \in B(Y, l_p)$ and $x = \langle x_n \rangle \in l_p^w(X)$ so that $E_x \in B(l_{p'})$, then $TE_x \in \Pi_p^d(l_{p'}, X)$. Thus, by Lemma 4.3, it follows that $STE_x \in \Pi_p^d(l_{p'}, l_p) = \Pi_p(l_{p'}, l_p)$. In other words, $\langle STx_n \rangle \in l_p^s(l_p)$ so that $ST \in \Pi_p(X, l_p)$. Thus, (3) also holds. Finally, we can trace back the proof to show that (3) implies (1). □

Remarks.

(1) If an operator $T \in \Pi_p(X, Y) \cup \Pi_p^d(X, Y)$, then T is sequentially p-limited.
(2) Every sequentially p-limited operator in $\mathcal{B}(X, l_p)$ is in $\Pi_p(X, l_p)$.

Now we prove the following sequential characterization of subspaces of $L_p(\mu)$ whose operator characterization was obtained by Kwapién [**15**].

THEOREM 4.5. *Let $1 \leq p < \infty$. For a Banach space X, the following are equivalent*:

(1) *Every operator p-summable sequence in X is norm p-summable.*
(2) *$\Pi_p^d(Y, X) \subset \Pi_p(Y, X)$ for every Banach space Y.*
(3) *$\Pi_p^d(l_{p'}, X) = \Pi_p(l_{p'}, X)$.*
(4) *X is a subspace of $L_p(\mu)$ for some Borel measure μ*[**15**].

Proof. The equivalence of (2) and (4) was proved by Kwapién [**15**].

Let (1) hold. Assume that $T \in \Pi_p^d(Y, X)$ for some Banach space Y. If $y = \langle y_n \rangle \in l_p^w(Y)$, then $E_y \in B(l_{p'}, Y)$. Thus, $TE_y \in \Pi_p^d(l_{p'}, X)$. Now by Lemma 4.3 and assumption (1), we get that $\langle Ty_n \rangle \in l_p^{op}(X) = l_p^s(X)$. It follows that $T \in \Pi_p(Y, X)$ so that (2) holds.

Since $\Pi_p(l_{p'}, X) \subset \Pi_p^d(l_{p'}, X)$ follows from [**22**], we may conclude that (2) implies (3).

Finally, assume that (3) holds. Let $x \in l_p^{op}(X)$. Then $E_x \in \Pi_p^d(l_{p'}, X) = \Pi_p(l_{p'}, X)$. Now that $x \in l_p^s(X)$ again follows from [**22**]. This completes the proof. $\square$

Recall that a Banach space X is said to have the Gelfand–Phillips property if every limited set in X is relatively compact. Recall further that every limited set in a Banach space is conditionally weakly compact [**2**]. We do not know about the 'p-version' of this result, possibly due to the absence of a p-prototype of a Rosenthal's l_1-theorem. At the same time, let us note that in a Banach space, in which any operator p-summable sequence is norm p-summable, a (relatively) weakly p-compact set is (relatively) p-compact if and only if it is p-limited. Thus, the condition that every operator p-summable sequence in a Banach space is norm p-summable can be seen as a p-version of the Gelfand–Phillips property.

An operator ideal. Let $1 \leq p < \infty$. For a pair of Banach spaces X and Y, consider the set $Lt_p^{\cdot}(X, Y)$ of all sequentially p-limited operators in $B(X, Y)$. For $T \in Lt_p^{\cdot}(X, Y)$, we define

$$lt_p(T) := sup\{\pi_p(ST) : S \in B(Y, l_p) and \|S\| \leq 1\}.$$

Then it is a routine to prove the following result.

Proposition 4.6. *For $1 \leq p < \infty$, $(Lt_p^{\cdot}, lt_p)$ is a normed operator ideal.*

Note. We have not been able to show whether in general $Lt_p^{\cdot}(X, Y)$ is lt_p-complete. We, however, adopted the following approach.

Let X and Y be Banach spaces and $T \in B(X, Y)$. For any $1 \leq p < \infty$, we can define $\varphi_T : B(Y, l_p) \to B(X, l_p)$ given by $\varphi_T(S) = ST$ for all $S \in B(Y, l_p)$. Now it is easy to show that $T \mapsto \varphi_T$ is a linear isometry from $B(X, Y)$ into $B(B(Y, l_p), B(X, l_p))$; $1 \leq p \leq \infty$. Moreover, if $1 \leq p < \infty$, it follows from Proposition 4.3 that $T \in Lt_p^{\cdot}(X, Y)$ if and only if $\varphi_T(B(Y, l_p)) \subset \Pi_p(X, l_p)$. In this case for all $T \in Lt_p^{\cdot}(X, Y)$, we have

$$lt_p(T) = \text{The operator norm of } \varphi_T \text{ in } B(B(Y, l_p), \Pi_p(X, l_p)).$$

We write for the completion of $\{\varphi_T : T \in Lt_p^{\cdot}(X, Y)\}$ in $B(B(Y, l_p), \Pi_p(X, l_p))$ by $Lt_p(X, Y)$ and denote the operator norm on $Lt_p(X, Y)$ again by $lt_p(.)$. Thus, proposition 4.6 may be re-investigated in the following manner.

Proposition 4.7. *(Lt_p, lt_p) is a Banach operator ideal,* $1 \leq p < \infty$.

Remarks.
(1) If $x \in l_p^w(X)$ is operator p-summable, then $\|x\|_p^w = \|E_x\| \leq lt_p(E_x) := lt_p(x)$. If $x \in l_p^s(X)$, then $lt_p(x) \leq \|x\|_p^s$.
(2) For $T \in \Pi_p(X, Y)$, $\|T\| \leq lt_p(T) \leq \pi(T)$. If $T \in \Pi_p(X, l_p)$, then $lt_p(T) = \pi_p(T)$.

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