A NOTE ON " SIMPLIFICATION AND SCALING "

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1. Introduction

In a recent paper Segel (**1**) points out that the diverse techniques (which " comprise the core of the applied mathematicians art (or craft) ") of the applied mathematician, although in general reliably proven, are " rarely explicitly delineated but rather are transmitted indirectly and informally ". In his article Segel aims to clarify two such techniques, namely:

(i) *Scaling*—or how to choose dimensionless variables in such a way that the relative size of the various terms in an equation is explicitly indicated by the magnitudes of the dimensionless parameters which precede them,

(ii) *Simplification*—a procedure in which a term is neglected under the assumption that it is small, and the consistency of the assumption checked later.

By studying relatively simple examples Segel shows how an inappropriate scaling can lead to a problem which is not meaningful, and furthermore how the simplification procedure can lead to " wretched consistent approximations " in which the approximation, although apparently good, is in fact a poor one.

The purpose of the present note is to show how when (i) is applied, perhaps naïvely but certainly plausibly, to a fairly complicated particular physical problem an ordinary differential equation results to which (ii) may be applied. An apparently consistent first-order solution is derived using the method of matched asymptotic expansions. Carrying the procedure to higher order reveals an anomaly, and we verify that the first-order solution is " wretchedly consistent ". The implications of this failure of the procedure (ii) for the original scaling are discussed.

2. Statement of the problem

We consider the high Reynolds number flow of a viscous, electrically conducting fluid past a semi-infinite flat plate in the presence of a magnetic field which, at infinity, is parallel to the plate. The magnetic Reynolds number is assumed to be small so that perturbations to the applied uniform magnetic field are small. We assume, as a first approximation, that the magnetic field is unperturbed which is equivalent to setting the magnetic Reynolds number to zero in the induction equation. Then, as in the non-magnetic case, the major disturbance to the uniform flow is due to the thin boundary layer which is associated with the no-slip condition at the plate. Accordingly we introduce the classical boundary-layer scaling into the Navier-Stokes equations and retain only the highest order terms. Thus we write the two-dimensional stream function as

$$\psi = (2\nu U_0 x)^{\frac{1}{2}} f(\eta), \quad \eta = (U_0/2\nu x)^{\frac{1}{2}} y. \tag{2.1}$$

Here, x, y are coordinates along and normal to the plate with origin at the leading edge, U_0 is the undisturbed stream speed and ν the kinematic viscosity of the fluid. The Lorentz force normal to the plate $y = 0$ is balanced by the pressure gradient across the boundary layer, and if the pressure gradient is eliminated from the boundary-layer equations we have the following problem for $f(\eta)$,

$$f''' + ff'' + \varepsilon(\eta f - \eta^2 f') = 0, \tag{2.2}$$

with

$$f(0) = f'(0) = 0, \tag{2.3}$$

and

$$f'(\infty) = 1. \tag{2.4}$$

where $\varepsilon = (\sigma\mu^2 H_0^2 \nu/\rho U_0^2)$. The constants σ, μ, H_0 and ρ represent the fluid conductivity, permeability, undisturbed field strength and density respectively. The parameter $\varepsilon^{\frac{1}{2}}$ may be interpreted as the ratio of the Hartmann number to the Reynolds number. We shall assume $\varepsilon \ll 1$.

3. Solution procedure

To simplify the problem further we now neglect the term $O(\varepsilon)$ in (2.2) so that the first term of an approximate solution satisfies

with
$$\left.\begin{aligned} f_0''' + f_0 f_0'' &= 0, \\ f_0(0) = f_0'(0) = 0, \quad f_0'(\infty) &= 1, \end{aligned}\right\} \tag{3.1}$$

with solution $f_0(\eta) = B(\eta)$, where B is the classical Blasius function.

We note that

$$B(\eta) - \eta \to c, \quad (c \neq 0), \quad \text{as} \quad \eta \to \infty, \tag{3.2}$$

whereas we may infer from equations (2.2) to (2.4) that

$$f(\eta) - \eta \to 0 \quad \text{as} \quad \eta \to \infty. \tag{3.3}$$

The solution f_0 then can only be thought of as an " inner " solution and there must be an " outer " region in which the magnetic interaction terms in (2.2) (that is the terms $O(\varepsilon)$) are of the same order as either the viscous or inertia terms. This requirement together with (3.2) leads to the scaling

$$\eta = \varepsilon^{-\frac{1}{2}}\xi, \quad f(\eta) = \varepsilon^{-\frac{1}{2}} F(\xi), \tag{3.4}$$

and (2.2) becomes

$$\varepsilon F''' + FF'' + \xi F - \xi^2 F' = 0, \tag{3.5}$$

and in this outer region the viscous term is now relegated to $O(\varepsilon)$. Simplifying the outer problem by neglecting the term $O(\varepsilon)$, the first term of an " outer " approximate solution satisfies

$$F_0F_0''+\xi F_0-\xi^2F_0'=0,$$

and the solution, which matches with f_0 and is consistent with (3.3), is simply $F_0=\xi$. Thus we have constructed an apparently consistent first-order solution by matching solutions which are valid in different domains. As Segel observes, once a problem has been correctly scaled one may exploit perturbation theory systematically to derive arbitrarily accurate approximations. In practice this may not always be possible, but the problem represented by equations (2.2) to (2.4) is sufficiently simple for us to proceed to higher-order terms. The outer expansion of the inner solution f_0 shows that we require, as $\xi\to 0$,

$$F\sim\xi+\varepsilon^{\frac{1}{2}}c,$$

so to continue the solutions we write

$$f=f_0+\varepsilon^{\frac{1}{2}}f_1+O(\varepsilon),\quad F=F_0+\varepsilon^{\frac{1}{2}}F_1+O(\varepsilon), \tag{3.6}$$

in the inner and outer regions respectively. The equation satisfied by F_1 is

$$F_1''-\xi F_1'+F_1=0.$$

One solution of this equation has $F_1''=C_1\exp(\frac{1}{2}\xi^2)$ and since with this solution we cannot satisfy the outer boundary condition we set $C_1=0$. The other solution $F_1=C_2\xi$ is also excluded by the outer boundary condition and we have, therefore, $F_1\equiv 0$, which cannot match with the inner solution. At this stage then the expansion scheme, which gives an apparently self-consistent first-order solution, fails.

The devotee of singular-perturbation techniques may at once suspect that the scaling (3.4) is incorrect or that the series (3.6) are incomplete. Neither resolves the paradox and we have in fact constructed a wretched consistent first approximation, as we now demonstrate by proving that (2.2) has no solution subject to (2.3), (2.4).

The non-existence proof, which is in the same spirit as that employed by Reuter and Stewartson (**2**) for the case of large magnetic Reynolds number is in three parts depending upon whether $f_0''(0)$ is positive, negative or zero, and we treat each case separately.

Case (i). $f''(0)=0$

With $f''(0)=0$ we see from (2.2), (2.3) that $f'''(0)=0$ and repeated differentiation of (2.2) gives $f^n(0)=0$ for all n. This implies, by the analyticity of solutions of (2.3), that $f\equiv 0$, which contradicts (2.4) showing that no solutions exist with $f''(0)=0$.

Case (ii). $f''(0)>0$

As we have observed in (3.3) $f \sim \eta + o(1)$ as $\eta \to \infty$ where in fact the error is exponentially small. Thus $f'' \to 0$ as $\eta \to \infty$. Suppose, however, that f'' first vanishes at a finite value, η_0 say, of η. Then we have $f''(\eta_0) = 0, f'(\eta_0) = a_1$ (say), where $a_1 > 0$ and, if the solution is to be regular $f'''(\eta_0) \leqq 0$. Thus for $0 \leqq \eta < \eta_0, f'(\eta) < a_1$ which shows, using (2.3), that $f(\eta_0) < a_1 \eta_0$. We then note that with $f'''(\eta_0) \leqq 0$ and $(\eta f - \eta^2 f')_{\eta = \eta_0} < 0$ we have a contradiction which shows that f'' does not vanish at a finite value of η. This result, along with (2.3), implies that $f' \to 1$ monotonically from below as $\eta \to \infty$. We now define a new quantity $g(\eta)$ as

$$g = f' - 1, \tag{3.7}$$

and we note that for $0 \leqq \eta < \infty$, $g < 0$. Consider now $\int_0^\eta g d\eta$. From (3.3) we see that this integral converges and hence, as $\eta \to \infty$,

$$\int_0^\eta g d\eta \sim b + g^*(\eta), \tag{3.8}$$

where b is a negative constant and g^* is exponentially small. Thus from (3.7), (3.8) we have

$$f \sim \eta + b + g^*,$$

which contradicts (2.2) and shows that no solutions exist with $f''(0) > 0$.

Case (iii). $f''(0)<0$

When $f''(0) < 0$ we see from (2.3) that $f'(0+) < 0$. This, together with (3.3) shows that there exists a value, η_0 say, of η for which $f'(\eta_0) = 0$. Invoking Rolle's theorem, we see that $f''(\eta)$ vanishes at least once in the interval $0 < \eta < \eta_0$. Suppose that f'' vanishes for the first time at $\eta = \eta_1$ where $0 < \eta_1 < \eta_0$. We then have $f''(\eta_1) = 0, f'(\eta_1) = -a_2 < 0$ and, for the solution to be regular $f'''(\eta_1) \geqq 0$. From (2.3) we see that $f(\eta_1) > -a_2 \eta_1$, which in turn shows that

$$(\eta f - \eta^2 f')_{\eta = \eta_1} > 0;$$

this result together with $f'''(\eta_1) \geqq 0$ leads, as before, to a contradiction showing that no solutions of (2.2) to (2.4) exist with $f''(0) < 0$.

This completes the proof, in spite of the fact that it is possible using the method of matched asymptotic expansions to construct a first-order solution, that no solution of (2.2) to (2.4) exist.

4. Discussion

In deriving equation (2.2) we recall that since the magnetic Reynolds number is small the induction equation has been simplified by neglecting convective terms. Further, a scaling has been introduced into the Navier-Stokes equations which reflects the conjecture that all departures from uniform conditions occur in a thin boundary layer close to the plate. Although we have constructed an apparently self-consistent first-order solution of (2.2) using the method of matched asymptotic expansions, we have amply demonstrated that care must be taken in applying these formal methods. This is true in particular for those problems which exhibit a multi-layer structure for which it is not a practical proposition to extend the solution beyond the first order. In the present case, failure of the simplification procedure at the second stage does not imply that the procedure has been misused but is directly attributable to the non-existence of solutions.

The final question which we must pose is: at what point does a flaw exist in our analysis? The answer to this question can be found in the work of Cole (**3**), who has carried out a careful analysis of this problem. Consider again the scaling which leads first to (2.2) and then, subsequently, to the outer equation (3.5). If R and R_m are the Reynolds number and magnetic Reynolds number, based upon distance from the leading edge of the plate, respectively, then

$$y/(2x)^{\frac{1}{2}} = \eta R^{-\frac{1}{2}} = (\varepsilon R)^{-\frac{1}{2}}\xi = (\beta R_m)^{-\frac{1}{2}}\xi, \tag{4.1}$$

where $\beta = \mu H_0^2/\rho U_0^2$ is the square of the ratio of the Alfvén speed to the undisturbed fluid speed. Now, implicit in our arguments is the assumption that with $\varepsilon \ll 1$, $R \gg 1$ then $\varepsilon R \gg 1$. Furthermore, we have assumed $R_m \ll 1$, and so for consistency β must be very large. However, unless $\beta < 1$ disturbances will propagate upstream, and the notion of disturbances confined to a boundary layer growing from the leading edge is invalid. This is the source of our difficulty. As may be seen from the work of Cole (**3**), the disturbances to the magnetic field penetrate to a distance $O(R_m^{-\frac{1}{2}})$ from the plate, as indicated in (4.1). Thus, as Cole points out, perturbations to the magnetic field cannot be completely ignored but that they take place over a very much greater distance than the boundary-layer scale. In other words, this particular problem exhibits not only the features of high Reynolds number flows but also those analogous to low Reynolds number situations, and it is these latter which were not accounted for in our original scaling of the basic equations.

REFERENCES

(**1**) L. A. SEGEL, Simplification and scaling, *SIAM Rev.* **14** (1972), 547.

(**2**) G. E. H. REUTER and K. STEWARTSON, A non-existence theorem in magneto-fluid dynamics, *Phys. Fluids* **4** (1961), 276.

(**3**) R. J. COLE, Aligned-field magnetohydrodynamic flow past a flat plate, *Quart. J. Mech. Appl. Math.* **24** (1971), 187.

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