POSITIVE VALUES OF INHOMOGENEOUS QUATERNARY QUADRATIC FORMS, I

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1. Introduction

Let $Q(x_1, \cdots, x_n)$ be an indefinite quadratic form in n-variables with real coefficients, determinant $D \neq 0$ and signature (r, s), $r+s = n$. Then it is known (e.g. see Blaney [2]) that there exist constants $\Gamma_{r,s}$ depending only on r and s such for any real numbers $c_1, \cdots, c_n$ we can find integers $x_1, \cdots, x_n$ satisfying

$$0 < Q(x_1+c_1, \cdots, x_n+c_n) \leqq (\Gamma_{r,s}|D|)^{1/n}. \tag{1.1}$$

Let $\Gamma_{r,s}$ denote the best possible constant for which (1.1) is valid. Davenport and Heilbronn [5] proved that $\Gamma_{1,1} = 4$. E. S. Barnes [1] has proved that $\Gamma_{2,1} = 4$. In a paper accepted for publication [6] I have proved that $\Gamma_{1,2} = 8$. The object of this paper is to prove that $\Gamma_{3,1} = \frac{16}{3}$. In the next paper we shall show that $\Gamma_{2,2} = 16$.

More precisely we prove:

THEOREM. *Let $Q(x, y, z, t)$ be an indefinite quaternary quadratic form of the type* $(3, 1)$ *and determinant* $D < 0$. *Then given any real numbers* x_0, y_0, z_0, t_0 *we can find integers* x, y, z, t *such that*

$$0 < Q(x+x_0, y+y_0, z+z_0, t+t_0) \leqq (\tfrac{16}{3}|D|)^{\frac{1}{4}}. \tag{1.2}$$

Equality is necessary if and only if either

$$Q(x, y, z, t) \sim \rho Q_1 = \rho(x^2+xy+y^2+zt); \quad or \tag{1.3}$$

$$Q(x, y, z, t) \sim \rho Q_2 = \rho(x^2+y^2+z^2-3t^2); \tag{1.4}$$

where $\rho > 0$. *For* Q_1 *equality occurs if and only if* $(x_0, y_0, z_0, t_0) \equiv (0, 0, 0, 0)$ *(mod* 1) *and for* Q_2 *if and only if* $(x_0, y_0, z_0, t_0) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ *(mod* 1).

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2. Some Lemmas

In the course of the proof we shall use the following Lemmas:

LEMMA 1. *Let $Q(x, y, z, t)$ be an indefinite quaternary quadratic form of the type* (3, 1) *and determinant D. Then there exist integers x_1, y_1, z_1, t_1 such that*

$$0 < Q(x_1, y_1, z_1, t_1) \leqq (\tfrac{16}{3}|D|)^{\frac{1}{4}}.$$

Equality occurs if and only if $Q \sim \rho Q_1$; $\rho > 0$.

This is Theorem 2 of Oppenheim [8].

LEMMA 2. *Let $\varphi(y, z, t)$ be an indefinite ternary quadratic form with determinant $D < 0$. Then there exist integers u, v, w such that*

$$0 < \varphi(u, v, w) \leqq (\tfrac{9}{4}|D|)^{\frac{1}{3}}$$

except when

$$\varphi(y, z, t) \sim \rho(y^2+zt),\ \rho > 0.$$

This is a theorem due to Oppenheim [7].

LEMMA 3. *Let $\varphi(y, z, t)$ be an indefinite ternary quadratic form with determinant $D < 0$. Then given any real numbers y_0, z_0, t_0 we can find $(y, z, t) \equiv (y_0, z_0, t_0)$ (mod 1) such that*

$$|\varphi(y, z, t)| \leqq (\tfrac{27}{100}|D|)^{\frac{1}{3}}.$$

This is a theorem due to Davenport [4].

LEMMA 4. *Let $\varphi(y, z, t)$ be an indefinite ternary quadratic form with determinant $D > 0$. Then given any real numbers y_0, z_0, t_0 we can find $(y, z, t) \equiv (y_0, z_0, t_0)$ (mod 1) such that*

$$0 < \varphi(y, z, t) \leqq (8|D|)^{\frac{1}{3}}.$$

This is a result due to author [6] accepted for publication.

LEMMA 5. *Let $\chi(z, t)$ be an indefinite binary quadratic form with discriminant $\Delta^2 > 0$ and let $\lambda > 0$ be a real number. Then for any real numbers z_0, t_0 we can find $(z, t) \equiv (z_0, t_0)$ (mod 1) satisfying*

$$-\frac{\Delta}{4\lambda} \leqq \chi(z, t) < \frac{\lambda\Delta}{4}.$$

Equality is needed if and only if $\lambda^2 = (m+2)/m$; $m = 1, 2, \cdots$ and

$$\chi(z, t) \sim c\chi_m(z, t) = c(z^2-m(m+2)t^2),\ c > 0. \tag{2.1}$$

For $\chi_m(z, t)$ equality occurs if and only if $(z_0, t_0) \equiv (m/2, \frac{1}{2})$ (mod 1).

This is Theorem 1 of Blaney [3].

LEMMA 6. *Let α, β, d be real numbers with $d \geqq 1$; then for any real number x_0 there exists $x \equiv x_0$ (mod 1) satisfying*

$$0 < (x+\alpha)^2-\beta^2 \leqq d \tag{2.2}$$

provided that

$$\beta^2 < \left(\frac{[d]}{2}\right)^2. \tag{2.3}$$

If d is not an integer, (2.2) *is true with strict inequality. If d is an integer a sufficient condition for* (2.2) *to be true with strict inequality is that*

$$\beta^2 < \left(\frac{d-1}{2}\right)^2.$$

PROOF. If $\beta^2 < (d-1)^2/4$, choose $x \equiv x_0$ (mod 1) with

$$|\beta| < x+\alpha \leqq |\beta|+1,$$

so that

$$0 < (x+\alpha)^2-\beta^2 \leqq 2\,|\beta|+1 < d.$$

If d is an integer and $\beta^2 = (d-1)^2/4$, then we proceed as above and get the result perhaps with equality.

Now suppose

$$\beta^2 \begin{cases} \geqq \left(\dfrac{d-1}{2}\right)^2 & \text{if } d \text{ is not an integer} \\ > \left(\dfrac{d-1}{2}\right)^2 & \text{if } d \text{ is an integer;} \end{cases}$$

so that in either case

$$\beta^2 > \left(\frac{[d]-1}{2}\right)^2. \tag{2.4}$$

Choose $x \equiv x_0$ (mod 1) to satisfy

$$\frac{[d]}{2} \leqq |x+\alpha| \leqq \frac{[d]+1}{2},$$

so that

$$0 < \left(\frac{[d]}{2}\right)^2-\beta^2 \leqq (x+\alpha)^2-\beta^2 < \left(\frac{[d]+1}{2}\right)^2-\left(\frac{[d]-1}{2}\right)^2$$
$$= [d] \leqq d$$

from (2.3) and (2.4).

This completes the proof of the lemma.

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3. Proof of the theorem

Let

$$m = \inf Q(x, y, z, t) \quad x, y, z, t \text{ integers} \quad Q(x, y, z, t) > 0. \tag{3.1}$$

3.1. Case $m = 0$

Lemma 7. *If $m = 0$, then the result is true.*

Proof. Since $m = 0$; given ε_0 $(0 < \varepsilon_0 < 1)$ we can find integers x_1, y_1, z_1, t_1 such that

$$0 < Q(x_1, y_1, z_1, t_1) = \varepsilon < \varepsilon_0, \ (x_1, y_1, z_1, t_1) = 1.$$

By replacing Q by an equivalent form we can suppose $Q(1, 0, 0, 0) = \varepsilon$. Then Q can be written as

$$Q(x, y, z, t) = \varepsilon(x+hy+gz+ut)^2+\varphi(y, z, t)$$

where $\varphi(y, z, t)$ is an indefinite ternary quadratic form with determinant $D/\varepsilon < 0$. By Lemma 4, we can find $(y, z, t) \equiv (y_0, z_0, t_0) \pmod 1$ such that

$$0 < \beta^2 = -\varphi(y, z, t) \leqq \left(\frac{8|D|}{\varepsilon}\right)^{\frac{1}{3}}.$$

Let $\alpha = hy+gz+ut$ and choose $x \equiv x_0 \pmod 1$ with

$$\frac{\beta}{\sqrt{\varepsilon}} < x+\alpha \leqq \frac{\beta}{\sqrt{\varepsilon}}+1,$$

so that

$$\begin{aligned} 0 < Q(x, y, z, t) = \varepsilon(x+\alpha)^2-\beta^2 &\leqq \varepsilon+2\beta\sqrt{\varepsilon} \\ &\leqq \varepsilon+2\left(\frac{8|D|}{\varepsilon}\right)^{\frac{1}{6}}\varepsilon^{\frac{1}{2}} \\ &= \varepsilon+A|D|^{\frac{1}{6}}\cdot\varepsilon^{\frac{1}{3}} \\ &< \varepsilon_0+A|D|^{\frac{1}{6}}\varepsilon_0^{\frac{1}{3}} \end{aligned} \tag{3.2}$$

where A is an absolute constant. Since ε_0 can be chosen arbitrarily small, the right hand side of (3.2) can be made as small as we please and the lemma follows.

3.2. Proof continued

We can now suppose $m > 0$.

Then given $0 < \varepsilon_0 < \frac{1}{16}$, we can find integers x_1, y_1, z_1, t_1 to satisfy

$$Q(x_1, y_1, z_1, t_1) = \frac{m}{1-\varepsilon},$$

where $0 \leqq \varepsilon < \varepsilon_0$. By Lemma 1 we can further suppose that

$$Q(x_1, y_1, z_1, t_1) = \frac{m}{1-\varepsilon} \leqq (\tfrac{16}{3}|D|)^{\frac{1}{4}}.$$

Since $0 \leqq \varepsilon < \varepsilon_0 < \frac{1}{16}$, by definition of m we must have $(x_1, y_1, z_1, t_1) = 1$. By applying a suitable transformation to Q we can suppose that $Q(1, 0, 0, 0) = m/1-\varepsilon$. $Q(x, y, z, t)$ can then be written as

$$Q(x, y, z, t) = \frac{m}{1-\varepsilon}\{(x+hy+gz+ut)^2+\varphi(y, z, t)\}$$

where $\varphi(y, z, t)$ is an indefinite ternary quadratic form with determinant

$$\frac{D}{\left(\dfrac{m}{1-\varepsilon}\right)^4} \leqq -\tfrac{3}{16};$$

with equality if and only if $\varepsilon = 0$, $Q \sim mQ_1$, by Lemma 1. Also by definition of m we have for any integers x, y, z, t either $Q(x, y, z, t) \leqq 0$ or $Q(x, y, z, t) \geqq m$. Because of homogeneity it suffices to prove:

THEOREM A. *Let*

$$Q(x, y, z, t) = (x+hy+gz+ut)^2+\varphi(y, z, t); \tag{3.3}$$

where $\varphi(y, z, t)$ is an indefinite ternary quadratic form with determinant

$$D \leqq -\tfrac{3}{16}, \tag{3.4}$$

$D = -\frac{3}{16}$ if and only if $Q \sim Q_1$. Suppose that for integer x, y, z, t we have either

$$Q(x, y, z, t) \leqq 0 \text{ or } Q(x, y, z, t) \geqq 1-\varepsilon \tag{3.5}$$

where $0 \leqq \varepsilon \leqq \frac{1}{16}$ is sufficiently small. Let

$$d = (\tfrac{16}{3}|D|)^{\frac{1}{4}}, \tag{3.6}$$

so that from (3.4) we have $d \geqq 1$ with $d = 1$ if and only if $Q \sim Q_1$. Then given any real numbers x_0, y_0, z_0, t_0 we can find $(x, y, z, t) \equiv (x_0, y_0, z_0, t_0)$ (mod 1) satisfying

$$0 < Q(x, y, z, t) \leqq d. \tag{3.7}$$

Equality holds in (3.7) if and only if either $Q \sim Q_1$ or Q_2.

3.3. PROOF OF THEOREM A

LEMMA 8. *If $Q(x, y, z, t)$ is given as in Theorem A, then for integers y, z, t we have either*

(3.8) $\qquad \varphi(y, z, t) = 0$ *or* $\varphi(y, z, t) \leqq -\frac{1}{4}$ *or* $\varphi(y, z, t) \geqq \frac{3}{4}-\varepsilon.$

PROOF. If $0 < \varphi(y, z, t) < \frac{3}{4}-\varepsilon$, we get a contradiction to (3.5) by choosing integer x with $|x+hy+gz+ut| \leqq \frac{1}{2}$. If $-\frac{1}{4} < \varphi(y, z, t) < -\varepsilon$, we again get a contradiction by choosing x with $\frac{1}{2} \leqq |x+hy+gz+ut| \leqq 1$. If $-\varepsilon \leqq \varphi(y, z, t) < 0$, then for a suitable integer n we have $-\frac{1}{4} < Q(ny, nz, nt) < -\varepsilon$, which is not possible. This proves the lemma.

LEMMA 9. *If* $d = 1$, *so that* $Q = Q_1 = x^2+xy+y^2+zt$, (3.7) *is true with strict inequality unless* $(x_0, y_0, z_0, t_0) \equiv (0, 0, 0, 0)$ (*mod* 1).

PROOF. If $(z_0, t_0) \not\equiv (0, 0) \pmod 1$; without loss of generality we can suppose that $z_0 \not\equiv 0 \pmod 1$. Choose $z \equiv z_0 \pmod 1$ with $0 < |z| \leqq \frac{1}{2}$. Choose $(x, y) \equiv (x_0, y_0) \pmod 1$ arbitrarily so that Q is of the form

$$Q(x, y, z, t) = A+zt.$$

Now choose $t \equiv t_0 \pmod 1$ with $0 < A+zt \leqq z \leqq \frac{1}{2} < 1 = d$. If $(z_0, t_0) \equiv (0, 0) \pmod 1$; take $z = t = 0$. Choose $y \equiv y_0 \pmod 1$ with $|y| \leqq \frac{1}{2}$. If $y_0 \not\equiv 0 \pmod 1$, choose $x \equiv x_0 \pmod 1$ with $|x+y/2| \leqq \frac{1}{2}$, so that

$$0 < Q(x, y, z, t) = \left(x+\frac{y}{2}\right)^2 + \tfrac{3}{4}y^2 \leqq \tfrac{1}{4}+\tfrac{3}{16} < 1 = d.$$

If $y = 0$, so that $y_0 \equiv 0 \pmod 1$ and $Q(x, y, z, t) = x^2$. Choose $x \equiv x_0 \pmod 1$ with $0 < x \leqq 1$; so that

$$0 < Q(x, y, z, t) = x^2 \leqq 1 = d.$$

Equality can occur only if $x_0 \equiv 0 \pmod 1$ i.e. only if $(x_0, y_0, z_0, t_0) \equiv (0, 0, 0, 0) \pmod 1$. Clearly equality is needed for Q_1 when $(x_0, y_0, z_0, t_0) \equiv (0, 0, 0, 0) \pmod 1$.

This proves the lemma.

We can now suppose $d > 1$.

LEMMA 10. *Let* $\nu_1 > 0$, $\nu_2 > 0$ *be defined by*

(3.9) $$\nu_1 = d-\tfrac{1}{4}$$

(3.10) $$\nu_2 = \begin{cases} \left(\dfrac{d-1}{2}\right)^2 & \text{if } d \text{ is an integer} \\ \left(\dfrac{[d]}{2}\right)^2 & \text{if } d \text{ is not an integer.} \end{cases}$$

Suppose we can find $(y, z, t) \equiv (y_0, z_0, t_0)$ (*mod* 1) *satisfying*

(3.11) $$-\nu_2 \leqq \phi(y, z, t) < \nu_1.$$

Then for any x_0 *there exists* $x \equiv x_0$ (*mod* 1) *such that*

$$(3.12) \qquad 0 < Q(x, y, z, t) \leqq d.$$

Further strict inequality in (3.11) *implies strict inequality in* (3.12).

PROOF. If $0 < \varphi(y, z, t) < \nu_1$, choose $x \equiv x_0 \pmod 1$ with

$$|x+hy+gz+ut| \leqq \tfrac{1}{2},$$

so that

$$0 < Q(x, y, z, t) = (x+hy+gz+ut)^2+\varphi(y, z, t) < \tfrac{1}{4}+\nu_1 = d.$$

If $-\nu_2 \leqq \varphi(y, z, t) \leqq 0$, then the result follows from Lemma 6 with

$$\alpha = hy+gz+ut, \ \beta^2 = -\varphi(y, z, t).$$

This completes the proof of the lemma.

LEMMA 11. *If $d > 12$, then* (3.7) *is true with strict inequality.*

PROOF. By Lemma 4, we can find $(y, z, t) \equiv (y_0, z_0, t_0) \pmod 1$ satisfying

$$-(\tfrac{3}{2}d^4)^{\frac{1}{3}} = -(8\,|D|)^{\frac{1}{3}} \leqq \varphi(y, z, t) < 0.$$

Therefore (3.11) is satisfied if we have

$$(\tfrac{3}{2}d^4)^{\frac{1}{3}} < \left(\frac{d-1}{2}\right)^2,$$

or

$$f(d) = \frac{(d-1)^6}{d^4} > 96.$$

Since $f(d)$ is an increasing function of d for $d > 1$ and $f(13) > 96$, the result follows from Lemma 10. Let now $12 < d < 13$, so that $[d] = 12$, d not an integer. In this case (3.11) holds if we have

$$(\tfrac{3}{2}d^4)^{\frac{1}{3}} < 36.$$

Since $(\tfrac{3}{2}d^4)^{\frac{1}{3}} \leqq (\tfrac{3}{2}\cdot 13^4)^{\frac{1}{3}} < 35\cdot 1 < 36$, for $d < 13$, the result again follows from Lemma 10.

LEMMA 12. *If* $4 < d \leqq 12$, then $(3\cdot 7)$ *is true with strict inequality.*

PROOF. By Lemma 3, we can find $(y, z, t) \equiv (y_0, z_0, t_0) \pmod 1$ such that

$$|\varphi(y, z, t)| \leqq (\tfrac{27}{100}\,|D|)^{\frac{1}{3}} = (\tfrac{81}{1600}d^4)^{\frac{1}{3}}.$$

Hence the result will follow from Lemma 10 if we

$$(\tfrac{81}{1600}d^4)^{\frac{1}{3}} < \min(\nu_1, \nu_2).$$

Now

$$(\tfrac{81}{1600}d^4)^{\frac{1}{3}} < \nu_1 = d-\tfrac{1}{4}, \quad \text{if}$$

$$f(d) = \frac{(4d-1)^3}{d^4} > \tfrac{81}{25}.$$

$f(d)$ is a decreasing function of d for $d > 1$ so that $f(d) \geqq f(12) > \frac{81}{25}$ for $d \leqq 12$. Also

$$(\tfrac{81}{1600} d^4)^{\frac{1}{3}} < \nu_2 = \begin{cases} \left(\dfrac{d-1}{2}\right)^2 & \text{if } 5 \leqq d \leqq 12 \\ 4 & \text{if } 4 < d < 5 \end{cases}$$

is easily seen to be true and the assertion of the lemma follows.

LEMMA 13. *If $\varphi(y, z, t) \sim \rho(y^2+zt)$, $\rho > 0$, $1 < d \leqq 4$, then again* (3.7) *is true with strict inequality.*

PROOF. Without loss of generality we can suppose that

$$Q(x, y, z, t) = (x+hy+gz+ut)^2+\rho(y^2+zt),$$

with $|h| \leqq \frac{1}{2}$, $|g| \leqq \frac{1}{2}$, $|u| \leqq \frac{1}{2}$.

We first assert that $h = g = u = 0$.

If $g \neq 0$, then

$$0 < Q(0, 0, 1, 0) = g^2 \leqq \tfrac{1}{4},$$

contrary to (3.5). Similarly $u = 0$. If $h \neq 0$, then

$$0 < Q(0, 1, 1, -1) = h^2 \leqq \tfrac{1}{4},$$

again contrary to (3.5). Thus

$$Q(x, y, z, t) = x^2+\rho(y^2+zt).$$

If $(z_0, t_0) \not\equiv (0, 0) \pmod 1$; without loss of generality suppose that $z_0 \not\equiv 0 \pmod 1$. Choose $z \equiv z_0 \pmod 1$ with $0 < |z| \leqq \frac{1}{2}$. Choose any $(x, y) \equiv (x_0, y_0) \pmod 1$ and then take $t \equiv t_0 \pmod 1$ to satisfy

$$0 < x^2+\rho(y^2+zt) \leqq \rho\,|z| \leqq \frac{\rho}{2} = (\tfrac{3}{32} d^4)^{\frac{1}{3}} < d; \text{ since } d \leqq 4.$$

Let now $(z_0, t_0) \equiv (0, 0) \pmod 1$. We now distinguish between the following two subcases:

(i) $y_0 \equiv 0 \pmod 1$

(ii) $y_0 \not\equiv 0 \pmod 1$.

Subcase (i): In this case take $y = 1$, $z = 1$, $t = -1$. Choose $x \equiv x_0 \pmod 1$ with $0 < x \leqq 1$, so that

$$0 < Q(x, y, z, t) = x^2 \leqq 1 < d.$$

Subcase (ii): In this case take $z = t = 0$. Choose $(x, y) \equiv (x_0, y_0) \pmod 1$ with $|x| \leqq \frac{1}{2}$, $0 < |y| \leqq \frac{1}{2}$, so that

$$0 < Q(x, y, z, t) = x^2+\rho y^2 \leqq \frac{1+\rho}{4} < d,$$

if $\rho < 4d-1$; or

$$f(d) = \frac{d^4}{(4d-1)^3} < \tfrac{4}{3}.$$

Since $f(d)$ is an increasing function of d for $d > 1$ and $f(4) = \frac{256}{15^3} < \frac{4}{3}$, the desired result follows.

3.4. Proof of theorem A continued

From now on we can suppose that

$$1 < d \leqq 4;\ \varphi(y, z, t) \nsim \rho(y^2+zt),\ \rho > 0.$$

By Lemma 2 we can find integers y_2, z_2, t_2 such that

$$0 < \varphi(y_2, z_2, t_2) = a \leqq (\tfrac{9}{4}|D|)^{\frac{1}{3}} = \tfrac{3}{4}d^{\frac{1}{3}};\ (y_2, z_2, t_2) = 1.$$

Also by (3.8) we have $a \geqq \frac{3}{4}-\varepsilon$. By a suitable unimodular transformation we can suppose that $\varphi(1, 0, 0) = a$, so that

(3.13) $$\varphi(y, z, t) = a\{(y+fz+vt)^2+\psi(z, t)\};$$

where

(3.14) $$\tfrac{3}{4}-\varepsilon \leqq a \leqq \tfrac{3}{4}d^{\frac{1}{3}};$$

and $\psi(z, t)$ is an indefinite binary quadratic form with discriminant

(3.15) $$\Delta^2 = \frac{4|D|}{a^3} \geqq \frac{16}{9}.$$

Let

(3.16) $$k = \frac{d-\frac{1}{4}}{a},$$

(3.17) $$\lambda = \frac{4k-1}{\Delta}.$$

Then from (3.16) we have

$$k \geqq \tfrac{4}{3}\cdot\frac{d-\frac{1}{4}}{d^{\frac{1}{3}}} = \frac{4d-1}{3d^{\frac{1}{3}}} \geqq \frac{4\cdot 4-1}{3\cdot 4^{\frac{1}{3}}} = \frac{5}{4^{\frac{1}{3}}} > \tfrac{1}{4};$$

for $1 < d \leqq 4$, so that $\lambda > 0$. By Lemma 5 we can find $(z, t) \equiv (z_0, t_0) \pmod 1$ such that

$$-\frac{\Delta}{4\lambda} \leqq \psi(z, t) < \frac{\lambda\Delta}{4} = k-\tfrac{1}{4}$$

or

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$$-\frac{3d^4}{16a^2(4d-1-a)} \leqq \psi(z, t) < k-\tfrac{1}{4}. \tag{3.18}$$

Therefore,

$$Q(x, y, z, t) = (x+hy+gz+ut)^2+a\{(y+fz+vt)^2+\psi(z, t)\};$$

where $\psi(z, t)$ satisfies (3.18).

Lemma 14. *If* $k-1 \leqq \psi(z, t) < k-\frac{1}{4}$, *then* (3.7) *is true with strict inequality.*

Proof. Choose $y \equiv y_0 \pmod 1$ with $|y+fz+vt| \leqq \frac{1}{2}$, so that

$$\begin{aligned} a(k-1) \leqq \varphi(y, z, t) = a\{(y+fz+vt)^2+\psi(z, y)\} &< a(\tfrac{1}{4}+k-\tfrac{1}{4}) \\ -(a-d+\tfrac{1}{4}) \leqq \varphi(y, z, t) &< d-\tfrac{1}{4}. \end{aligned}$$

Thus (3.11) is satisfied if we have

$$a-d+\tfrac{1}{4} < \left(\frac{d-1}{2}\right)^2;$$

or

$$4a < d^2+2d.$$

This is satisfied if

$$4a \leqq 3d^{\frac{4}{3}} < d(d+2),$$

or

$$f(d) = \frac{(d+2)^3}{d} > 27.$$

$f(d)$ is an increasing function of d for $d > 1$. Thus for $d > 1$, $f(d) > f(1) = 27$, (3.11) holds and the result follows from Lemma 10.

From now we can suppose

$$1-k < \beta = -\psi(z, t) \leqq \frac{3d^4}{16a^2(4d-1-a)}. \tag{3.19}$$

Lemma 15. *If* $2 < d \leqq 4$, *then* (3.7) *is true with strict inequality.*

Proof. Choose $y \equiv y_0 \pmod 1$ to satisfy

$$\sqrt{\beta+k}-1 \leqq y+fz+vt < \sqrt{\beta+k}$$

so that

$$-(2a\sqrt{\beta+k}+\tfrac{1}{4}-a-d) \leqq \varphi(y, z, t) < ak = d-\tfrac{1}{4}.$$

The result will follow from Lemma 10 if we have

$$2a\sqrt{\beta+k}+\tfrac{1}{4}-a-d < \begin{cases} \frac{9}{4} & \text{if } 3 < d \leqq 4. \quad (3.20)\\ 1 & \text{if } 2 < d \leqq 3. \quad (3.21)\end{cases}$$

We take the two cases separately.

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Subcase (i) $3 < d \leqq 4$.

In this case by using (3.19), (3.20) will be satisfied if

$$4a^2 \cdot \frac{3d^4}{16a^2(4d-1-a)} + a(4d-1) < (a+d+2)^2$$

or

$$f(a, d) = 4a\{a-3(d-1)\}^2 + 3d^4 - 4(4d-1)(d+2)^2 < 0 \tag{3.22}$$

$$\frac{\partial f}{\partial a} = 12(a-3d+3)(a-d+1).$$

Since $a \leqq \frac{3}{4}d^{\frac{4}{3}} < 3(d-1)$; for $3 \leqq d \leqq 4$, maximum of $f(a, d)$ in the proper range occurs at $a = d-1$. Therefore

$$\begin{aligned} f(a, d) \leqq f(d-1, d) &= 16(d-1)^3 + 3d^4 - 4(4d-1)(d+2)^2 \\ &= 3d^2(d^2-36) \\ &< 0 \quad (\text{since } d \leqq 4). \end{aligned}$$

Thus (3.22) is true and the result follows in this case.

Subcase (ii) $2 < d \leqq 3$.

In this case (3.21) is satisfied if we have

$$4a^2(\beta+k) < (d+a+\tfrac{3}{4})^2.$$

This will be so if

$$4a^2 \cdot \frac{3d^4}{16a^2(4d-1-a)} + a(4d-1) < \frac{(4d+4a+3)^2}{16}$$

or

$$f(a, d) = 16a\left(a - \frac{12d-7}{4}\right)^2 + 12d^4 - (4d-1)(4d+3)^2 < 0 \tag{3.23}$$

$$\frac{\partial f}{\partial a} = 48\left(a - \frac{12d-7}{4}\right)\left(a - \frac{12d-7}{12}\right).$$

Since $a \leqq \frac{3}{4}d^{\frac{4}{3}} < (12d-7/4)$ for $2 < d \leqq 3$, we have

$$\begin{aligned} \max f(a, d) &= f\left(\frac{12d-7}{12}, d\right) \\ \tfrac{3}{4}-\varepsilon \leqq a \leqq \tfrac{3}{4}d^{\frac{4}{3}} &= 12d^4 - 192d^2 + \tfrac{160}{3}d - \tfrac{100}{27} \\ &= 12d^2(d^2-9) + \tfrac{80}{3}d(2-d) - \tfrac{172}{3}d^2 - \tfrac{100}{27} \\ &< 0 \ \text{ for } \ 2 < d \leqq 3. \end{aligned}$$

Hence (3.23) is satisfied and the result follows.

LEMMA 16. *If* $1 < d \leqq 2$, *then again* (3.7) *is true.*

PROOF. If $\beta+k > \frac{9}{4}$, choose $y \equiv y_0 \pmod 1$ with

$$1 \leqq |y+fz+vt| \leqq \tfrac{3}{2},$$

so that

$$-a(\beta-1) \leqq \varphi(y, z, t) \leqq a(\tfrac{9}{4}-\beta) < ak = d-\tfrac{1}{4}.$$

The result will follow from Lemma 10 with strict inequality if we have

$$a(\beta-1) < \tfrac{1}{4}.$$

This will be so if

$$\frac{3d^4}{16a(4d-1-a)} < a+\tfrac{1}{4}$$

i.e.

$$f(a, d) = 4(4a^2+a)(4d-1-a)-3d^4 > 0.$$

For

$$\tfrac{3}{4}-\varepsilon \leqq a \leqq \tfrac{3}{4}d^{\frac{4}{3}} < 2; \text{ for } 1 < d \leqq 2,$$

we have

$$\begin{aligned} f(a, d) &\geqq 4(\tfrac{9}{4}+\tfrac{3}{4})(4d-3)-3d^4+0(\varepsilon) \\ &= 3(16d-12-d^4)+0(\varepsilon) \\ &> 0 \text{ for } 1 < d \leqq 2 \end{aligned}$$

as is easily verified by the rule of signs.

Let now $1 < \beta+k \leqq \frac{9}{4}$, choose $y \equiv y_0 \pmod 1$ with $\frac{1}{2} \leqq |y+fz+vt| \leqq 1$, so that

$$-a(\beta-\tfrac{1}{4}) \leqq \varphi(y, z, t) \leqq a(1-\beta) < ak = d-\tfrac{1}{4}.$$

Thus $\varphi(y, z, t)$ satisfies (3.11) if we have

$$a\beta \leqq \frac{a+1}{4} \tag{3.24}$$

We shall now distinguish the following two subcases:

(i) $a < d/2$.

(ii) $a \geqq d/2$.

Subcase (i) If $a < d/2$, then since $\beta+k \leqq \frac{9}{4}$, we have

$$\alpha\beta \leqq a(\tfrac{9}{4}-k) = \frac{9a}{4} - d + \tfrac{1}{4} < \frac{a}{4} + \tfrac{1}{4};$$

and the result follows.

Subcase (ii) If $a \geqq d/2$, then $a\beta \leqq (a+1)/4$ is satisfied if we have

$$a\beta \leqq \frac{3d^4}{16a(4d-1-a)} \leqq \frac{a+1}{4},$$

or

$$\text{(3.25)}\qquad f(a,d) = 4(a^2+a)(4d-1-a)-3d^4 \geqq 0$$

$$\frac{\partial f}{\partial a} = 4\{-3a^2+4a(2d-1)+4d-1\}$$

$$= 12(a+\alpha_1)(\alpha_2-a),\ \alpha_1 > 0$$

$$\alpha_2 = \frac{4d-2+\{(4d-2)^2+3(4d-1)\}^{\frac{1}{2}}}{3}$$

$$> \tfrac{3}{4}d^{\frac{4}{3}} \geqq a \quad \text{if } 1 < d \leqq 2,$$

so that $\partial f/\partial a > 0$. Therefore,

$$\text{(3.26)}\qquad f(a,d) \geqq f\left(\frac{d}{2}, d\right) = \tfrac{1}{2}d(2-d)(6d^2+5d-2)$$

$$\geqq 0 \quad \text{for } 1 < d \leqq 2.$$

Thus (3.25) is satisfied and the result follows from Lemma 10. This completes the proof of the lemma.

4. Case of equality

LEMMA 17. *Equality is needed in* (3.7) *if and only if* ($Q(x, y, z, t) \sim Q_2$ *and* $(x_0, y_0, z_0, t_0) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ (*mod* 1).

PROOF. Equality can occur only if we have equality in (3.25). This from (3.26) will be so only if

$$d = 2, \quad a = \frac{d}{2} = 1.$$

From (3.6), (3.15), (3.16) and (3.17) we have

$$|D| = 3, \quad k = \tfrac{7}{4}, \quad \Delta^2 = 12, \quad \lambda^2 = 3.$$

Also we must have equality in Lemma 5. Since $\lambda^2 = 3 = (1+2)/1$ ($m = 1$), we must have

$$\psi(z, t) \sim c\psi_1 = c(z^2-3t^2), \quad c > 0$$

and $(z_0, t_0) \equiv (\frac{1}{2}, \frac{1}{2})$ (mod 1).

Since $\Delta^2 = 12$, we get $c = 1$. Thus we can take

$$\text{(4.1)}\qquad \varphi(y, z, t) = (y+fz+vt)^2+z^2-3t^2.$$

By a suitable unimodular transformation we can suppose

$$\text{(4.2)}\qquad |h| \leqq \tfrac{1}{2}, \quad |g| \leqq \tfrac{1}{2}, \quad |u| \leqq \tfrac{1}{2}, \quad |f| \leqq \tfrac{1}{2}, \quad |v| \leqq \tfrac{1}{2}.$$

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Again for equality to occur, the inequalities

$$-\tfrac{1}{4} < F(y, z, t) = \left(y+fz+vt+\frac{f}{2}+\frac{v}{2}+y_0\right)^2+(z+\tfrac{1}{2})^2-3(t+\tfrac{1}{2})^2 < \tfrac{7}{4}$$

should have no solution in integers y, z, t.

$$-\tfrac{1}{4} < F(y, 1, 0) = \left(y+\frac{3f}{2}+\frac{v}{2}+y_0\right)^2+\tfrac{3}{2} < \tfrac{7}{4}$$

is solvable for integer y unless

$$\frac{3f}{2}+\frac{v}{2}+y_0 \equiv \tfrac{1}{2} \pmod 1. \tag{4.3}$$

Similarly by considering $F(y, 1, -1)$ and $F(y, 0, 0)$ we see that if equality is to occur we must have

$$\frac{3f}{2}-\frac{v}{2}+y_0 \equiv \tfrac{1}{2} \pmod 1 \tag{4.4}$$

and

$$\frac{f}{2}+\frac{v}{2}+y_0 \equiv \tfrac{1}{2} \pmod 1. \tag{4.5}$$

From (4.3), (4.4), (4.5) and (4.2) we get

$$f = v = 0, \quad y_0 = \tfrac{1}{2} \pmod 1.$$

Therefore,

$$\varphi(y, z, t) = y^2+z^2-3t^2,$$

and $(y_0, z_0, t_0) \equiv (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) \pmod 1$.

Again if equality is to occur, the inequalities

$$0 < G(x, y, z, t) = \left(x+hy+gz+ut+\frac{h}{2}+\frac{g}{2}+\frac{u}{2}+x_0\right)^2$$
$$+(y+\tfrac{1}{2})^2+(z+\tfrac{1}{2})^2-3(t+\tfrac{1}{2})^2 < 2,$$

should have no solution in integers x, y, z, t.

$$0 < G(x, 0, 0, 0) = \left(x+\frac{h}{2}+\frac{g}{2}+\frac{u}{2}+x_0\right)^2-\tfrac{1}{4} < 2$$

is solvable for integer x unless

$$\frac{h}{2}+\frac{g}{2}+\frac{u}{2}+x_0 \equiv \tfrac{1}{2} \pmod 1. \tag{4.6}$$

Similarly by considering $G(x, -1, 0, 0)$, $G(x, 0, -1, 0)$ and $G(x, 0, 0, -1)$ we see that if equality is to occur we must have

$$-\frac{h}{2}+\frac{g}{2}+\frac{u}{2}+x_0 \equiv \tfrac{1}{2} \pmod 1 \tag{4.7}$$

$$\frac{h}{2}-\frac{g}{2}+\frac{u}{2}+x_0 \equiv \tfrac{1}{2} \pmod 1 \tag{4.8}$$

$$\frac{h}{2}+\frac{g}{2}-\frac{u}{2}+x_0 \equiv \tfrac{1}{2} \pmod 1. \tag{4.9}$$

From the last four congruences and (4.2) we get

$$h = g = u = 0,\ x_0 \equiv \tfrac{1}{2} \pmod 1.$$

Thus equality can occur only if $Q(x, y, z, t) = x^2+y^2+z^2-3t^2 = Q_2$ and $(x_0, y_0, z_0, t_0) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2},)$ mod 1). We next show that equality is needed for Q_2 when $(x_0, y_0, z_0, t_0) \equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \pmod 1$. For this it suffices to show that for integers x, y, z, t we have either

$$(x+\tfrac{1}{2})^2+(y+\tfrac{1}{2})^2+(z+\tfrac{1}{2})^2-3(t+\tfrac{1}{2})^2 \leqq 0 \text{ or } \geqq 2.$$

It is enough to prove that $x^2+y^2+z^2-3t^2 \leqq 0$ or $\geqq 8$ for odd integers x, y, z, t. This is clear since $x^2+y^2+z^2-3t^2 \equiv 0 \pmod 8$ for odd x, y, z, t. This completes the proof of the Lemma.

Theorem A follows from Lemmas 8 to 17, and the theorem is proved.

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