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Structure of perfect rings

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In the present note, we offer a simple characterization of perfect rings in terms of their components and socle sequences, which is subsequently used to establish a one-to-one correspondence between perfect rings and certain finite additive categories. This correspondence is effected by means of a matrix representation, which describes the way in which perfect rings are built from local perfect rings.

The concept of a perfect ring was introduced by S. Eilenberg in [2]; later, in his paper [1], H. Bass characterized perfect rings in several ways. As our starting point, refer to Theorem P (1) of [1] and call a ring R (right) perfect if

(a) $R/\text{Rad } R$ is artinian (i.e. completely reducible)

and

(b) $\text{Rad } R$ is T-nilpotent in the sense that, given any sequence $\{\rho_i\}$ of elements of $\text{Rad } R$, there exists an n such that
$\rho_n \rho_{n-1} \cdots \rho_2 \rho_1 = 0$.

In what follows, R denotes a ring with unity; by a module M we always understand a (left unital) R-module. The symbol $\text{Rad } M$ stands for the intersection of all maximal submodules of M if there are any; otherwise $\text{Rad } M = M$. Dually, if M has minimal submodules, $\text{Soc } M$ denotes their union; if M has no minimal submodules, $\text{Soc } M = 0$. In a ring R, define the (left transfinite) socle sequence

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$$0 = S^{(0)} \subseteq S^{(1)} \subseteq \ldots \subseteq S^{(\alpha)} \subseteq \ldots \subseteq R$$

of two-sided ideals $S^{(\alpha)}$ by

$$S^{(\alpha)}/S^{(\alpha-1)} = \text{Soc } R/S^{(\alpha-1)} \quad \text{for all non-limit}$$

and

$$S^{(\alpha)} = \bigcup_{\beta<\alpha} S^{(\beta)} \quad \text{for all limit ordinals } 1 \leq \alpha .$$

If $R = S^{(\delta)}$ for a certain δ, R is said to *have a socle sequence*. It is easy to see that R has a socle sequence if and only if Soc $M \neq 0$ for every R-module $M \neq 0$. This, in turn, is equivalent to the fact that every non-zero monogenic R-module possesses a minimal submodule, i.e. that, for every proper left ideal L of R, the R-module R/L contains a minimal submodule.

PROPOSITION 1. *A ring R has a socle sequence if and only if $R/\text{Rad } R$ has a socle sequence and* Rad R *is T-nilpotent. Thus, in particular, if R has a socle sequence, then* Rad R *is nil.*

Proof. Using the argument of H. Bass [1] p. 470, the implication "if" follows easily. In order to prove the opposite assertion, consider a proper left ideal L of R and notice that Soc $R/L \neq 0$ if Rad $R \subseteq L$. Also, R/L has obviously a minimal submodule provided that $R/L \cap \text{Rad } R$ has one. Hence, we may assume that $L \subsetneq \text{Rad } R$.

Suppose that R/L has no minimal submodule. Then, we can construct a sequence $\{\rho_i\}$ of elements of Rad R in the following way: Take $\rho_1 \in \text{Rad } R \setminus L$ and assume that we have already chosen $\rho_2, \ldots, \rho_n \in \text{Rad } R$ such that

$$\alpha_n = \rho_n \rho_{n-1} \ldots \rho_2 \rho_1 \notin L .$$

Thus $R\alpha_n \nsubseteq L$; moreover, we can show that also $(\text{Rad } R)\alpha_n \nsubseteq L$. For, if $(\text{Rad } R)\alpha_n \subseteq L$, then the non-zero submodule $R\alpha_n + L/L$ of R/L which is isomorphic to $R\alpha_n/R\alpha_n \cap L$ would be a homomorphic image of $R\alpha_n/(\text{Rad } R)\alpha_n$ and thus a homomorphic image of $R/\text{Rad } R$. Therefore R/L would possess

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a minimal submodule. Hence, we may choose $\rho_{n+1} \in \text{Rad } R$ such that $\alpha_{n+1} = \rho_{n+1}\alpha_n \notin L$. However, the existence of such a sequence $\{\rho_i\}$ contradicts the T-nilpotence of Rad R . The proof is completed.

Notice that using Proposition 1 we can deduce easily that R is perfect if and only if (a) holds and R has a socle sequence (cf. J.P. Jans [3]).

PROPOSITION 2. *Let* $R = L \oplus T$ *with an indecomposable left ideal* L *and let* R *have a socle sequence. Then* L *contains a unique left ideal* K *of* R *maximal in* L *. Thus, in particular,* R *is an indecomposable ring which has a socle sequence if and only if* R *is a local (i.e. possessing a unique maximal left ideal) perfect ring.*

Proof. Let $\{\lambda, \tau\}$ be a complete set of orthogonal idempotents of R corresponding to the decomposition $R = L \oplus T$, and $\{\bar{\lambda}, \bar{\tau}\}$ the respective set of idempotents of $R/\text{Rad } R$. Obviously, $K = \text{Rad } L \subsetneqq L$. Take a left ideal S of R , $K \subset S \subseteq L$, such that S/K is a minimal submodule of L/K . By definition of K , there is a left ideal W of R , $K \subseteq W \subset L$, maximal in L and such that $S \cap W = K$. Our Proposition will be proved if we show that $W = K$, i.e. that $S = L$. Assume that $W \neq K$. Then $R/\text{Rad } R \cong L/K \oplus T/\text{Rad } T = S/K \oplus W/K \oplus T/\text{Rad } T$; moreover, S/K contains an idempotent $\bar{\sigma}_0$. Consider the set consisting of the idempotents $\bar{\sigma} = \bar{\lambda}\bar{\sigma}$, $\bar{\omega} = \bar{\lambda} - \bar{\sigma}$ and $\bar{\tau}$; arguments of a routine nature yield that it is a complete set of orthogonal idempotents of $R/\text{Rad } R$. In view of Proposition 1, we can lift these idempotents modulo Rad R and we get

$$R = R\sigma \oplus R\omega \oplus T ,$$

contradicting the indecomposability of L . The proof is completed.

THEOREM 1. *A ring* R *is (right) perfect if and only if*

(a)* $R = \bigoplus_{i=1}^{r} L_i$ *with indecomposable (left) ideals* L_i , $1 \le i \le r$,

and

(b)* R *has a socle sequence.*

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A ring R *satisfying (a*) and (b*) has the property that each component* L_i *contains a unique left ideal* K_i *of* R *maximal in* L_i *,*

$$\text{Rad } R = \bigoplus_{i=1}^{r} K_i ,$$

$$R/\text{Rad } R \cong \bigoplus_{i=1}^{r} L_i/K_i$$

and thus, the decomposition of R *in (a*) is unique up to an isomorphism and order of the components.*

Proof. In view of our definition of a perfect ring, both *(a*)* and *(b*)* follow easily from Proposition 1 and the fact that idempotents modulo Rad R can be lifted.

Now, if R satisfies *(a*)* and *(b*)*, then in view of Proposition 2 each L_i contains a unique maximal left ideal K_i ,

$\text{Rad } R = \bigoplus_{i=1}^{r} \text{Rad } L_i = \bigoplus_{i=1}^{r} K_i$ and the uniqueness of the decomposition in *(a*)* follows from the uniqueness of the decomposition $\bigoplus_{i=1}^{r} L_i/K_i$ of $R/\text{Rad } R$ combined with the fact that $L_i/K_i \simeq L_j/K_j$ for $1 \le i$, $j \le r$ implies $L_i \simeq L_j$. In particular, R satisfies (a) and is therefore perfect.

PROPOSITION 3. *Let* $R = \bigoplus_{i=1}^{r} L_i$ *be an indecomposable decomposition of a perfect ring* R *. Then, for each* i *,* $1 \le i \le r$ *, the endomorphism ring* $\text{End}_R(L_i)$ *of* L_i *is a local perfect ring.*

Proof. Without loss of generality, take $i = 1$ and denote by K_1 the unique left ideal of R maximal in L_1 . First, notice that

$$\{\varphi \mid \varphi \in \text{End}_R(L_1) \text{ and } L_1\varphi \subseteq K_1\}$$

is the unique (left) maximal ideal of $E_1 = \text{End}_R(L_1)$ and hence E_1 is local.

In order to establish that E_1 satisfies *(b*)* , let us take an

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arbitrary left ideal A of E_1 and show that the E_1-module E_1/A has a minimal E_1-submodule. To this end, consider the "standard" matrix representation Δ of the ring R corresponding to our decomposition: R is isomorphic to the ring $R\Delta$ of all $r \times r$ matrices (x_{ij}) with $x_{ij} \in \mathrm{Hom}_R(L_i, L_j)$. Now, denote, for $2 \le i \le r$, by N_{i1} the submodule of $\mathrm{Hom}_R(L_i, L_1)$ of all a_{i1} such that

$$x_{1i}a_{i1} \in A \quad \text{for all} \quad x_{1i} \in \mathrm{Hom}_R(L_1, L_i) .$$

And, observe that the set X of all matrices $(a_{ij}) \in R\Delta$ such that $a_{ij} = 0$ for $j \ge 2$, $a_{11} \in A$ and $a_{i1} \in N_{i1}$ for $2 \le i \le r$, is a left ideal of $R\Delta$. Since R is perfect, there is a left ideal Y of $R\Delta$ such that $X \subseteq Y$ and Y/X is a simple $R\Delta$-module. Obviously,

$$A \subsetneqq \left\{x_{11} \mid (x_{ij}) \in Y\right\} = B \subseteq E_1 ;$$

moreover, B is a left ideal of E_1. And finally, B/A is a simple E_1-module. For, otherwise there would be a left ideal C of E_1 such that $A \subsetneqq C \subsetneqq B$, and thus the left ideal Z of $R\Delta$ of all matrices (c_{ij}) such that $c_{ij} = 0$ for $j \ge 2$, $c_{11} \in C$ and $c_{i1} \in N_{i1}$ for $2 \le i \le r$, would satisfy $X \subsetneqq Z \subsetneqq Y$. The proof is completed.

Now, given a perfect ring R, consider its indecomposable decomposition $R = \bigoplus_{i=1}^{r} L_i$ and denote by $R\Phi$ the finite additive category whose objects are R-modules $L_1, L_2, \ldots, L_r$ and whose morphisms are all homomorphisms belonging to $\mathrm{Hom}_R(L_i, L_j)$, $1 \le i$, $j \le r$. Notice that the mapping Φ of the class of all perfect rings into the class of all finite additive categories is, in view of uniqueness of decomposition, well-defined. The image $R\Phi$ of every perfect ring R is, moreover, a category such that the endomorphism rings of its objects are local perfect rings. For the sake of brevity, let us call such finite additive categories *perfect*.

On the other hand, let C be a finite additive category; denote by

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$C_1, C_2, \ldots, C_r$ its objects. Define the ring $C\Psi$ in the following way: $C\Psi$ is the ring of all $r \times r$ matrices (x_{ij}) such that

$$x_{ij} \in [C_i, C_j] \text{ for all } 1 \le i , j \le r ,$$

with respect to matrix addition and multiplication.

PROPOSITION 4. *If* C *is a perfect category, then* $C\Psi$ *is a perfect ring.*

Proof. For each i, $1 \le i \le r$, denote by W_i the unique maximal (left) ideal of the ring $[C_i, C_i]$ and by $L_i \subseteq C\Psi$ the subset of all matrices (x_{ij}) such that

$$x_{ij} = 0 \text{ for } j \neq i ,$$

which is obviously a left ideal of $C\Psi$. One can verify readily that the subset $K_i \subseteq L_i$ of those matrices (x_{ij}) which satisfy

$$[C_i, C_k] \times x_{ki} \subseteq W_i \text{ for all } k , 1 \le k \le r ,$$

is a unique left ideal of $C\Psi$ maximal in L_i. Hence L_i are indecomposable and $C\Psi$ has property *(a*)*.

In order to verify *(b*)*, it is sufficient to show that, for every i, $1 \le i \le r$, and every left ideal X of $C\Psi$ contained properly in L_i, there exists a left ideal Y, $X \subset Y \subseteq L_i$ such that Y/X is a simple $C\Psi$-module. The latter is trivial for $r = 1$. Thus, assume that $r > 1$ and, without loss of generality, present a proof for $i = 1$. The left ideal $X \subsetneqq L_1$ consists evidently of all matrices $(x_{kj}) \in L_1$ such that x_{11} belongs to a certain left ideal X_{11} of $[C_1, C_1]$ and for each k, $2 \le k \le r$, x_{k1} belongs to a certain submodule X_{k1} of $[C_k, C_1]$. Since $[C_r, C_r]$ is perfect, the $[C_r, C_r]$-module $[C_r, C_r]/X_{r1}$ has a simple submodule Y_{r1}/X_{r1}. Denote by $X^{(r)}$ the left ideal of $C\Psi$ generated by the set of all matrices $(x_{kj}) \in L_1$ with $x_{k1} \in X_{k1}$ for $1 \le k \le r-1$ and $x_{r1} \in Y_{r1}$. Obviously, $X \subsetneqq X^{(r)} \subseteq L_1$.

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Furthermore, writing

$$Z_{k1}^{(r)} = X_{k1} + [C_k, C_r] \times Y_{r1} \ , \quad 1 \le k \le r \ ,$$

we can see easily that $X^{(r)}$ consists of all matrices $(x_{kj}) \in L_1$ such that $x_{k1} \in Z_{k1}^{(r)}$.

Now, if $Z_{k1}^{(r)} = X_{k1}$ for all $k \le r-1$, then $Y = X^{(r)}$ satisfies the property that Y/X is a simple $C\Psi$-module, and the proof is completed. Otherwise, denote by x the greatest index $\le r-1$ such that

$$X_{s1} \subsetneqq Z_{s1}^{(r)} \ ,$$

and by Y_{s1} the $[C_s, C_s]$-submodule of $Z_{s1}^{(r)}$ containing X_{s1} such that Y_{s1}/X_{s1} is simple; such a submodule exists because the ring $[C_s, C_s]$ is perfect. Furthermore, let $X^{(s)}$ be the left ideal of $C\Psi$ generated by the set of all matrices $(x_{k1}) \in L_1$ with $x_{k1} \in X_{k1}$ for $1 \le k \le r$, $k \neq s$ and $x_{s1} \in Y_{s1}$. Obviously,

$$X \subsetneqq X^{(s)} \subseteq X^{(r)} \subseteq L_1 \ .$$

Again, write

$$Z_{k1}^{(s)} = X_{k1} + [C_k, C_s] \times Y_{s1} \quad \text{for} \quad 1 \le t \le s \ ,$$

and repeat the above argument. After a finite number of steps, we reach a lèft ideal $X^{(q)}$ of $C\Psi$ such that

$$X \subsetneqq X^{(q)} \subseteq \ldots \subseteq X^{(s)} \subseteq X^{(r)} \subseteq L_1$$

and such that $q = 1$ or the corresponding

$$Z_{k1}^{(q)} = X_{k1} \quad \text{for all} \quad k \le q-1 \ .$$

In either case, $Y = X^{(q)}$ has the required property. The proof is completed.

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Now, we can formulate

THEOREM 2. *There is a one-to-one correspondence between the non-isomorphic perfect rings and non-isomorphic perfect categories. This correspondence is effected by a matrix representation which describes the way in which perfect rings are built from local perfect rings.*

Proof. Given a perfect ring R, the matrix ring $R\Phi\Psi$ is the "standard" matrix representation of R and is thus isomorphic to R. Also, given a perfect category C and expressing $C\Psi = \bigoplus_{i=1}^{r} L_i$ as the direct sum of the column vectors L_i, we check easily that $C\Psi\Phi$ is isomorphic to the category C. The theorem follows.

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