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GROUPS WITH A NILPOTENT TRIPLE FACTORISATION

BERNHARD AMBERG, SILVANA FRANCIOSI
AND FRANCESCO DE GIOVANNI

In the investigation of factorised groups one often encounters groups $G = AB = AK = BK$ which have a triple factorisation as a product of two subgroups A and B and a normal subgroup K of G. It is of particular interest to know whether G satisfies some nilpotency requirement whenever the three subgroups A, B and K satisfy this same nilpotency requirement. A positive answer to this problem for the classes of nilpotent, hypercentral and locally nilpotent groups is given under the hypothesis that K is a minimax group or G has finite abelian section rank. The results become false if K has only finite Prüfer rank. Some applications of the main theorems are pointed out.

1. INTRODUCTION

If N is a normal subgroup of a factorised group $G = AB$, where A and B are subgroups of G, the factoriser $X(N) = AN \cap BN$ of N can be written as

$$X(N) = N(A \cap BN) = N(B \cap AN) = (A \cap BN)(B \cap AN)$$

(see [**1**], Theorem 1.7). Therefore the investigation of a factorised group very often reduces to the consideration of a triple factorised group

$$G = AB = AK = BK$$

where K is a normal subgroup of G. Groups with such a triple factorisation have played a rôle in almost every paper on factorised groups, in particular in [**2**], [**8**], [**9**], [**16**], [**21**].

In the following we are interested in the case that A, B and K satisfy some nilpotency requirement. Under certain conditions it will then be shown that the triple factorised group G satisfies the same nilpotency requirement.

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THEOREM A. *Let the group $G = AB = AK = BK$ be the product of two subgroups A and B and a normal minimax subgroup K of G.*

(a) *If A, B and K are nilpotent, then G is nilpotent,*

(b) *If A, B, and K are hypercentral, then G is hypercentral,*

(c) *If A, B and K are locally nilpotent, then G is locally nilpotent.*

Here, parts (b) and (c) of Theorem A even hold if K has finite abelian section rank and K/T is a minimax group, where T is the torsion subgroup of K.

However, the condition that K is a minimax group cannot be weakened to the condition that K has only finite Prüfer rank. This can be seen from an example given in Sysak [**18**], p. 29 of a torsion-free non-locally-nilpotent group $G = AB = AK = BK$ where A and B are abelian subgroups with infinite Prüfer rank and K is an abelian normal subgroup of G with Prüfer rank 1.

Note also that part (a) of Theorem A becomes false if K is locally finite with finite abelian section rank. This can be seen from the following:

EXAMPLE: There exists a locally finite non-nilpotent group G with finite Prüfer rank which has a triple factorisastion $G = AB = AK = BK$ where A, B and K are abelian subgroups and K is normal in G. In fact, for each odd prime p let G_p be a metacyclic p-group of class $\geqslant p$ which has a triple factorisation

$$G_p = A_pB_p = A_pK_p = B_pK_p,$$

where A_p, B_p and K_p are cyclic and K_p is normal in G_p (see [**21**], Example 1). The direct product $G = Dr_pG_p$ can be written as

$$G = AB = AK = BK,$$

where $A = Dr_pA_p$, $B = Dr_pB_p$ and $K = Dr_pK_p$. It is easy to see that G satisfies the required conditions, is not nilpotent and has Prüfer rank 2.

THEOREM B. *Let the group $G = AB = AK = BK$ with finite abelian section rank be the product of two subgroups A and B and a normal subgroup K of G.*

(a) *If A, B and K are nilpotent and the torsion subgroup $T(K)$ of K is a Černikov group, then G is nilpotent.*

(b) *If A, B and K are locally nilpotent, then G is locally nilpotent and hence hypercentral.*

The example above shows that in part (a) of Theorem B the requirement that $T(K)$ is a Černikov group cannot be omitted. From this example one can also see that in parts (a) of Theorems A and B, the nilpotency class of the group G cannot be bounded by the nilpotency classes of A and B.

Theorem B also holds in the case that the group G is soluble and the subgroups A and B have finite abelian section rank. This follows from the Theorem in [**7**].

Theorem A is a generalisation of Theorem 2 of Zaicev [**21**] for the product of two abelian groups, and Theorem B is a generalisation of Theorem 5 of Robinson [**16**] for the product of two nilpotent groups; see also [**8**], Theorem 2.

Theorem B has the following consequence.

COROLLARY. *If the radical group $G = AB$ with finite abelian section rank is the product of two locally nilpotent subgroups A and B, then each term of the ascending Hirsch-Plotkin series of G is factorised; in particular the Hirsch-Plotkin radical of G is factorised.*

Using the well-known theorem of Kegel and Wielandt ([**12**] and [**19**]), it is easy to see that the corollary even holds for a group G such that every non-trivial epimorphic image of G contains a non-trivial finite or locally nilpotent normal subgroup. The corollary generalises parts of Theorem 5.7 of [**1**], Satz 4.1 of [**3**] and Theorem 2.5 of [**4**]; for the finite case, see also [**13**]. Some further consequences of our theorems can be found in Section 5 below.

In the proofs in this paper, some cohomological arguments are decisive; see in particular Robinson [**17**].

Notation.

The notation is standard and can for instance be found in [**15**]. We note in particular:

A group G has *finite abelian section rank* if it has no infinite elementary abelian p-sections for every prime p.

G has *finite Prüfer rank* if there exists a positive integer r such that every finitely generated subgroup of G can be generated by at most r elements.

A soluble group G is a *minimax group* if it has a finite series whose factors are finite or infinite cyclic or quasicyclic of type p^∞; the number of infinite factors in such a series is called the *minimax rank of* G.

The *ascending Hirsch-Plotkin series* of a group G is defined in the following way:

$R_O(G) = 1$,

$R_{\alpha+1}(G)/R_\alpha(G) =$ Hirsch-Plotkin radical of $G/R_\alpha(G)$ for every ordinal α,

$R_\gamma(G) = \bigcup_{\sigma<\gamma} R_\sigma(G)$ for limit ordinals γ.

G is called *radical* if $G = R_\tau(G)$ for some ordinal τ.

A subgroup S of a factorised group $G = AB$ is called *factorised* if $S = (A \cap S)(B \cap S)$ and $A \cap B \leqslant S$ (see [**19**] or [**1**]).

2. Some lemmas

The first lemma gives a useful criterion for a group with a nilpotent triple factorisation to be nilpotent.

Lemma 2.1. (Robinson [**16**], Lemma 3). *Let the group $G = AB = AK = BK$ be the product of three nilpotent subgroups A, B and K, where K is normal in G, and assume that the Baer radical of G is nilpotent. If there exists a normal subgroup N of G such that the factoriser $X(N)$ and the factor group G/N are nilpotent, then G is nilpotent.*

The following lemma about Baer groups is probably already known. It ensures that in the proof of parts (a) of Theorems A and B, Lemma 2.1 is applicable.

Lemma 2.2. *Let N be a normal subgroup with finite Prüfer rank r of the Baer group G.*

(a) If N is a radicable abelian p-group, then N is contained in the r-th term $Z_r(G)$ of the upper central series of G.

(b) If the torsion subgroup of N is a Černikov group and the factor group G/N is nilpotent, then G is nilpotent.

Proof: (a) If H is a finitely generated subgroup of G, then H is a nilpotent subnormal subgroup of G. Write $N_i = [N, \underbrace{H, \dots, H}_{i}]$ for every positive integer i. Then $N_t = 1$ for some t. Since every N_i is radicable, it follows that N_i is a direct factor of N_{i-1} for all $i \leqslant t$. Hence, $t \leqslant r$ and thus $N_r = 1$. Therefore also $[N, \underbrace{G, \dots, G}_{r}] = 1$ and so $N \leqslant Z_r(G)$.

(b) Since the torsion subgroup T of N is a Černikov group, its finite residual J is a radicable abelian torsion group with finite Prüfer rank, and T/J is finite. Clearly, N/T is a torsion-free nilpotent normal subgroup of G/T with finite Prüfer rank, and so $N/T \leqslant Z_s(G/T)$ for some positive integer s (see [**15**], Part 2, p. 35) and G/T is nilpotent. Thus G/J is finite-by-nilpotent and hence nilpotent. By (a) we have that $J \leqslant Z_r(G)$, so that G is nilpotent. ∎

Essential use will be made of the following cohomological result. As usual, $H^i(Q,M)$ and $H_i(Q,M)$ denote the i-th cohomology group and the i-th homology group of the group Q with coefficients in the Q-module M, respectively.

Lemma 2.3. *Let Q be a locally nilpotent group and M a Q-module such that $Q/C_Q(M)$ is hypercentral and $H^0(Q,M) = 0$.*

(a) If M is an artinian Q-module, then $H^n(Q,M) = 0$ for every non-negative integer n.

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(b) If N is an artinian Q-submodule of M, then $H^0(Q, M/N) = 0$.

PROOF: For statement (a) see [**17**], Theorem 3.5, or also [**10**], Satz 3.2.9.

For the proof of (b) note that $H^0(Q,N) = 0$ implies $H^1(Q,N) = 0$ by (a). The long exact cohomology sequence gives the exact sequence

$$H^0(Q,M) \longrightarrow H^0(Q,M/N) \longrightarrow H^1(Q,N),$$

and hence $H^0(Q,M/N) = 0$. ∎

3. PROOF OF THEOREM A

Proof of statement (a). Assume that this is false, and choose among the counterexamples with K of minimal minimax rank a group G for which the sum of the nilpotency classes of A and B is minimal. We may assume that K is abelian (see [**14**]).

(i) The case: K is a torsion group.

In this case K is a Černikov group. There exists a finite G-invariant subgroup F of K such that K/F is radicable. If G/F is nilpotent, then G is finite-by-nilpotent and hence $|G : Z_n(G)|$ is finite for some non-negative integer n by a result of P. Hall (see [**15**], Part 1, p. 117). Since statement (a) holds for a finite group G (see for instance [**2**], Satz 3.3), it follows that G is nilpotent. Therefore we may assume that K is radicable.

Let H be a non-trivial radicable G-invariant subgroup of K. If $H < K$, then the group G/H and the factoriser $X(H)$ of H in $G = AB$ are nilpotent. By Lemma 2.2(b), the Baer radical of G is nilpotent, so that G is nilpotent by Lemma 2.1. This contradiction shows that every proper G-invariant subgroup of K is finite.

Clearly the normal subgroups $A \cap K$ and $B \cap K$ of G are properly contained in K, so that $C = (A \cap K)(B \cap K)$ is a finite normal subgroup of G. If G/C is nilpotent, then G is finite-by-nilpotent and it follows as above that G is nilpotent. Therefore G/C is not nilpotent and we may assume that $A \cap K = B \cap K = 1$. For every $a \in Z(A)$, the group $[K,a]$ is a normal subgroup of G which is properly contained in K (see[**6**], Lemma 1.2). Since K is radicable, we have that $[K,a] = 1$. This shows that $Z(A) \leqslant Z(G)$, so that $G/Z(G)$ is nilpotent. This contradiction proves that G is nilpotent when K is a torsion group.

(ii) The general case.

The factoriser $X(T)$ of the torsion subgroup T of K is nilpotent by case (i). By Lemma 2.2 (b) the Baer radical of G is nilpotent, so that G/T is not nilpotent by Lemma 2.1. Hence we may assume that K is torsion-free.

Since $Z(A) \cap K \leqslant Z(G)$, the group $G/(Z(A) \cap K)$ is not nilpotent, so that $Z(A) \cap K = 1$ and hence even $A \cap K = 1$. Since K is a torsion-free minimax group, the abelian

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subgroups of $A/C_A(K)$ are minimax groups, so that the nilpotent group $A/C_A(K)$ is a minimax group by theorems of Baer (see [**15**], Part 2, pp. 171–173). Clearly $C_A(K)$ is even normal in G and the factor group $G/C_A(K)$ is also a minimax group, so that it is nilpotent by Theorem 4 of [**16**]. Since $C_A(K) \cap K = 1$, it follows that G is nilpotent. This contradiction proves Theorem A(a).

The following lemmas deal with special situations needed in the proof of part (c) of Theorem A.

LEMMA 3.1. *Let the group $G = AK$ be the product of two hypercentral subgroups A and K, where K is normal in G, and let J be a radicable abelian normal torsion subgroup of G such that G/J is hypercentral. If $Z(G) = 1$ and every proper G-invariant subgroup of J is finite, then $A \cap J = 1$ and $C_G(J) = J$.*

PROOF: Since the socle of J is finite, J satisfies the minimum condition on subgroups. From $H^0(G/J, J) = 0$ it follows that $H^2(G/J, J) = 0$ by Lemma 2.3(a). Hence $G = L \ltimes J$ for some hypercentral subgroup L of G. Then $C_{Z(L)}(J) \leqslant Z(G) = 1$ and so $C_L(J) = 1$ and $C_G(J) = J$. For each positive integer n the group $JZ_n(K)$ is nilpotent, so that $[J, Z_n(K)] < J$. Since J is radicable, $[J, Z_n(K)] = 1$. Therefore $Z_n(K) \leqslant C_G(J) = J$ and thus also $Z_\omega(K) = \bigcup_n Z_n(K) \leqslant J$.

If $Z_\omega(K) < J$, then K is nilpotent and $K \leqslant J$. In this case $Z(A) \cap J \leqslant Z(G) = 1$ and hence even $A \cap J = 1$. Assume now that $Z_\omega(K) = J$. Suppose that a is a non-trivial element of $Z(A) \cap J$, and let n be the least positive integer such that $a \in Z_n(K)$. If $\bar{G} = G/Z_{n-1}(K)$, then $\bar{a} \in Z(\bar{G}) \cap \bar{J}$, so that $Z(\bar{G}) \cap \bar{J} \neq 1$. This contradicts Lemma 2.3(b). Therefore also in this case $Z(A) \cap J = 1$ and $A \cap J = 1$. This, then proves the lemma. ∎

LEMMA 3.2. *Let the group $G = AB$ be the product of two nilpotent subgroups A and B, and let J be a radicable abelian normal torsion subgroup of G such that $C_G(J) = J$ and the factor group G/J is nilpotent. If every proper G-invariant subgroup of J is finite, then J is factorised.*

PROOF: Since the socle of J is finite, J satisfies the minimum condition on subgroups. By Lemma 2.2(b) the Baer radical V of G is nilpotent. It follows that $[J, V]$ is a proper G-invariant subgroup of J and since J is radicable, $[J, V] = 1$. Then $V \leqslant C_G(J) = J$ and hence $J = V$. The factoriser $X = X(J)$ of J is nilpotent by Theorem A(a). Therefore $X \leqslant V = J$ and $J = X$ is factorised. ∎

Proof of statements (b) and (c) of Theorem A. Part (b) will follow immediately from part (c), since if G is locally nilpotent then K lies in the hypercentre of G (see [**15**], Part 2, p. 39).

Assume that statement (c) is false, and let G be a counterexample such that the torsion-free rank of K is minimal. We may suppose that the torsion subgroup T of K is a p-group. Among all such counterexamples choose one G for which the finite residual J of T has minimal Prüfer rank.

(i) The case: K nilpotent.

In this case we may suppose that K is abelian (see [**14**]). Since the hypercentre factor group $G/\bar{Z}(G)$ is not locally nilpotent, without loss of generality we may take $Z(G)=1$. But $A\cap K$ lies in the hypercentre $\bar{Z}(A)$ and hence also in $\bar{Z}(G)$. Thus $A\cap K=1$, and similarly $B\cap K=1$.

If $\bar{G}=G/T$, the locally nilpotent group $\bar{A}/C_{\bar{A}}(\bar{K})$ is isomorphic with a group of automorphisms of the torsion-free abelian group of finite Prüfer rank $\bar{K}$ and is therefore nilpotent (see [**15**], Part 2, p. 31). Moreover its abelian subgroups are minimax groups and it is itself a minimax group by results of Baer (see [**15**], Part 2, pp. 171–173). Clearly $C_{\bar{A}}(\bar{K})$ is even normal in $\bar{G}$ and $\bar{G}/C_{\bar{A}}(\bar{K})$ is a minimax group. Now

$$(\bar{B}C_{\bar{A}}(\bar{K})/C_{\bar{A}}(\bar{K}))\cap(\bar{K}C_{\bar{A}}(\bar{K})/C_{\bar{A}}(\bar{K}))$$

is contained in some term with finite ordinal type of the upper central series of $\bar{B}C_{\bar{A}}(\bar{K})/C_{\bar{A}}(\bar{K})$ (see [**15**], Part 2, p. 35). Therefore $\bar{B}C_{\bar{A}}(\bar{K})/C_{\bar{A}}(\bar{K})$ is nilpotent. This implies that $\bar{G}/C_{\bar{A}}(\bar{K})$ is hypercentral by Lemma 4 of [**16**]. Since $\bar{A}\cap\bar{K}=1$, the factor group $\bar{G}=G/T$ is locally nilpotent.

Since T is a Černikov p-group, there exists a finite G-invariant subgroup F of T with $FJ=T$. The locally nilpotent group $G/C_G(J)$ is hypercentral since it is linear over the field of p-adic numbers (see [**15**], Part 2, p.31). The finite factor group $G/C_G(F)$ is obviously nilpotent. Since $C_G(T)=C_G(F)\cap C_G(J)$, the factor group $G/C_G(T)$ is hypercentral. From $H^0(G/T,T)=0$ it follows that $H^1(G/T,T)=H^2(G/T,T)=0$ by Lemma 2.3(a). In particular this means that $G=L\ltimes T$ for some subgroup L of G. Then $K=(K\cap L)\times T$ and $K\cap L$ is normal in G. If $K\cap L\neq 1$, the factor group $G/(K\cap L)$ is locally nilpotent by the minimality of the torsion-free rank of K. Then also G is locally nilpotent. This contradiction shows that $K\cap L=1$ and $K=T$. Therefore $H^1(G/K,K)=0$ and the complements A and B of K in G are conjugate, so that $A=B=G$ is locally nilpotent. This proves that G is locally nilpotent when K is nilpotent.

(ii) The general case.

Now let K be an arbitrary locally nilpotent minimax group whose torsion subgroup T is a p-group. Here K/J is nilpotent and hence G/J is locally nilpotent by case (i). Again we may assume that $Z(G)=1$. The locally nilpotent group $G/C_G(J)$ is linear over the field of p-adic numbers, and then it is nilpotent since its periodic subgroups are finite (see [**15**], Part 1, p. 85 and Part 2, p. 31). If S is an infinite G-invariant

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subgroup of J then $H^0(G/J, J/S) = 0$ by Lemma 2.3(b). But the minimality of the Prüfer rank of J ensures that G/S is locally nilpotent and so $J/S \leqslant \bar{Z}(G/S)$. Then $J = S$. We have shown that every proper G-invariant subgroup of J is finite.

From $H^0(G/J., J) = 0$ it follows that $H^2(G/J, J) = 0$ by Lemma 2.3(a). Therefore $G = L \ltimes J$ for some locally nilpotent subgroup L of G. If $G^\star = G/C_L(J)$, then $C_{G^\star}(J^\star) = J^\star$ and $G^\star/J^\star$ is nilpotent. But $A^\star \cap J^\star$ lies in the hypercentre of $A^\star$ and so $A^\star$ is hypercentral. Since $Z(G^\star) = 1$ we can apply Lemma 3.1. Therefore $A^\star \cap J^\star = B^\star \cap J^\star = 1$. Thus $A^\star$ and $B^\star$ are nilpotent and Lemma 3.2 says that $J^\star = (A^\star \cap J^\star)(B^\star \cap J^\star) = 1$. Thus $J = 1$. This contradiction proves the theorem.

4. Proof of Theorem B

Proof of statement (a). Since the torsion subgroup T of K is a Černikov group, its factoriser $X(T)$ in $G = AB$ is nilpotent by Theorem A(a). Clearly K/T is torsion-free, and so G/T is nilpotent by Theorem 4 of [**16**]. The Baer radical of G is nilpotent by Lemma 2.2(b), so that G is nilpotent by Lemma 2.1. This proves statement (a) of Theorem B.

Proof of statement (b). This runs along the same lines as that of Theorem A(c). Note that in this case $\bar{G}/C_{\bar{A}}(\bar{K})$ has finite abelian section rank by supposition, enabling the application of Lemma 4 of [**16**].

Proof of the Corollary. Let $G = AB$ be a radical group and let

$$1 = R_0 \leqslant R_1 \leqslant \ldots \leqslant R_\tau = G$$

be the ascending Hirsch-Plotkin series of G. For every ordinal $\alpha \leqslant \tau$ let X_α be the factoriser of R_α in G. Then for every $\alpha < \tau$ the subgroup $X_{\alpha+1}/R_\alpha$ is the factoriser of $R_{\alpha+1}/R_\alpha$ in G/R_α, so that $X_{\alpha+1}/R_\alpha$ is hypercentral by Theorem B(b). Since $R_\alpha \leqslant X_\alpha \leqslant X_{\alpha+1}$, the subgroup X_α is ascendant in $X_{\alpha+1}$. If γ is a limit ordinal, also $R_\gamma \leqslant \bigcup_{\beta<\gamma} X_\beta \leqslant X_\gamma$ and so $X_\gamma = \bigcup_{\beta<\gamma} X_\beta$. It follows that $X_1 = X(R_1)$ is a hypercentral ascendant subgroup of G, so that the Hirsch-Plotkin radical $R = X(R)$ of G is factorised.

Since the hypotheses of the corollary are inherited by factor groups, each term of the ascending Hirsch-Plotkin series of G is factorised. This proves the corollary.

5. Some further results

Theorem A has the following consequences.

Corollary 5.1. *Let the group $G = AB$ be the product of an abelian subgroup A and and a hypercentral subgroup B. If the Hirsch-Plotkin radical R of G is a minimax group, then R is factorised.*

PROOF: Let $X = X(R)$ be the factoriser of R in G, and write $\bar{G} = G/R$. Then $\bar{X} = \bar{A} \cap \bar{B}$ is normal in $\bar{A}$ and ascendant in $\bar{B}$. Thus $\bar{X}^{\bar{G}} = \bar{X}^{\bar{B}} \leqslant \bar{B}$ and hence $\bar{X}$ is ascendant in $\bar{G}$. Therefore X is ascendant in G and so $X \leqslant R$, since X is locally nilpotent by Theorem A(c). This yields that $R = X$ is factorised. ∎

COROLLARY 5.2. *Let the group $G = AB$ be the product of two abelian subgroups A and B. If the Fitting subgroup F of G is a minimax group then F is factorised.*

PROOF: The factoriser $X(N)$ of a nilpotent normal subgroup N of G is a nilpotent normal subgroup of G by Theorem A(a), so that $X(N) \leqslant F$. Therefore F is generated by factorised normal subgroups of G and hence is itself factorised. ∎

COROLLARY 5.3. *Let the group $G = AB \neq 1$ be the product of two abelian subgroups A and B. If the commutator subgroup G' of G is a minimax group, then at least one of the subgroups A and B contains a non-trivial normal subgroup of G.*

PROOF: Assume that this is false. Then for instance by condition (+) in the proof of the Proposition in [5] it follows that the factoriser $X = X(G')$ of G' has trivial centre. But X is nilpotent by Theorem A(a), since G' is abelian by a well-known theorem of Itô (see [**11**]). This contradiction proves Corollary 5.3. ∎

Remarks. (a) Note that Corollaries 5.1 and 5.2 are improvements of Theorems 2.5 and 2.4 of [**4**]. Also, corollary 5.3 improves the Theorem in [**5**].
(b) The example of Sysak [**18**] mentioned in the introduction shows that Corollary 5.3 does not hold when the commutator subgroup G' is only (torsion-free) with finite Prüfer rank.

In our last result we note a situation in which the hypothesis of parts (b) and (c) of Theorem A can be weakened.

PROPOSITION 5.4. *Let the group $G = AB = AK = BK$ be the product of two subgroups A and B and a radicable normal subgroup K of G with finite abelian section rank.*

(a) If A, B and K are hypercentral, then G is hypercentral.

(b) If A, B and K are locally nilpotent, then G is locally nilpotent.

PROOF: Statement (a) will follow from statement (b), since if G is locally nilpotent then K lies in the hypercentre of G (see [**15**], Part 2, p.39).

Assume that (b) is false, and let G be a counterexample such that K has minimal torsion-free rank. If T is the torsion subgroup of K, then K/T is nilpotent. Since K is hypercentral and radicable, it is nilpotent (see [**15**], Part 2, p. 125). We may suppose that K is abelian (see [**14**]), and that its torsion subgroup is a p-group for some prime p.

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The factor group $G/(K \cap \bar{Z}(G))$ is not locally nilpotent, so that $K \cap \bar{Z}(G)$ is periodic and hence $G/(K \cap \bar{Z}(G))$ is also a minimal counterexample. Therefore it can be assumed that $K \cap Z(G) = 1$. The normal subgroup $A \cap K$ of the locally nilpotent group A is contained in the hypercentre of A (see [**15**], Part 2, p.39). It follows that $A \cap K \leqslant K \cap \bar{Z}(G) = 1$. Similarly it follows that also $B \cap K = 1$.

Write $\bar{G} = G/T$. If $\bar{K} \cap Z(\bar{G}) \neq 1$, then $\bar{G}/(\bar{K} \cap Z(\bar{G}))$ is locally nilpotent and hence also $\bar{G}$ is locally nilpotent. If $\bar{K} \cap Z(\bar{G}) = 1$, then $H^0(\bar{G}/\bar{K}, \bar{K}) = 0$ and therefore $H_0(\bar{G}/\bar{K}, \bar{K})$ has finite exponent by Proposition 4.1 of [**17**]. It follows that $H_0(\bar{G}/\bar{K}, \bar{K}) = 0$ since $\bar{K}$ is radicable. Then $H^1(\bar{G}/\bar{K}, \bar{K}) = 0$ by Theorem 4.5 of [**17**]. Thus the complements $\bar{A}$ and $\bar{B}$ of $\bar{K}$ in $\bar{G}$ are conjugate, so that $\bar{G} = \bar{A} = \bar{B}$ is locally nilpotent. Therefore in both cases G/T and hence also $G/C_G(T)$ are locally nilpotent.

As $G/C_G(T)$ is also linear, it must be hypercentral (see [**15**], Part 2, p. 31). From $H^0(G/T, T) = 0$ it follows that $H^2(G/T, T) = 0$ by Lemma 2.3(a). Therefore $G = L \ltimes T$ for some locally nilpotent subgroup L of G. Then $K = (K \cap L) \times T$, where the subgroup $K \cap L$ is normal in G. The group $G/(K \cap L)$ is locally nilpotent by Theorem A(c). Hence also G is locally nilpotent. This contradiction proves the proposition. ∎

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Fachbereich Mathematik,
Universität Mainz,
Saarstraße 21,
D-6500 Mainz,
West Germany.

Dipartimento di Matematica,
Università di Napoli,
via Mezzocannone 8,
I - 80134 Napoli,
Italy.

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