



# Cotangent Sums Related to the Riemann Hypothesis for Various Shifts of the Argument

Helmut Maier and Michael Th. Rassias

*Abstract.* One of the approaches to the Riemann Hypothesis is the Nyman–Beurling criterion. Cotangent sums play a significant role in this criterion. Here we investigate the values of these cotangent sums for various shifts of the argument.

## 1 Introduction

In several papers ([7–11]) the authors have investigated the distribution of the cotangent sums

$$c_0\left(\frac{r}{b}\right) := - \sum_{m=1}^{b-1} \frac{m}{b} \cot\left(\frac{\pi mr}{b}\right),$$

where  $r, b \in \mathbb{N}$ ,  $b \geq 2$ ,  $1 \leq r \leq b$ , and  $(r, b) = 1$ .

They could establish a link with the function  $g(\alpha)$ , defined as follows.

**Definition 1.1** Let

$$g(\alpha) := \sum_{l=1}^{+\infty} \frac{1 - 2\{l\alpha\}}{l}, \quad \alpha \in (0, 1), \quad \text{where } \{u\} := u - [u], u \in \mathbb{R},$$

which is convergent for almost all  $\alpha$  (see [5]).

**Definition 1.2** For  $z \in \mathbb{R}$ , let

$$F(z) := \text{meas}\{\alpha \in (0, 1) : g(\alpha) \leq z\},$$

where “meas” denotes the Lebesgue measure.

Let  $\mu$  be the uniquely defined positive measure on  $\mathbb{R}$  with the following property. For  $\alpha < \beta \in \mathbb{R}$ , we have  $\mu([\alpha, \beta]) = F(\beta) - F(\alpha)$ . We set

$$C_0(\mathbb{R}) := \left\{ f \in C(\mathbb{R}) : \forall \epsilon > 0, \exists \text{ a compact set } \mathcal{K} \subset \mathbb{R}, \right. \\ \left. \text{such that } |f(x)| < \epsilon, \forall x \notin \mathcal{K} \right\}.$$

---

Received by the editors January 16, 2019; revised June 23, 2019.

Published online on Cambridge Core October 15, 2019.

M. Th. Rassias was co-funded by the John S. Latsis Public Benefit Foundation and the University of Zürich.

AMS subject classification: 33B10, 41A60, 11M06, 28A25, 60E10.

Keywords: cotangent sum, equidistribution, moment, asymptotics, Estermann zeta function, Riemann zeta function, Riemann Hypothesis, exponential sums in finite fields.

The second author established the following result in his thesis [13] (see also [8, Theorem 1.2]). Let  $A_0, A_1$  be fixed constants, such that  $1/2 < A_0 < A_1 < 1$ . For all  $f \in C_0(\mathbb{R})$ , we have

$$(1.1) \quad \lim_{b \rightarrow +\infty} \frac{1}{(A_1 - A_0)\phi(b)} \sum_{\substack{(r,b)=1 \\ A_0 b \leq r \leq A_1 b}} f\left(\frac{1}{b} c_0\left(\frac{r}{b}\right)\right) = \int f d\mu,$$

where  $\phi(\cdot)$  denotes the Euler phi-function.

Later, S. Bettin [3] replaced the inequality  $1/2 < A_0 < A_1 < 1$  by  $0 < A_0 < A_1 \leq 1$ . In [12] the authors considered the distribution of the values of  $c_0$  for rational numbers with primes as numerator and a fixed prime as denominator and proved a result analogous to (1.1) ([12, Theorem 1.3]). Namely, they proved that for all  $f \in C_0(\mathbb{R})$ , the following holds true:

$$\lim_{\substack{q \rightarrow +\infty \\ q \text{ prime}}} \frac{\log q}{(A_1 - A_0)q} \sum_{\substack{A_0 q \leq p \leq A_1 q \\ p \text{ prime}}} f\left(\frac{1}{q} c_0\left(\frac{p}{q}\right)\right) = \int f d\mu.$$

The cotangent sum  $c_0(r/b)$  has gained importance in the Nyman–Beurling criterion for the Riemann Hypothesis through its relation with the Vasyunin sum, which is defined by

$$V\left(\frac{r}{b}\right) := \sum_{m=1}^{b-1} \left\{ \frac{mr}{b} \right\} \cot\left(\frac{\pi mr}{b}\right).$$

In this paper, we simultaneously consider the values of  $c_0$  for various shifts of the argument. We consider general numerators as well as prime numerators. For simplicity, the denominator will be a fixed prime  $q$ . Our main results are the following.

**Theorem 1.3** *Let  $A_0, A_1$  be fixed constants, such that  $1/2 < A_0 < A_1 < 1$ . Let  $a_1, a_2, \dots, a_L$  be distinct non-negative integers. Let  $f_1, f_2, \dots, f_L \in C_0(\mathbb{R})$  ( $L \in \mathbb{N}$ ). Then we have*

(i)

$$\lim_{\substack{q \rightarrow +\infty \\ q \text{ prime}}} \frac{1}{\phi(q)} \sum_{A_0 q \leq r \leq A_1 q} \frac{1}{(A_1 - A_0)^L} \prod_{l=1}^L f_l\left(c_0\left(\frac{r + a_l}{q}\right)\right) = \prod_{l=1}^L \left(\int f_l d\mu\right),$$

(ii)

$$\lim_{\substack{q \rightarrow +\infty \\ q \text{ prime}}} \frac{\log q}{q} \sum_{\substack{A_0 q \leq p \leq A_1 q \\ p \text{ prime}}} \frac{1}{(A_1 - A_0)^L} \prod_{l=1}^L f_l\left(c_0\left(\frac{p + a_l}{q}\right)\right) = \prod_{l=1}^L \left(\int f_l d\mu\right).$$

We give a detailed proof of (i) and sketch the changes needed for the proof of (ii).

## 2 Outline of the Proof

Several fundamental ideas appear in the paper [8] and in the thesis [13]. The key to the treatment of the sum  $c_0(r/q)$  lies in its relation to the sum

$$Q\left(\frac{r}{q}\right) := \sum_{m=1}^{q-1} \cot\left(\frac{\pi mr}{q}\right) \left\lfloor \frac{mr}{q} \right\rfloor.$$

The second author established the following representation in his thesis [13] (see also [8, Proposition 1.8]):

$$c_0\left(\frac{r}{q}\right) = \frac{1}{r} c_0\left(\frac{1}{q}\right) - \frac{1}{r} Q\left(\frac{r}{q}\right).$$

We now have to consider simultaneously the values

$$f_1\left(Q\left(\frac{r+a_1}{q}\right)\right), \dots, f_L\left(Q\left(\frac{r+a_L}{q}\right)\right).$$

By the Weierstrass approximation theorem, this question can be reduced to the study of the joint distribution of the products

$$(2.1) \quad \prod(k_1, \dots, k_L) := Q\left(\frac{r+a_1}{q}\right)^{k_1} \dots Q\left(\frac{r+a_L}{q}\right)^{k_L}$$

for  $L$ -tuplets  $(k_1, \dots, k_L)$  of non-negative integers.

In a similar fashion as in the previous papers, we will break up the range of summation into subintervals in which

$$\left\lfloor \frac{(r+a_l)m}{q} \right\rfloor$$

assumes constant values.

**Definition 2.1** For  $j \in \mathbb{N}$ ,  $l \in \{1, \dots, L\}$ , we set

$$S_j^{(l)} := \{(r+a_l)m : qj \leq (r+a_l)m < q(j+1)\}$$

and write

$$S_j^{(l)} := \{qj + s_j^{(l)}, qj + s_j^{(l)} + (r+a_l), \dots, qj + s_j^{(l)} + (r+a_l)d_j^{(l)}\}.$$

We also define  $t_j^{(l)}$  by

$$qj + s_j^{(l)} + d_j^{(l)}(r+a_l) + t_j^{(l)} := q(j+1).$$

In [8, 12] the map

$$j \longrightarrow s_j$$

and its inverse

$$s \longrightarrow j(s)$$

were very important. They are now replaced by  $L$  maps

$$j \longrightarrow s_j^{(l)} \quad (1 \leq l \leq L)$$

and their inverses

$$s \longrightarrow j_l(s).$$

We also have  $L$  pairs of congruences

$$(2.2) \quad s_j^{(l)} \equiv -qj \pmod{(r + a_l)} \quad \text{and} \quad t_j^{(l)} \equiv q(j + 1) \pmod{(r + a_l)}.$$

Each of the sums

$$Q\left(\frac{r + a_l}{q}\right)$$

is dominated by small values of  $s_j^{(l)}$  and  $t_j^{(l)}$ , because of the poles of the function  $\cot(\pi x)$  at  $x = 0$  and  $x = 1$ . We denote this partial sum by

$$Q_0\left(\frac{r + a_l}{q}\right),$$

and thus we obtain the decomposition

$$Q\left(\frac{r + a_l}{q}\right) = Q_0\left(\frac{r + a_l}{q}\right) + Q_1\left(\frac{r + a_l}{q}\right), \quad (\text{see Definition 5.1}).$$

The function  $\cot(\pi x)$  is antisymmetric

$$\cot(\pi(1 - x)) = -\cot(\pi x).$$

Therefore, there will be considerable cancellation in the sums

$$Q_1\left(\frac{r + a_l}{q}\right),$$

which will thus be small. By the binomial theorem, the products

$$\prod(k_1, \dots, k_L)$$

in (2.1) will be linear combinations of products of the form

$$(2.3) \quad Q_{\epsilon_1}\left(\frac{r + a_{l_1}}{q}\right)^{h_1} \cdots Q_{\epsilon_M}\left(\frac{r + a_{l_M}}{q}\right)^{h_M}$$

with  $\epsilon_g \in \{0, 1\}$ .

We will show that only the products with  $\epsilon_g = 0$  for  $1 \leq g \leq M$  will give a substantial contribution.

The asymptotic size of these products will be determined by localising the solutions of the congruences (2.1) simultaneously for all  $l$ . Whereas in [8] Kloosterman sums could be used for this localisation, here we need results on more general exponential sums in finite fields, due to Weil [15, 16], as well as Fouvry and Michel [6].

The contribution of the other products in (2.3) is small, since at least one  $\epsilon_g = 1$ .

For the discussion of these factors,

$$Q_1\left(\frac{r + a_{l_g}}{q}\right)^{h_g},$$

we can refer to the results of [8].

### 3 Exponential Sums in Finite Fields

**Lemma 3.1** *Let  $q$  be a prime number. Let  $a_1, \dots, a_L$  be distinct non-negative integers,  $n, m_1, \dots, m_L$  integers not all 0,*

$$R(x) = nx + \frac{m_1}{x + a_1} + \dots + \frac{m_L}{x + a_L}.$$

*Then we have*

$$\sum_{\substack{x=1 \\ x \neq -a_l, 1 \leq l \leq L}}^{q-1} e\left(\frac{R(x)}{q}\right) = O(q^{1/2}).$$

*The constant implied by the  $O$ -symbol may depend on  $L$ .*

**Proof** This follows from the work of Weil [15,16]. ■

**Lemma 3.2** *Let  $\mathbb{F}_r$  be the finite field with  $r$  elements and let  $\psi$  be a non-trivial additive character over  $\mathbb{F}_r$ ,  $f$  a rational function of the form*

$$f(x) = \frac{P(x)}{Q(x)},$$

*$P$  and  $Q$  relatively prime monic non-constant polynomials,*

$$S(f; r, x) := \sum_{p \leq x} \psi(f(p)).$$

*( $p$  denotes the  $p$ -fold sum of the element 1 in  $\mathbb{F}_r$ ). Then we have*

$$S(f; r, x) \ll r^{3/16+\epsilon} x^{25/32}.$$

*The implied constant depends only on  $\epsilon$  and the degrees of  $P$  and  $Q$ .*

**Proof** This is due to Fouvry and Michel [6]. ■

### 4 Localizations of the Solutions of the Congruences

We now come to the localization of the solutions of the congruences in (2.2). This will be achieved in Lemma 4.1, which is a multidimensional version of [8, Lemma 4.3]. As in [8, Lemma 4.3], as well as our Lemma 4.1 the localization of variables is achieved by Fourier analysis. Thus the calculations in this paper bear a close resemblance to those of [8].

**Lemma 4.1** *Let  $q$  be prime. Let  $1/2 < A_0 < A_1 < 1$  and  $r \in \mathbb{N}$ . Let  $q^*(r; l)$  be defined by  $qq^*(r; l) \equiv 1 \pmod{(r + a_l)}$ . Let  $\alpha_1, \dots, \alpha_L \in (0, 1)$ ,  $\delta > 0$ , such that*

$$\alpha_l + \delta < 1 \text{ for } 1 \leq l \leq L.$$

*Then we have*

$$\begin{aligned} N(\alpha_1, \dots, \alpha_L, \delta) &:= \left| \left\{ r : r \in \mathbb{N}, A_0q \leq r \leq A_1q, \alpha_l \leq \frac{q^*(r; l)}{r} \leq \alpha_l + \delta \right\} \right| \\ &= \delta^L (A_1 - A_0)q(1 + o(1)), \quad q \longrightarrow \infty. \end{aligned}$$

**Proof** In the sequel, we assume  $1 \leq l \leq L$ . If  $q$  is sufficiently large, we have

$$1 \leq r + a_1 < \dots < r + a_L < q,$$

and therefore  $(r + a_l, q) = 1$  for  $1 \leq l \leq L$ . We let  $(r + a_l)^*$  be determined by

$$(r + a_l)(r + a_l)^* \equiv 1 \pmod{q}.$$

The Diophantine equation  $qx + (r + a_l)y = 1$  has exactly one solution  $(x_{0,l}, y_{0,l})$  with

$$-\left\lfloor \frac{r + a_l}{2} \right\rfloor < x_{0,l} \leq \left\lfloor \frac{r + a_l}{2} \right\rfloor, \quad -\left\lfloor \frac{q}{2} \right\rfloor < y_{0,l} \leq \frac{q}{2}.$$

We have

$$\begin{aligned} q^*(r; l) &\equiv x_{0,l} \pmod{(r + a_l)}, \\ (r + a_l)^* &\equiv y_{0,l} \pmod{q}. \end{aligned}$$

Therefore, for  $\beta_l \in (-1/2, 1/2)$  and  $\delta > 0$  with

$$\beta_l + \delta < \frac{1}{2} \quad \text{and} \quad \beta_l - \delta > -\frac{1}{2} \quad \text{for} \quad 1 \leq l \leq L,$$

we have

$$\begin{aligned} (4.1) \quad &\left\{ \left\{ r : A_0q \leq r \leq A_1q, \frac{y_{0,l}}{q} \in [\beta_l, \beta_l + \delta] \right\} \right\} \\ &= \left\{ \left\{ r : A_0q \leq r \leq A_1q, \frac{x_{0,l}}{r} \in [-(\beta_l + \delta), -\beta_l] \right\} \right\} + O(1) \\ &= \left\{ \left\{ r : A_0q \leq r \leq A_1q, \frac{q^*(r, l)}{r} \pmod{1} \in [-(\beta_l + \delta), -\beta_l] \right\} \right\} + O(1), \end{aligned}$$

where

$$\frac{q^*(r, l)}{r} \pmod{1} \in [-(\beta_l + \delta), -\beta_l]$$

stands for

$$\frac{q^*(r, l)}{r} \in \begin{cases} [1 - (\beta_l + \delta), 1 - \beta_l], & \text{if } \beta_l \geq 0, \\ [-(\beta_l + \delta), -\beta_l], & \text{if } \beta_l < 0. \end{cases}$$

Let  $\Delta > 0$ , such that  $\beta_l + \delta + \Delta \leq \frac{1}{2}, 0 \leq v_l \leq \Delta$ . We define the function

$$(4.2) \quad \chi_{1,l}(u, v) := \begin{cases} 1 & \text{if } u \in [\beta_l + \Delta - v, \beta_l + \delta - \Delta + v), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(4.3) \quad \chi_{2,l}(u, v) := \begin{cases} 1 & \text{if } u \in [\beta_l - \Delta + v, \beta_l + \Delta - v), \\ 0 & \text{otherwise.} \end{cases}$$

as well as the functions  $\lambda_{1,l}, \lambda_{2,l}$  by

$$\lambda_{i,l}(u) := \Delta^{-1} \int_0^\Delta \chi_{i,l}(u, v) \, dv \quad \text{for } i = 1, 2.$$

Let the function

$$(4.4) \quad \tilde{\chi}_l(r, \beta) := \begin{cases} 1 & \text{if } \frac{(r+a_l)^*}{q} \in [\beta_l, \beta_l + \delta] \\ 0 & \text{otherwise.} \end{cases}$$

Since  $\lambda_{i,l}$  for  $i = 1, 2$  is obtained from  $\chi_{i,l}$  by averaging over  $r$  and since  $0 \leq \chi_{i,l}(u, \nu) \leq 1$  it follows that  $0 \leq \lambda_{i,l}(u) \leq 1$  for  $i = 1, 2$ . From (4.2), we have

$$\lambda_{1,l}\left(\frac{(r + a_l)^*}{q}\right) = 0, \quad \text{if } \frac{(r + a_l)^*}{q} \notin [\beta_l, \beta_l + \delta).$$

Similarly, from (4.3), we have

$$\lambda_{2,l}\left(\frac{(r + a_l)^*}{q}\right) = 1, \quad \text{if } \frac{(r + a_l)^*}{q} \notin [\beta_l, \beta_l + \delta).$$

Thus, we obtain

$$(4.5) \quad \lambda_{1,l}\left(\frac{(r + a_l)^*}{q}\right) \leq \tilde{\chi}_l(r, \beta) \leq \lambda_{2,l}\left(\frac{(r + a_l)^*}{q}\right).$$

We have the Fourier expansion

$$\lambda_{i,l}(u) = \sum_{n=-\infty}^{\infty} a_l(n)e(nu).$$

The Fourier coefficients  $a_l(n)$  are computed as follows.

For  $i = 1$ ,

$$a_l(0) = \Delta^{-1} \int_0^\Delta \left( \int_{\beta_l + \Delta - \nu}^{\beta_l + \delta - \Delta + \nu} 1 \, du \right) d\nu = \delta - \Delta,$$

as well as

$$\begin{aligned} a_l(n) &= \Delta^{-1} \int_0^\Delta \left( \int_{\beta_l + \Delta - \nu}^{\beta_l + \delta - \Delta + \nu} e(-nu) \, du \right) d\nu \\ &= \Delta^{-1} \int_0^\Delta -\frac{1}{2\pi i n} \left( e(-n(\beta_l + \delta - \Delta + \nu)) - e(-n(\beta_l + \Delta - \nu)) \right) d\nu \\ &= -\frac{1}{4\pi^2 n^2} \Delta^{-1} \left( e(-n(\beta_l + \delta)) - e(-n(\beta_l + \delta - \Delta)) \right. \\ &\quad \left. - e(-n\beta_l) + e(-n(\beta_l + \Delta)) \right). \end{aligned}$$

From the above and an analogous computation, for  $i = 2$ , we obtain

$$a_l(0) = \delta + R_{1,l}, \text{ where } |R_{1,l}| \leq \Delta$$

and

$$a_l(n) = \begin{cases} O(\Delta) & \text{if } |n| \leq \Delta^{-1}, \\ O(\Delta^{-1}n^2) & \text{if } |n| > \Delta^{-1}. \end{cases}$$

Let  $\Delta_1 > 0$ , such that  $A_0 - \Delta_1 > 1/2$ ,  $A_1 + \Delta_1 < 1$  and  $0 \leq \nu \leq \Delta_1$ .

We define the functions

$$(4.6) \quad \chi_3(u, \nu) := \begin{cases} 1 & \text{if } u \in [A_0 + \nu - \Delta_1, A_1 - \nu + \Delta_1], \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(4.7) \quad \chi_4(u, \nu) := \begin{cases} 1 & \text{if } u \in [A_0 + \Delta_1 - \nu, A_1 + \Delta_1 + \nu], \\ 0 & \text{otherwise.} \end{cases}$$

as well as the functions  $\lambda_3, \lambda_4$  by

$$\lambda_i(u) := \Delta_1^{-1} \int_0^{\Delta_1} \chi_i(u, v) dv \text{ for } i = 3, 4.$$

Let the function

$$(4.8) \quad \chi^*(r, \beta) := \begin{cases} 1 & \text{if } A_0 \leq \frac{r}{q} \leq A_1, \\ 0 & \text{otherwise.} \end{cases}$$

Since  $\lambda_i$  for  $i = 3, 4$  is obtained from  $\chi_i$  by averaging over  $r$ , and since

$$0 \leq \chi_i(u, v) \leq 1 \text{ for } i = 3, 4$$

we obtain  $0 \leq \lambda_i(u) \leq 1$  for  $i = 3, 4$ .

From (4.6), we have

$$\lambda_3\left(\frac{r}{q}\right) = 0, \quad \text{if } \frac{r}{q} \notin (A_0, A_1).$$

From (4.7), we have

$$\lambda_4\left(\frac{r}{q}\right) = 1, \quad \text{if } \frac{r}{q} \in (A_0, A_1).$$

Therefore, we obtain

$$(4.9) \quad \lambda_3\left(\frac{r}{q}\right) \leq \chi^*\left(r, \beta\right) \leq \lambda_4\left(\frac{r}{q}\right).$$

By an analogous computation as for  $\lambda_1, \lambda_2$ , we obtain the Fourier expansions

$$\lambda_i(u) = \sum_{n=-\infty}^{\infty} c(n)e(nu), \quad \text{for } i = 3, 4,$$

with  $c(0) = A_1 - A_0 + R_2$ , where  $|R_2| \leq \Delta_1$  and

$$c(n) := \begin{cases} O(1), & \text{if } |n| \leq \Delta_1^{-1}, \\ O(\Delta_1^{-1}n^{-2}), & \text{if } |n| > \Delta_1^{-1}. \end{cases}$$

From (4.1), (4.4), (4.5), (4.8), and (4.9), setting  $\beta = -\alpha$ , we get the following

$$\begin{aligned} & \sum_{1 \leq r \leq q-1} \left( \prod_{l=1}^L \lambda_{1,l} \left( \frac{(r+a_l)^*}{q} \right) \right) \lambda_3\left(\frac{r}{q}\right) \\ & \leq N(\alpha_1, \dots, \alpha_L, \delta) \\ & \leq \sum_{1 \leq r \leq q-1} \left( \prod_{l=1}^L \lambda_{2,l} \left( \frac{(r+a_l)^*}{q} \right) \right) \lambda_4\left(\frac{r}{q}\right). \end{aligned}$$

We obtain

$$(4.10) \quad \sum_{1 \leq r \leq q-1} \left( \prod_{l=1}^L \lambda_{1,l} \left( \frac{(r+a_l)^*}{q} \right) \right) \lambda_3\left(\frac{r}{q}\right) = \sum_{m_1, \dots, m_L, n=-\infty}^{\infty} a(m_1)a(m_2) \cdots a(m_L)c(n)E(n, m_1, \dots, m_L, q),$$



with

$$E(n, m_1, \dots, m_L, q) := \sum_{1 \leq r \leq q-1} e\left(\frac{nr + m_1(r + a_1)^* + \dots + m_L(r + a_L)^*}{q}\right).$$

For  $(n, m_1, \dots, m_L) \neq (0, 0, \dots, 0)$ , we estimate  $E(n, m_1, \dots, m_L, q)$  by Lemma 3.1 and obtain

$$E(n, m_1, \dots, m_L, q) = O(q^{1/2}).$$

From (4.10), we get

$$(4.11) \quad \sum_{1 \leq r \leq q-1} \left( \prod_{l=1}^L \lambda_{1,l} \left( \frac{(r + a_l)^*}{q} \right) \right) \lambda_3 \left( \frac{r}{q} \right) = (\delta + R_1)^L (A_1 - A_0 + R_2)q + o(q),$$

for  $|R_1| \leq \Delta$  and  $|R_2| \leq \Delta$ . The same computation also gives

$$(4.12) \quad \sum_{1 \leq r \leq q-1} \left( \prod_{l=1}^L \lambda_{2,l} \left( \frac{(r + a_l)^*}{q} \right) \right) \lambda_4 \left( \frac{r}{q} \right) = (\delta + R_1)^L (A_1 - A_0 + R_2)q + o(q).$$

Since  $\Delta$  and  $\Delta_1$  can be chosen to be arbitrarily small, it follows that (4.11) and (4.12) imply Lemma 4.1. ■

### 5 Decomposition of the Sums Q

We start from the decompositions

$$Q\left(\frac{r + a_l}{q}\right) = \sum_{j=1}^{r-1} j \sum_{h=0}^{d_j} \cot\left(\pi \frac{s_j^{(l)} + hr}{q}\right).$$

We further decompose  $Q\left(\frac{r+a_l}{q}\right)$  as in the following definition.

**Definition 5.1** We have

$$Q\left(\frac{r + a_l}{q}\right) := Q_0\left(\frac{r + a_l}{q}\right) + Q_1\left(\frac{r + a_l}{q}\right)$$

with

$$Q_0\left(\frac{r + a_l}{q}\right) := \sum_{j=1}^{q-1} {}^* j \sum_{h=0}^{d_j^{(l)}} \cot\left(\pi \frac{s_j^{(l)} + hr}{q}\right),$$

where  $\sum_{j=1}^{q-1} {}^*$  means that the sum is extended over all values of  $j$ , for which

$$\left\{ \frac{\theta j q}{r + a_j} \right\} \leq q^{-1} 2^{m_1}$$

for either  $\theta = 1$  or  $\theta = -1$ ,

$$Q_1\left(\frac{r + a_l}{q}\right) = Q\left(\frac{r + a_l}{q}\right) - Q_0\left(\frac{r + a_l}{q}\right),$$

$m_1$  is a fixed positive integer.

(For the conclusion of the proof we let  $m_1 \rightarrow \infty$ ).

The size of  $Q\left(\frac{r+a_l}{q}\right)$  and also of  $c_0\left(\frac{r+a_l}{q}\right)$  is essentially determined by  $Q_0$ , since  $Q_1$  is small, as we shall see in Section 7.

### 6 Comparison of $Q_0$ and $g$

**Definition 6.1** Let  $A_0q \leq r \leq A_1q$ . We set

$$\alpha^{(l)} := \alpha^{(l)}(r, q) = \frac{q_l^*}{r + a_l},$$

where

$$\begin{aligned} q_l^* q &\equiv 1 \pmod{(r + a_l)}, \\ g(\alpha; m_1) &:= \sum_{s=1}^{2^{m_1}} \frac{1 - 2\{s\alpha\}}{s}, \\ Q(r, q, m_1, l) &:= \frac{rq}{\pi} g(\alpha^{(l)}; m_1). \end{aligned}$$

The next lemma shows, that  $Q_0\left(\frac{r+a_l}{q}\right)$  is well approximated by  $Q(r, q, m_1, l)$ .

**Lemma 6.2** We have

$$Q_0\left(\frac{r + a_l}{q}\right) = Q(r, q, m_1, l) + O(q2^{m_1}).$$

**Proof** This follows from the result in the thesis of the second author [13] and in the paper [8, step 1 of the proof of Theorem 4.15], if  $r$  is replaced by  $r + a_l$ . ■

### 7 The Estimate of $Q_1(p/q)$

**Lemma 7.1** We have

$$Q_1\left(\frac{r + a_l}{q}\right) = O(q^2 2^{-m_1}).$$

**Proof** From [8, (4.113)], we have

$$Q_1\left(\frac{r}{q}\right) = O(q^2 2^{-m_1}).$$

Lemma 7.1 follows, if we replace  $r$  by  $r + a_l$ . ■

### 8 The Joint Moments of the Sums $Q(r, q, m_1, l)$ and $c_0\left(\frac{r+a_l}{q}\right)$

**Lemma 8.1** We have

$$\begin{aligned} \lim_{\substack{q \rightarrow \infty \\ q \text{ prime}}} (A_1 - A_0)^{-L} q^{-L} \sum_{A_0q \leq r \leq A_1q} \prod_{l=1}^L Q(r, q, m_1, l)^{k_l} = \\ \prod_{l=1}^L \left( \int_0^1 g(\alpha, m_1)^{k_l} d\alpha \right). \end{aligned}$$

**Proof** We choose a partition  $\mathcal{P}$  of the interval  $[0, 1]$ :

$$(8.1) \quad 0 = \alpha_0 < \alpha_1 < \dots < \alpha_{n-1} < \alpha_n = 1$$

and consider upper and lower Riemann sums of the  $L$ -dimensional Riemann integral

$$I := \int_0^1 \dots \int_0^1 g(x^{(1)}, m_1)^{k_1} \dots g(x^{(L)}, m_1)^{k_L} dx^{(1)} \dots dx^{(L)},$$

the upper sum

$$\begin{aligned} &\mathcal{U}(g(x^{(1)}, m_1)^{k_1}, \dots, g(x^{(L)}, m_1)^{k_L}; \mathcal{P}) \\ &:= \sum_{i_1=0}^{n-1} \dots \sum_{i_L=0}^{n-1} \sup_{(\alpha^{(i_1)}, \dots, \alpha^{(i_L)}) \in X_{l=1}^L[\alpha_{i_l}, \alpha_{i_l+1}]} (g(\alpha^{(i_1)}, m_1)^{k_1} \dots g(\alpha^{(i_L)}, m_1)^{k_L}) \\ &\quad \times \prod_{l=1}^L (\alpha_{i_l+1} - \alpha_{i_l}), \end{aligned}$$

and the lower sum

$$\begin{aligned} &\mathcal{L}(g(x^{(1)}, m_1)^{k_1}, \dots, g(x^{(L)}, m_1)^{k_L}; \mathcal{P}) \\ &:= \sum_{i_1=0}^{n-1} \dots \sum_{i_L=0}^{n-1} \inf_{(\alpha^{(i_1)}, \dots, \alpha^{(i_L)}) \in X_{l=1}^L[\alpha_{i_l}, \alpha_{i_l+1}]} (g(\alpha^{(i_1)}, m_1)^{k_1} \dots g(\alpha^{(i_L)}, m_1)^{k_L}) \\ &\quad \times \prod_{l=1}^L (a_{i_l+1} - a_{i_l}). \end{aligned}$$

The function  $g(x, m_1)$  is piecewise linear. Therefore, the integral  $I$  exists. It is well known from the Theory of the Riemann-integral, that for given  $\epsilon > 0$ , there is a partition  $\mathcal{P}_\epsilon$  of the form (8.1), such that

$$\mathcal{L} \leq \prod_{l=1}^L \left( \int_0^1 g(x, m_1)^{k_l} dx \right) \leq \mathcal{U} \leq \mathcal{L} + \epsilon.$$

We now let

$$N_{i_1, \dots, i_L} := \left\{ r : A_0 q \leq r \leq A_1 q, \frac{q^*(r + a_l)}{r} \in [\alpha_{i_l}, \alpha_{i_l+1}] \text{ for } 1 \leq l \leq L \right\}.$$

By Lemma 4.1, we have

$$N_{i_1, \dots, i_L} = \delta^L(A_1 - A_0)q(1 + o(1)).$$

From the asymptotics

$$\begin{aligned} \cot\left(\frac{\pi(qj + s_j^{(l)})}{q}\right) &= \frac{q}{\pi s_j^{(l)}}(1 + o(1)), \\ \cot\left(\frac{\pi(qj + d_j^{(l)}(r + a_l) + t_j^{(l)})}{q}\right) &= -\frac{q}{\pi t_j^{(l)}}(1 + o(1)), \end{aligned}$$

we obtain (using the notation (2.2)):

$$\begin{aligned} & \sum_{1 \leq r \leq q-1} \prod_{l=1}^L Q(r, q, m_1, l)^{k_l} \\ &= \sum_{i_1=0}^{n-1} \cdots \sum_{i_L=0}^{n-1} \prod_{l=1}^L \left( \sum_{\substack{j: \{\frac{\theta_j q}{r+a_l}\} \in [\alpha_j, \alpha_{j+1}] \\ \theta \in \{1, -1\}}} \cot\left(\frac{\pi u_j^{(l)}}{q}\right)^{k_l} \right) \\ & \quad \text{(where we set } u_j^{(l)} = s_j^{(l)} \text{ if } \theta = 1, \text{ and } u_j^{(l)} = -t_j^{(l)} \text{ if } \theta = -1) \\ &= \left( \prod_{l=1}^L \left( \int_0^1 g(\alpha, m_1)^{k_l} d\alpha \right) \right) q^L (1 + o(1)), \end{aligned}$$

which proves Lemma 8.1. ■

**Lemma 8.2** We have

$$\lim_{\substack{q \rightarrow \infty \\ q \text{ prime}}} (A_1 - A_0)^{-L} q^{-L} \sum_{A_0 q \leq r \leq A_1 q} \prod_{l=1}^L Q\left(\frac{r+a_l}{q}\right)^{k_l} = \prod_{l=1}^L \left( \int_0^1 g(\alpha)^{k_l} d\alpha \right).$$

**Proof** The proof is a modification of the procedure of Step 7 of the proof of [8, Theorem 4.15].

We use the decomposition

$$Q\left(\frac{r+a_l}{q}\right) = Q_0\left(\frac{r+a_l}{q}\right) + Q_1\left(\frac{r+a_l}{q}\right).$$

From Lemma 6.2, we obtain

$$Q_0\left(\frac{r+a_l}{q}\right) = Q(r, q, m_1, l) + O(q2^{m_1}).$$

From Lemma 7.1, we have

$$Q\left(\frac{r+a_l}{q}\right) = Q(r, q, m_1, l) + Q_1\left(\frac{r+a_l}{q}\right) + O(q2^{m_1}).$$

We then expand the factors of the product

$$\prod_{l=1}^L Q\left(\frac{r+a_l}{q}\right)^{k_l}$$

by applications of the Multinomial Theorem, which leads to a sum of products of the type

$$\prod_{l=1}^L Q(r, q, m_1, l)^{h_{1,l}} Q_1\left(\frac{r+a_l}{q}\right)^{h_{2,l}}.$$

Only products with  $h_{1,l} = k_l, h_{2,l} = 0$  contribute substantially, the contributions being given in Lemma 8.1.

The other terms can be estimated by the application of Hölder’s inequality as in Step 7 of the proof of [8, Theorem 4.15]. ■

**Lemma 8.3** *We have*

$$\lim_{\substack{q \rightarrow \infty \\ q \text{ prime}}} \frac{1}{\phi(q)} \sum_{A_0 q \leq r \leq A_1 q} (A_1 - A_0)^{-(k_1 + \dots + k_L)} \prod_{l=1}^L c_0 \left( \frac{r + a_l}{q} \right)^{k_l} \\ = \prod_{l=1}^L \left( \int_0^1 g(x)^{k_l} dx \right).$$

**Proof** In [8, Theorem 5.2 (c)] a more general one-dimensional version of Lemma 8.3 has been proved. The proof of Lemma 8.3 has been obtained by modifying this proof to obtain results for monomials

$$\prod_{l=1}^L c_0 \left( \frac{r + a_l}{q} \right)^{k_l}$$

in several variables. ■

## 9 Conclusion of the Proof and Concluding Remarks

By the Weierstrass approximation theorem, each of the functions  $f_l$  in Theorem 1.3 can be approximated arbitrarily closely by a polynomial

$$p_l(x) = \sum_{j=0}^{C_l} e_{j,l} x^j,$$

where  $C_l \in \mathbb{N}_0$ . The product

$$\prod_{l=1}^L f_l \left( c_0 \left( \frac{r + a_l}{q} \right) \right)$$

then becomes the sum of products as considered in Lemma 8.3.

The Theorem follows from the fact that  $g$  has a continuous distribution function ([8, Theorem 5.2]).

The proof of Theorem 1.3(ii) can be obtained from the proof of (i) by applying Lemma 3.2 instead of Lemma 3.1 and making the obvious changes otherwise.

**Open Problem** Can the result of Theorem 1.3(i) be modified, such that the primality of  $q$  is not requested?

**Acknowledgments** M. Th. Rassias would like to express his gratitude to the John S. Latsis Public Benefit Foundation for their financial support provided under the auspices of his current “Latsis Foundation Senior Fellowship” position.

## References

- [1] B. Bagchi, *On Nyman, Beurling and Baez-Duarte’s Hilbert space reformulation of the Riemann hypothesis*. Proc. Indian Acad. Sci. Math. 116(2006), 137–146. <https://doi.org/10.1007/BF02829783>
- [2] S. Bettin, *A generalization of Rademacher’s reciprocity law*. Acta Arith. 159(2013), 363–374. <https://doi.org/10.4064/aa159-4-5>
- [3] S. Bettin, *On the distribution of a cotangent sum*. IMRN 2015 no. 21, <https://doi.org/10.1093/imrn/mv036>
- [4] S. Bettin and B. Conrey, *Period functions and cotangent sums*. Algebra Number Theory 7(2013), 215–242. <https://doi.org/10.2140/ant.2013.7.215>

- [5] R. de la Bretèche and G. Tenenbaum, *Séries trigonométriques à coefficients arithmétiques*. J. Anal. Math. 92(2004), 1–79. <https://doi.org/10.1007/BF02787756>
- [6] E. Fouvry and Ph. Michel, *Sur certaines sommes d'exponentielles sur les nombres premiers*. Ann. Sci. École Norm. Sup. (4) 31(1998), 93–130. [https://doi.org/10.1016/S0012-9593\(98\)80019-0](https://doi.org/10.1016/S0012-9593(98)80019-0)
- [7] H. Maier and M. Th. Rassias, *The order of magnitude for moments for certain cotangent sums*. J. Math. Anal. Appl. 429(2015), 576–590. <https://doi.org/10.1016/j.jmaa.2015.04.036>
- [8] H. Maier and M. Th. Rassias, *Generalizations of a cotangent sum associated to the Estermann zeta function*. Commun. Contemp. Math. 18(2016), no. 1, 1550078. <https://doi.org/10.1142/S0219199715500789>
- [9] H. Maier and M. Th. Rassias, *The rate of growth of moments of certain cotangent sums*. Aequationes Math. 90(2016), 581–595. <https://doi.org/10.1007/s00010-015-0361-3>
- [10] H. Maier and M. Th. Rassias, *Asymptotics for moments of certain cotangent sums*. Houston J. Math. 43(2017), 207–222.
- [11] H. Maier and M. Th. Rassias, *The maximum of cotangent sums related to Estermann's zeta function in rational numbers in short intervals*. Appl. Anal. Discrete Math. 11(2017), 166–176. <https://doi.org/10.2298/AADM1701166M>
- [12] H. Maier and M. Th. Rassias, *Distribution of a cotangent sum related to the Nyman–Beurling criterion for the Riemann Hypothesis*. Appl. Math. Comput. 363(2019), 124589. <https://doi.org/10.1016/j.amc.2019.124589>
- [13] M. Th. Rassias, *Analytic investigation of cotangent sums related to the Riemann zeta function*, Doctoral Dissertation, ETH–Zürich, Switzerland, 2014.
- [14] M. Th. Rassias, *On a cotangent sum related to zeros of the Estermann zeta function*. Appl. Math. Comput. 240(2014), 161–167. <https://doi.org/10.1016/j.amc.2014.04.086>
- [15] A. Weil, *On some exponential sums*. Proc. Natl. Acad. Sci. USA 34(1948), 204–207. <https://doi.org/10.1073/pnas.34.5.204>
- [16] A. Weil, *Sur les courbes algébriques et les variétés qui s'en déduisent*. Actualités Sci. Ind. 1041. Publ. Inst. Math. Univ. Strasbourg 7(1945). Paris, Hermann 1948.

*Department of Mathematics, University of Ulm, Helmholtzstrasse 18, 89081 Ulm, Germany*  
*e-mail: [helmut.maier@uni-ulm.de](mailto:helmut.maier@uni-ulm.de)*

*Institute of Mathematics, University of Zurich, CH-8057, Zurich, Switzerland*

*Moscow Institute of Physics and Technology, 141700 Dolgoprudny, Institutskiy per, d. 9, Russia*

*Institute for Advanced Study, Program in Interdisciplinary Studies, 1 Einstein Dr, Princeton, NJ 08540, USA*  
*e-mail: [mikhail.rassias@math.uzh.ch](mailto:mikhail.rassias@math.uzh.ch)*