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TEMPERED REPRESENTATIONS AND THE THETA CORRESPONDENCE

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ABSTRACT. Let V be an even dimensional nondegenerate symmetric bilinear space over a nonarchimedean local field F of characteristic zero, and let n be a nonnegative integer. Suppose that $\sigma \in \mathrm{Irr}(\mathrm{O}(V))$ and $\pi \in \mathrm{Irr}(\mathrm{Sp}(n, F))$ correspond under the theta correspondence. Assuming that σ is tempered, we investigate the problem of determining the Langlands quotient data for π.

Let F be a nonarchimedean local field of characteristic zero, let V_m be a nondegenerate symmetric bilinear space over F of Witt index m and even dimension l, and let n be a nonnegative integer. Fix a nontrivial additive character ψ of F. Let $\omega_{m,n}$ be the smooth Weil representation of $\mathrm{O}(V_m) \times \mathrm{Sp}(n, F)$ associated to ψ, where $\mathrm{Sp}(n, F)$ is the isometry group of the nondegenerate symplectic bilinear space of dimension $2n$. Let $R_n\big(\mathrm{O}(V_m)\big)$ be the set of $\sigma \in \mathrm{Irr}\big(\mathrm{O}(V_m)\big)$ such that σ is a nonzero quotient of $\omega_{m,n}$, and define $R_m\big(\mathrm{Sp}(n, F)\big)$ similarly. The Howe duality conjecture states that the set known as the theta correspondence

$$\left\{(\sigma, \pi) \in R_n\big(\mathrm{O}(V_m)\big) \times R_m\big(\mathrm{Sp}(n, F)\big) : \mathrm{Hom}_{\mathrm{O}(V_m)\times \mathrm{Sp}(n,F)}(\omega_{m,n}, \sigma \otimes \pi) \neq 0\right\}$$

is the graph of a bijection between $R_n\big(\mathrm{O}(V_m)\big)$ and $R_m\big(\mathrm{Sp}(n, F)\big)$. The conjecture holds if the residual characteristic of F is odd [W]. If $\sigma \in \mathrm{Irr}\big(\mathrm{O}(V_m)\big)$ and $\pi \in \mathrm{Irr}\big(\mathrm{Sp}(n, F)\big)$ correspond, *i.e.*, the above homomorphism space is nonzero, then one can ask how properties of σ carry over to properties of π. In the range $\dim_F V_m = l \leq 2n$, if σ is unramified, and the L-group parameter of σ is given, then π is unramified, and the L-group parameter of π is known [K-R2]. Also, if parabolic inducing data for σ is known, then parabolic inducing data for π is known [K1]. In this note, given that σ is tempered, we investigate the problem of determining the Langlands quotient data for π. Throughout the note, we *do not* assume the Howe duality conjecture or that the residual characteristic of F is odd.

Our first main result addresses the question of when π is tempered. In the following theorem, if k a positive integer, then $\mathbf{St}_k$ is the Steinberg representation of $\mathrm{Gl}(k, F)$, and χ is the quadratic character of $F^\times$ defined by $\chi(t) = \big(t, \mathrm{disc}(V_m)\big)_F$; for the definition of the Langlands quotient, see Section 3.

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THEOREM 4.2. *Suppose that* $\sigma \in \mathrm{Irr}\big(\mathrm{O}(V_m)\big)$ *is tempered, and* $\pi \in \mathrm{Irr}\big(\mathrm{Sp}(n,F)\big)$ *is such that*

$$\mathrm{Hom}_{\mathrm{O}(V_m)\times \mathrm{Sp}(n,F)}(\omega_{m,n}, \sigma\otimes\pi) \neq 0.$$

(1) If $2n \leq \dim_F V_m = l$*, then* π *is tempered;*
(2) If $2n > \dim_F V_m = l$*,* $\sigma \notin R_{n-1}\big(\mathrm{O}(V_m)\big)$*, and* π *is not tempered, then*

$$\pi = L(\delta_1 \otimes \cdots \otimes \delta_t \otimes \tau),$$

where

$$\delta_1 \cong \mathrm{Ind}^{\mathrm{Gl}(n_1,F)}_{P^{\mathrm{Gl}}_{p_1,\ldots,p_s}}(\mathbf{St}_{p_1} \otimes \cdots \otimes \mathbf{St}_{p_s}) \otimes \chi(\det)|\det|^{n-l/2-c/2},$$

with $\min(p_1,\ldots,p_s) \geq n-l/2+1$ *and* c *an integer such that* $\min(p_1,\ldots,p_s)-1 \geq c > 0$.

It would be very interesting to determine whether π as in (2) of Theorem 4.2 actually exist, *i.e.*, whether for $2n > \dim_F V_m = l$ there exist $\sigma \in \mathrm{Irr}\big(\mathrm{O}(V_m)\big)$ and $\pi \in \mathrm{Irr}\big(\mathrm{Sp}(n,F)\big)$ such that σ and π correspond, σ is tempered, $\sigma \notin R_{n-1}\big(\mathrm{O}(V_m)\big)$, but π is not tempered.

Our second main result considers the situation when n varies. Assume now that $\sigma \in \mathrm{Irr}\big(\mathrm{O}(V_m)\big)$ is pre-unitary, corresponds to π, and that π is tempered. Suppose that $n' \geq n$, and that σ also corresponds to $\pi' \in \mathrm{Irr}\big(\mathrm{Sp}(n',F)\big)$. Then the Langlands quotient data for π' is determined by π in the range $2n' \geq 2n \geq \dim_F V_m = l$:

THEOREM 4.4. *Suppose that* $\sigma \in \mathrm{Irr}\big(\mathrm{O}(V_m)\big)$ *is pre-unitary. Let* n *and* n' *be positive integers such that* $2n' \geq 2n \geq \dim_F V_m = l$*. Let* $\pi \in \mathrm{Irr}\big(\mathrm{Sp}(n,F)\big)$ *and* $\pi' \in \mathrm{Irr}\big(\mathrm{Sp}(n',F)\big)$*. If* π *is tempered and*

$$\mathrm{Hom}_{\mathrm{O}(V_m)\times \mathrm{Sp}(n,F)}(\omega_{m,n}, \sigma\otimes\pi) \neq 0, \quad \mathrm{Hom}_{\mathrm{O}(V_m)\times \mathrm{Sp}(n',F)}(\omega_{m,n'}, \sigma\otimes\pi') \neq 0,$$

then

$$\pi' = L(\chi|\ |^{n'-l/2} \otimes \chi|\ |^{n'-1-l/2} \otimes \cdots \otimes \chi|\ |^{n+1-l/2} \otimes \pi).$$

Of course, one expects analogous results if one begins instead with an element of $\mathrm{Irr}\big(\mathrm{Sp}(n,F)\big)$. One obstacle, however, to extending Theorem 4.4 to this case is the key result Proposition 4.3, from [K-R1].

One application which these results might make possible is the computation of the standard L-functions of theta lifts π' as in Theorem 4.4, and in particular the determination of poles of these L-functions. Indeed, this was our motivation for this note. We plan to return to this topic on a later occasion.

NOTATION. We will use the following notation. Let G be a group of td-type, as in [Car], with a countable basis. Let $\mathcal{S}(G)$ be the $\mathbb{C}$ vector space of locally constant, compactly supported $\mathbb{C}$ valued functions on G. Let $\mathrm{Irr}(G)$ be the set of equivalence classes of smooth admissible irreducible representations of G. We let $\mathbf{1}$ denote the trivial representation of G. Let π be a smooth representation of G. The smooth contragredient representation of π is $\pi^\vee$ and if π admits a central character, we denote it by ω_π. If G is

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contained as a normal subgroup in G', then for $g' \in G'$, we let $g' \cdot \pi$ be the representation of G with the same space as π and action $(g' \cdot \pi)(g) = \pi(g'^{-1}gg')$. A representation π of G is pre-unitary if there is a nondegenerate G invariant Hermitian form on the space of π. Suppose G is unimodular, and M and N are closed subgroups of G such that M normalizes N, $M \cap N = 1$, $P = MN$ is closed in G, N is unimodular and $P\backslash G$ is compact. Fix a Haar measure dn on N, and for $m \in M$, let $\delta(m)$ be the positive number such that all $f \in \mathcal{S}(N)$,

$$\int_N f(m^{-1}nm)\,dn = \delta(m)\int_N f(n)\,dn.$$

The normalized Jacquet module $\mathrm{R}_N(\pi)$ of π is the smooth representation of M defined by $\mathrm{R}_N(\pi) = \pi_N \otimes \delta^{-1/2}$, where π_N is the quotient of π by the $\mathbb{C}$ subspace generated by the vectors $v - \pi(n)v$, for $v \in \pi$ and $n \in N$. We define $\overline{\mathrm{R}}_N(\pi) = \mathrm{R}_N(\pi^\vee)^\vee$. Suppose that σ is a smooth representation of M. Then $\mathrm{Ind}_P^G \sigma$ is the representation of G by right translation on the $\mathbb{C}$ vector space of smooth functions f on G with values in σ such that $f(mng) = \delta(m)^{1/2}\sigma(m)f(g)$ for $m \in M$, $n \in N$ and $g \in G$. We have Frobenius reciprocity: $\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G \sigma) \cong \mathrm{Hom}_M\big(\mathrm{R}_N(\pi), \sigma\big)$ and $\mathrm{Hom}_G(\mathrm{Ind}_P^G \sigma, \pi^\vee) \cong \mathrm{Hom}_M\big(\sigma, \mathrm{R}_N(\pi)^\vee\big)$. If π is admissible we have $\mathrm{Hom}_G(\mathrm{Ind}_P^G \sigma, \pi) \cong \mathrm{Hom}_M\big(\sigma, \overline{\mathrm{R}}_N(\pi)\big)$. Throughout the paper, F is a nonarchimedean local field of characteristic zero, and $(\ ,\ )_F$ is the Hilbert symbol of F. We let $|\ |$ denote the valuation on F such that if μ is an additive Haar measure on F, then $\mu(xA) = |x|\mu(A)$ for $x \in F$ and $A \subset F$. If $\pi \in \mathrm{Irr}\big(\mathrm{Gl}(q, F)\big)$, then we let $e(\pi)$ be the unique real number such that the central character of $\pi \otimes |\det|^{-e(\pi)}$ is unitary. If n is a positive integer, then an ordered partition of n is a k-tuple $(n_1, \ldots, n_k)$ of positive integers such that $n = n_1 + \cdots + n_k$. If G is the group of F-points of a connected reductive algebraic group defined over F then $\pi \in \mathrm{Irr}(G)$ is tempered if and only if ω_π is unitary and every matrix coefficient of π lies in $\mathrm{L}^{2+\epsilon}\big(G/\,\mathrm{Z}(G)\big)$ for all $\epsilon > 0$. If $\pi \in \mathrm{Irr}\big(\mathrm{Gl}(n, F)\big)$, then π is essentially tempered or essentially square integrable if $\pi \otimes |\det|^{-e(\pi)}$ is tempered or square integrable, respectively. The algebraic closure of F is $\overline{F}$. In this note, all functions act on the left, and composition of functions is taken from right to left. In this paper we *do not* make assumptions about the residual characteristic of F or assume Howe duality.

1. **The groups.** Let q be a positive integer. We use the standard notation for $\mathrm{Gl}(q)$. As a maximal F split torus of $\mathrm{Gl}(q)$ we take the subgroup of diagonal matrices. As a base for the positive roots of $\mathrm{Gl}(q)$ we take $\Delta^{\mathrm{Gl}} = \{e_1 - e_2, e_2 - e_3, \ldots, e_{q-1} - e_q\}$. For an ordered partition $q = q_1 + \cdots + q_n$ of q we let $P^{\mathrm{Gl}}_{q_1,\ldots,q_n}$ be the F-points of the parabolic subgroup defined by $\Delta^{\mathrm{Gl}} - \{e_{q_1} - e_{q_1+1}, e_{q_1+q_2} - e_{q_1+q_2+1}, \ldots, e_{q_1+\cdots+q_{n-1}} - e_{q_1+\cdots+q_{n-1}+1}\}$. We let $M^{\mathrm{Gl}}_{q_1,\ldots,q_n}$ and $N^{\mathrm{Gl}}_{q_1,\ldots,q_n}$ denote the Levi factor and unipotent radical of $P^{\mathrm{Gl}}_{q_1,\ldots,q_n}$, respectively. We also define $P^{\mathrm{Gl}}_{0,q} = P^{\mathrm{Gl}}_{q,0} = M^{\mathrm{Gl}}_{0,q} = M^{\mathrm{Gl}}_{q,0} = \mathrm{Gl}(q, F)$ and $N^{\mathrm{Gl}}_{0,q} = N^{\mathrm{Gl}}_{q,0} = 1$. As usual, we identify the center of $\mathrm{Gl}(q, F)$ with $F^\times$ via $t \longmapsto t \cdot I_q$.

Let $\mathbf{V}_0$ be a vector space over $\overline{F}$ of dimension d endowed with an F-structure and a nondegenerate symmetric bilinear form $(\ ,\ )_0$ defined over F which is anisotropic over F. Let $V_0 = \mathbf{V}_0(F)$. For m a nonnegative integer, let

$$\mathbf{V}_m = (\overline{F}x_1 \oplus \cdots \oplus \overline{F}x_m) \oplus \mathbf{V}_0 \oplus (\overline{F}x_1' \oplus \cdots \oplus \overline{F}x_m'),$$

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and define the symmetric bilinear form $(\ ,\ )_m$ on $\mathbf{V}_m$ as the direct sum of $(\ ,\ )_0$ and the form defined by $(x_i, x_j) = (x_i', x_j') = 0$ and $(x_i, x_j') = \delta_{ij}$ for $1 \le i, j \le n$. The Witt index of $V_m = \mathbf{V}_m(F)$ is m. Let $l = \dim_F V_m$. We fix the maximal F-split torus of $\mathrm{SO}(\mathbf{V}_m)$ whose elements are the maps $\mathrm{diag}(t_1, \ldots, t_m)$ which send x_i to $t_i x_i$, are the identity on V_0, and send x_i' to $t_i^{-1} x_i'$. As a base for the F-roots of $\mathrm{SO}(\mathbf{V}_m)$ with respect to our torus we take $\Delta = \{e_1 - e_2, e_2 - e_3, \ldots, e_{m-1} - e_m, e_{m-1} + e_m\}$ if $d = 0$ and $\{e_1 - e_2, e_2 - e_3, \ldots, e_{m-1} - e_m, e_m\}$ otherwise. For $1 \le k \le m$, consider the parabolic subgroup defined by the complement of the k-th element of Δ. Then the group P_k^{SO} of F points of this parabolic subgroup is the stabilizer in $\mathrm{SO}(V_m)$ of $Fx_1 \oplus \cdots \oplus Fx_k$. We shall write the elements of P_k^{SO} with respect to the decomposition

$$V_m = X_k \oplus V_{m-k} \oplus Y_k,$$

where $X_k = Fx_1 \oplus \cdots \oplus Fx_k$ and $Y_k = Fx_1' \oplus \cdots \oplus Fx_k'$. Then the elements of P_k^{SO} have the form

$$\begin{pmatrix} h & * & * \\ 0 & g & * \\ 0 & 0 & h^{*-1} \end{pmatrix}$$

where h^* is the unique element of $\mathrm{Gl}(Y_k)$ such that $(hx, y) = (x, h^* y)$ for $x \in X_k$ and $y \in Y_k$, and $g \in \mathrm{SO}(V_{m-k})$. Often, we will identify $\mathrm{Gl}(X_k)$ and $\mathrm{Gl}(Y_k)$ with $\mathrm{Gl}(k, F)$ via our choice of bases. Then $h^* = {}^t h$. Let M_k^{SO} be the Levi component of P_k and let N_k be the unipotent radical of P_k^{SO}. Via the isomorphism $\mathrm{Gl}(X_k) \cong \mathrm{Gl}(k, F)$, we have an isomorphism $M_k^{\mathrm{SO}} \cong \mathrm{Gl}(k, F) \times \mathrm{SO}(V_{m-k})$. In addition, let P_k be the subgroup of elements of $\mathrm{O}(V_m)$ which stabilize X_k, *i.e.*, the elements of the above form, with $g \in \mathrm{O}(V_{m-k})$. Let M_k be the subgroup of elements of P_k of the form

$$\begin{pmatrix} h & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & h^{*-1} \end{pmatrix}.$$

Then M_k normalizes N_k, $M_k \cap N_k = 1$, and $P_k = M_k N_k$. Also, $M_k \cong \mathrm{Gl}(k, F) \times \mathrm{O}(V_{m-k})$. We let $P_0 = M_0 = \mathrm{O}(V_m)$ and $N_0 = 1$.

For n a nonnegative integer, let

$$\mathbf{W}_n = \overline{F} y_1 \oplus \cdots \oplus \overline{F} y_n \oplus \overline{F} y_1' \oplus \cdots \oplus \overline{F} y_n',$$

and endow $\mathbf{W}_n$ with the symplectic bilinear form such that $\langle y_i, y_j \rangle = \langle y_i', y_j' \rangle = 0$ and $\langle y_i, y_j' \rangle = \delta_{ij}$ for $1 \le i, j \le n$. Let $W_n = \mathbf{W}_n(F)$. We fix the maximal F-split torus of $\mathrm{Sp}(\mathbf{W}_n)$ whose elements are the maps $\mathrm{diag}(t_1, \cdots, t_n)$ which send y_i to $t_i y_i$ and y_i' to $t_i^{-1} y_i'$. As a base for the F-roots of $\mathrm{Sp}(\mathbf{W}_n)$ with respect to our torus we take $\Delta' = \{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n, 2e_n\}$. For $1 \le j \le n$ consider the parabolic subgroup defined by the complement of the j-th element of Δ'. Then the group P_j' of F points of this parabolic subgroup is the stabilizer in $\mathrm{Sp}(W_n)$ of $Fy_1 \oplus \cdots \oplus Fy_j$. We will write the elements of P_j' with respect to the decomposition

$$W_n = X_j' \oplus W_{n-j} \oplus Y_j',$$

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where $X'_j = Fy_1 \oplus \cdots \oplus Fy_j$ and $Y'_j = Fy'_1 \oplus \cdots \oplus Fy'_j$. Then the elements of P'_j have the form

$$\begin{pmatrix} h' & * & * \\ 0 & g' & * \\ 0 & 0 & h'^{*-1} \end{pmatrix},$$

where h'^* is defined as the unique element of $\mathrm{Gl}(Y'_j)$ such that $\langle hx, y\rangle = \langle x, h'^* y\rangle$ for $x \in X'_j$ and $y \in Y'_j$, and $g' \in \mathrm{Sp}(W_{n-j})$. Let M'_j be the Levi component of P'_j and let N'_j be the unipotent radical of P'_j. Then $M'_j \cong \mathrm{Gl}(j, F) \times \mathrm{Sp}(W_{n-j})$. More generally, if $j = j_1 + \cdots + j_a$ is an ordered partition of j, we let $P'_{j_1,\ldots,j_a}$ be the subgroup of P'_j whose elements are of the above form with $h' \in P^{\mathrm{Gl}}_{j_1,\ldots,j_a}$. We also let $P'_0 = M'_0 = \mathrm{Sp}(n, F)$ and $N'_0 = 1$.

2**. Results on square integrable and tempered representations.** In this section we recall some known results on square integrable and tempered representations. We also prove some results that will be used in the proof of part (2) of Theorem 4.2. Let $\sigma \in \mathrm{Irr}\big(\mathrm{O}(V_m)\big)$. Then $\sigma|_{\mathrm{SO}(V_m)} = \sigma_1 \oplus \cdots \oplus \sigma_t$, for some $\sigma_i \in \mathrm{Irr}\big(\mathrm{SO}(V_m)\big)$, $1 \leq i \leq t$. We say that σ is *tempered* if and only if every σ_i is tempered.

For the following theorem, see [C], Corollary 4.4.5 and Theorem 4.4.6.

THEOREM 2.1. *Let $1 \leq k \leq m$ and $1 \leq j \leq n$. Let $\sigma \in \mathrm{Irr}\big(\mathrm{O}(V_m)\big)$ and $\pi \in \mathrm{Irr}\big(\mathrm{Sp}(n, F)\big)$ be tempered. Suppose that $\overline{\mathrm{R}}_{N_k}(\sigma)$ and $\overline{\mathrm{R}}_{N'_j}(\pi)$ are nonzero, and $\sigma_1 \otimes \sigma_2 \in \mathrm{Irr}\big(\mathrm{Gl}(k, F) \times \mathrm{O}(V_{m-k})\big)$ and $\pi_1 \otimes \pi_2 \in \mathrm{Irr}\big(\mathrm{Gl}(j, F) \times \mathrm{Sp}(n-j, F)\big)$ are nonzero irreducible subquotients of $\overline{\mathrm{R}}_{N_k}(\sigma)$ and $\overline{\mathrm{R}}_{N'_j}(\pi)$, respectively. Then*

$$1 \leq |\omega_{\sigma_1}(t)|, \quad 1 \leq |\omega_{\pi_1}(t)|$$

for $|t| < 1$.

Let $p = p_1 + p_2$ be an ordered partition of p. Let $\delta \in \mathrm{Irr}\big(\mathrm{Gl}(p, F)\big)$ be square integrable. Suppose that $\overline{\mathrm{R}}_{N^{\mathrm{Gl}}_{p_1,p_2}}(\delta)$ is nonzero, and $\delta_1 \otimes \delta_2 \in \mathrm{Irr}\big(\mathrm{Gl}(p_1, F) \times \mathrm{Gl}(p_2, F)\big)$ is a nonzero irreducible subquotient of $\overline{\mathrm{R}}_{N^{\mathrm{Gl}}_{p_1,p_2}}(\delta)$. Then

$$1 < |\omega_{\delta_1}(t_1)|\, |\omega_{\delta_2}(t_2)|$$

for $|t_1| < |t_2|$.

PROPOSITION 2.2. *Let $\rho \in \mathrm{Irr}\big(\mathrm{Gl}(q, F)\big)$ be essentially square integrable. Let $q = q_1 + q_2$ be an ordered partition of q. Assume that $\mathrm{R}_{N^{\mathrm{Gl}}_{q_1,q_2}}(\rho)$ is nonzero, and that $\rho_1 \otimes \rho_2 \in \mathrm{Irr}\big(\mathrm{Gl}(q_1, F) \times \mathrm{Gl}(q_2, F)\big)$ is a nonzero irreducible subquotient of $\mathrm{R}_{N^{\mathrm{Gl}}_{q_1,q_2}}(\rho)$. If $\rho_2 = \alpha \circ \det$ is one dimensional, then $\rho \cong \mathbf{St}_q \otimes \beta(\det)$, where $\beta = \alpha |\ |^{q_1/2-a}$ and a is an integer such that $0 \leq a \leq q_1$.*

PROOF. By the classification of essentially square integrable representations, ρ is a quotient of $I = \mathrm{I}^{\mathrm{Gl}(q,F)}_{P^{\mathrm{Gl}}_{n,\ldots,n}}\big(\gamma \otimes (\gamma \otimes |\det|) \otimes \cdots \otimes (\gamma \otimes |\det|^{q/n-1})\big)$, where n divides q and $\gamma \in \mathrm{Irr}\big(\mathrm{Gl}(q/n, F)\big)$ is supercuspidal. Since $\mathrm{R}_{N^{\mathrm{Gl}}_{q_1,q_2}}$ is exact, it follows that $\mathrm{R}_{N^{\mathrm{Gl}}_{q_1,q_2}}(\rho)$

is a quotient of $\mathrm{R}_{N^{\mathrm{Gl}}_{q_1,q_2}}(I)$. Hence, $\rho_1 \otimes \rho_2$ is an irreducible subquotient of $\mathrm{R}_{N^{\mathrm{Gl}}_{q_1,q_2}}(I)$. By, for example, the summary [Rod], Proposition 3, p. 204, for some permutation z of $\{1,\dots,q/n\}$, $\rho_1 \otimes \rho_2$ is a subquotient of

$$\mathrm{Ind}^{\mathrm{Gl}(q_1,F)\times\mathrm{Gl}(q_2,F)}_{P^{\mathrm{Gl}}_{n,\dots,n}\cap(\mathrm{Gl}(q_1,F)\times\mathrm{Gl}(q_2,F))}\big((\gamma\otimes|\det|^{z(1)-1})\otimes\cdots\otimes(\gamma\otimes|\det|^{z(q/n)-1})\big),$$

where the partition of q defined by z, namely $q = n+\cdots+n$, is a refinement of the partition $q = q_1+q_2$, and $z(i) < z(i+1)$ if in is not q_1. This implies that q_1 and q_2 are divisible by n, and that

$$z(1) < \cdots < z(q_1/n), \quad z(q_1/n+1) < \cdots < z(q/n).$$

It follows that ρ_2 is an irreducible subquotient of

$$\mathrm{Ind}^{\mathrm{Gl}(q_2,F)}_{P^{\mathrm{Gl}}_{n,\dots,n}}\big((\gamma\otimes|\det|^{z(q_1/n+1)-1})\otimes\cdots\otimes(\gamma\otimes|\det|^{z(q/n)-1})\big).$$

On the other hand, $\rho_2 = \alpha\circ\det$ embeds in

$$\mathrm{Ind}^{\mathrm{Gl}(q_2,F)}_{P^{\mathrm{Gl}}_{1,\dots,1}}(\alpha|\ |^{(1-q_2)/2}\otimes\alpha|\ |^{(3-q_2)/2}\otimes\cdots\otimes\alpha|\ |^{(q_2-1)/2}).$$

See [K2]. By, for example, [Rod], Proposition 5, p. 206, $n = 1$, and

$$\alpha|\ |^{(1-q_2)/2},\dots,\alpha|\ |^{(2i+1-q_2)/2},\dots,\alpha|\ |^{(q_2-1)/2}$$

is a permutation of

$$\gamma|\ |^{z(q_1+1)-1},\dots,\gamma|\ |^{z(q_1+1+i)-1},\dots,\gamma|\ |^{z(q)-1}.$$

As $z(q_1+1) < \cdots < z(q)$, it follows that $\gamma|\ |^{z(q_1+i+1)-1} = \alpha|\ |^{(2i+1-q_2)/2}$ for $0 \le i \le q_2-1$. Hence, $\gamma = \alpha|\ |^{(1-q_2)/2+i+1-z(q_1+i+1)}$ for $0 \le i \le q_2-1$, and we see that $i+1-z(q_1+i+1)$ for $0 \le i \le q_2-1$ is constant. Thus, $z(q_1+i+1) = i+1+z(q)-q_2$ for $0 \le i \le q_2-1$. This implies $q \ge z(q) \ge q_2$. So $\gamma = \alpha|\ |^{(1-q_2)/2+q_2-z(q)} = \alpha|\ |^{-(q_2-1)/2-a}$, where $a = z(q)-q_2$. ∎

COROLLARY 2.3. *Let $\rho \in \mathrm{Irr}\big(\mathrm{Gl}(q,F)\big)$ be essentially square integrable. Let $q = q_1+q_2$ be an ordered partition of q. Assume that $\overline{\mathrm{R}}_{N^{\mathrm{Gl}}_{q_2,q_1}}(\rho)$ is nonzero, and that $\rho_2\otimes\rho_1 \in \mathrm{Irr}\big(\mathrm{Gl}(q_2,F)\times\mathrm{Gl}(q_1,F)\big)$ is a nonzero irreducible subquotient of $\overline{\mathrm{R}}_{N^{\mathrm{Gl}}_{q_2,q_1}}(\rho)$. If $\rho_2 = \alpha\circ\det$ is one dimensional, then $\rho \cong \mathbf{St}_q\otimes\beta(\det)$, where $\beta = \alpha|\ |^{q_1/2-a}$ and a is an integer such that $0 \le a \le q_1$.*

PROOF. We have by Corollary 4.2.5 of [C] and a straightforward isomorphism, $\overline{\mathrm{R}}_{N^{\mathrm{Gl}}_{q_2,q_1}}(\rho) = \mathrm{R}_{N^{\mathrm{Gl}}_{q_2,q_1}}(\rho^\vee)^\vee \cong \mathrm{R}_{\overline{N^{\mathrm{Gl}}_{q_2,q_1}}}(\rho) \cong \mathrm{R}_{N^{\mathrm{Gl}}_{q_1,q_2}}(\rho)$. The corollary now follows from Proposition 2.2. ∎

COROLLARY 2.4. *Let the notation be as in Corollary 2.3. Then $a < q_1/2$.*

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PROOF. Let $e = e(\rho)$. Since $\rho_2 \otimes \rho_1$ is a nonzero irreducible subquotient of $\overline{\mathrm{R}}_{N^{\mathrm{Gl}}_{q_2,q_1}}(\rho)$, it follows that $(\rho_2 \otimes |\det|^{-e}) \otimes (\rho_1 \otimes |\det|^{-e})$ is a nonzero irreducible subquotient of $\overline{\mathrm{R}}_{N^{\mathrm{Gl}}_{q_2,q_1}}(\rho \otimes |\det|^{-e})$. Since $\rho \otimes |\det|^{-e}$ is square integrable, taking $t_1 = t$ and $t_2 = 1$ in Theorem 2.1, we have for $|t| < 1$,

$$\begin{aligned} 1 &< |\omega_{\rho_2}(t)||t|^{-eq_2} \\ 1 &< |\alpha(t)|^{q_2}|t|^{-eq_2} \\ 1 &< |\beta(t)|^{q_2}|t|^{-(q_1/2-a)q_2-eq_2} \\ 1 &< |t|^{-(q_1/2-a)q_2}. \end{aligned}$$

Here we have used that $\rho \cong \mathbf{St}_q \otimes \beta(\det)$, so that $|\beta(t)|^{q_2} = |t|^{eq_2}$. This implies that $a < q_1/2$. ∎

3. **The Langlands classification for** $\mathrm{Sp}(n, F)$**.** We recall the Langlands classification of $\mathrm{Irr}\big(\mathrm{Sp}(n, F)\big)$ in terms of tempered representations. See [T], Section 6.

THEOREM 3.1 (LANGLANDS CLASSIFICATION). *Let $n = n_1 + \cdots + n_t + n_0$, where $n_1, \ldots, n_t, n_0$ are nonnegative integers, with $n_1, \ldots, n_t$ positive if $t > 0$. Let $\delta_i \in \mathrm{Irr}\big(\mathrm{Gl}(n_i, F)\big)$ for $1 \le i \le t$ and $\tau \in \mathrm{Irr}\big(\mathrm{Sp}(n_0, F)\big)$ be such that:*

(1) $\delta_1, \ldots, \delta_t$ are essentially tempered and

$$e(\delta_1) > \cdots > e(\delta_t) > 0;$$

(2) τ is tempered.

Then the representation $\mathrm{Ind}^{\mathrm{Sp}(n,F)}_{P'_{n_1,\ldots,n_t}}(\delta_1 \otimes \cdots \otimes \delta_t \otimes \tau)$ has a unique nonzero irreducible quotient $L(\delta_1 \otimes \cdots \otimes \delta_t \otimes \tau)$, $\mathrm{Ind}^{\mathrm{Sp}(n,F)}_{P'_{n_1,\ldots,n_t}}(\delta_1^\vee \otimes \cdots \otimes \delta_t^\vee \otimes \tau)$ has a unique nonzero irreducible subrepresentation $S(\delta_1 \otimes \cdots \otimes \delta_t \otimes \tau)$, and $L(\delta_1 \otimes \cdots \otimes \delta_t \otimes \tau) \cong S(\delta_1 \otimes \cdots \otimes \delta_t \otimes \tau)$. Moreover, if $\pi \in \mathrm{Irr}\big(\mathrm{Sp}(n, F)\big)$ then there exist a unique decomposition $n = n_1 + \cdots + n_t + n_0$ as above, unique $\delta_i \in \mathrm{Irr}\big(\mathrm{Gl}(n_i, F)\big)$, $1 \le i \le t$, and unique $\tau \in \mathrm{Irr}\big(\mathrm{Sp}(n_0, F)\big)$ such that (1) and (2) hold and $\pi \cong L(\delta_1 \otimes \cdots \otimes \delta_t \otimes \tau)$.

4. **Applications to the theta correspondence.** In this final section we give the proofs of Theorems 4.2 and 4.4. Fix a nontrivial additive character ψ of F. We let $\omega_{m,n}$ denote the Weil representation of $\mathrm{O}(V_m) \times \mathrm{Sp}(n, F)$ on $S(V^n)$ associated to ψ. Explicitly, $\omega_{m,n}$ is given by the following formulas.

$$\begin{aligned} \omega_{m,n}(g, 1)\varphi(v) &= \varphi(g^{-1}v), \\ \omega_{m,n}\left(1, \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}\right)\varphi(v) &= \chi\big(\det(a)\big)|\det(a)|^{l/2}\varphi(va), \\ \omega_{m,n}\left(1, \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right)\varphi(v) &= \psi\left(\frac{1}{2}\operatorname{tr}(bv, v)\right)\varphi(v), \\ \omega_{m,n}\left(1, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\varphi(v) &= \gamma\hat{\varphi}(v). \end{aligned}$$

Here, $\hat{\varphi}$ is the Fourier transform defined by

$$\hat{\varphi}(v) = \int_{V^n} \varphi(v')\psi\big(\mathrm{tr}(v, v')\big)\, dv',$$

where the Haar measure is such that $\hat{\hat{\varphi}}(v) = \varphi(-v)$ for $\varphi \in S(V^n)$ and $v \in V^n$, and γ is a fourth root of unity that depends only on V_0, n and ψ. If $g \in \mathrm{O}(V_m)$, $a \in \mathrm{Gl}(n, F)$, $b \in \mathrm{M}_n(F)$, ${}^t b = b$ and $v = (v_1, \ldots, v_n)$, $v' = (v'_1, \ldots, v'_n) \in V^n$, we write $g^{-1}v = (g^{-1}v_1, \ldots, g^{-1}v_n)$, $va = (v_1, \ldots, v_n)(a_{ij})$, $(v, v') = ((v_i, v'_j))$, $bv = b^t(v_1, \ldots, v_n)$. Also, χ is the quadratic character of $F^\times$ defined by $\chi(t) = \big(t, \mathrm{disc}(V_m)\big)_F$. We note that χ does not depend on m. We see that if $V_m = 0$, then $\omega_{m,n} = \mathbf{1}$ and if $n = 0$, then $\omega_{m,n} = \mathbf{1}$.

We let $R_n\big(\mathrm{O}(V_m)\big)$ be the set of $\sigma \in \mathrm{Irr}\big(\mathrm{O}(V_m)\big)$ such that σ is a nonzero quotient of $\omega_{m,n}$ restricted to $\mathrm{O}(V_m)$. We define $R_m\big(\mathrm{Sp}(n, F)\big)$ similarly.

We recall the computation of the Jacquet modules of $\omega_{m,n}$ from [K1]. We need some notation. For $1 \le j \le n$ and $0 \le k \le r = \min(m, j)$ let Q'_{jk} be the subgroup of M'_j of elements of the form

$$\begin{pmatrix} h'' & * & 0 & 0 & 0 \\ 0 & h' & 0 & 0 & 0 \\ 0 & 0 & g' & 0 & 0 \\ 0 & 0 & 0 & {}^t h''^{-1} & 0 \\ 0 & 0 & 0 & * & {}^t h'^{-1} \end{pmatrix}$$

where $h'' \in \mathrm{Gl}(j-k, F)$ and $h' \in \mathrm{Gl}(k, F)$. Also, define a representation $\boldsymbol{\sigma}_k$ of $\mathrm{Gl}(k, F) \times \mathrm{Gl}(k, F)$ on $S\big(\mathrm{Gl}(k, F)\big)$ by $\boldsymbol{\sigma}_k(h, h')\phi(x) = \phi(h^{-1}xh')$.

THEOREM 4.1 (KUDLA). *Let $1 \le j \le n$ and $r = \min(m, j)$. Then the representation $\mathrm{R}_{N'_j}(\omega_{m,n})$ of $\mathrm{O}(V_m) \times M'_j$ has a filtration*

$$0 = F^{r+1} \subset F^r \subset F^{r-1} \subset \cdots \subset F^1 \subset F^0 = \mathrm{R}_{N'_j}(\omega_{m,n})$$

such that for $0 \le k \le r$,

$$F^k/F^{k+1} \cong \mathrm{Ind}_{P_k \times Q'_{jk}}^{\mathrm{O}(V_m) \times M'_j} \xi_k \xi'_k \xi''_k \boldsymbol{\sigma}_k \otimes \omega_{m-k, n-j}.$$

Here, the induction is normalized, and the action is defined by

$$\left(\begin{pmatrix} h & * & * \\ 0 & g & * \\ 0 & 0 & {}^t h^{-1} \end{pmatrix}, \begin{pmatrix} h'' & * & 0 & 0 & 0 \\ 0 & h' & 0 & 0 & 0 \\ 0 & 0 & g' & 0 & 0 \\ 0 & 0 & 0 & {}^t h''^{-1} & 0 \\ 0 & 0 & 0 & * & {}^t h'^{-1} \end{pmatrix} \right) (\phi \otimes \varphi)$$
$$= \xi_k(\det h)\xi'_k(\det h')\xi''_k(\det h'')\boldsymbol{\sigma}_k(h, h')\phi \otimes \omega_{m-k,n-j}(g, g')\varphi,$$

where

$$\xi_k = |\ |^{-(l-k-1)/2}, \quad \xi'_k = \chi|\ |^{(l-k-1)/2}, \quad \xi''_k = \chi|\ |^{l/2-n+(j-k-1)/2}.$$

The statement of Theorem 4.1 was obtained by repeating the proof of the corresponding theorem in [K1], adjusting for our conventions. See also [MVW], Chapitre 3. We can now give the proof of Theorem 4.2:

PROOF OF THEOREM 4.2. First we make a preliminary observation. Let $1 \le j \le n$, and let ρ and ρ' be smooth admissible representations of finite length of $\mathrm{Gl}(j,F)$ and $\mathrm{Sp}(n-j,F)$, respectively. Assume that

$$\mathrm{Hom}_{\mathrm{O}(V_m)\times\mathrm{Sp}(n,F)}\big(\omega_{m,n}, \sigma\otimes \mathrm{Ind}_{P'_j}^{\mathrm{Sp}(n,F)}(\rho\otimes\rho')\big)\neq 0.$$

Then by Frobenius reciprocity,

$$\mathrm{Hom}_{\mathrm{O}(V_m)\times\mathrm{Gl}(j,F)\times\mathrm{Sp}(n-j,F)}\big(\mathrm{R}_{N'_j}(\omega_{m,n}), \sigma\otimes\rho\otimes\rho'\big)\neq 0.$$

By Theorem 4.1, this implies that for some k with $0\le k\le \min(m,j)$,

$$\mathrm{Hom}_{\mathrm{O}(V_m)\times\mathrm{Gl}(j,F)\times\mathrm{Sp}(n-j,F)}(\mathrm{Ind}_{P_k\times Q'_{jk}}^{\mathrm{O}(V_m)\times M'_j}\,\xi_k\xi'_k\xi''_k\boldsymbol{\sigma}_k\otimes\omega_{m-k,n-j}, \sigma\otimes\rho\otimes\rho')\neq 0.$$

By Frobenius reciprocity again, there is a nonzero $\mathrm{Gl}(k,F)\times\mathrm{O}(V_{m-k})\times\mathrm{Gl}(j-k,F)\times\mathrm{Gl}(k,F)\times\mathrm{Sp}(n-j,F)$ map

$$(*)\qquad \xi_k\xi'_k\xi''_k\boldsymbol{\sigma}_k\otimes\omega_{m-k,n-j}\longrightarrow \overline{\mathrm{R}}_{N_k}(\sigma)\otimes\overline{\mathrm{R}}_{N^{\mathrm{Gl}}_{j-k,k}}(\rho)\otimes\rho'.$$

Proof of (1). Assume that $n\le l/2$ and π is not tempered. As in Theorem 3.1, $\pi\cong S(\delta_1\otimes\cdots\otimes\delta_t\otimes\tau)$, where $n=n_1+\cdots+n_t+n_0$ with $t>0, \delta_i\in\mathrm{Irr}\big(\mathrm{Gl}(n_i,F)\big), 1\le i\le t$, are essentially tempered such that $e(\delta_1)>\cdots>e(\delta_t)>0$ and $\tau\in\mathrm{Irr}\big(\mathrm{Sp}(n_0,F)\big)$ is tempered. Let $j=n_1$, $\rho=\delta_1^\vee$ and

$$\rho'=\mathrm{Ind}_{P'_{n_2,\dots,n_t}}^{\mathrm{Sp}(n-j,F)}(\delta_2^\vee\otimes\cdots\otimes\delta_t^\vee\otimes\tau).$$

Since π is isomorphic to a subrepresentation of $\mathrm{Ind}_{P'_j}^{\mathrm{Sp}(n,F)}(\rho\otimes\rho')$, we have

$$\mathrm{Hom}_{\mathrm{O}(V_m)\times\mathrm{Sp}(n,F)}\big(\omega_{m,n}, \sigma\otimes \mathrm{Ind}_{P'_j}^{\mathrm{Sp}(n,F)}(\rho\otimes\rho')\big)\neq 0.$$

So there is a nonzero map as in (*), and hence there exist nonzero irreducible subquotients $\sigma_1\otimes\sigma_2\in\mathrm{Irr}\big(\mathrm{Gl}(k,F)\times\mathrm{O}(V_{m-k})\big)$ of $\overline{\mathrm{R}}_{N_k}(\sigma)$ and $\rho_2\otimes\rho_1\in\mathrm{Irr}\big(\mathrm{Gl}(j-k,F)\times\mathrm{Gl}(k,F)\big)$ of $\overline{\mathrm{R}}_{N^{\mathrm{Gl}}_{j-k,k}}(\rho)$ such that there is a nonzero $\mathrm{Gl}(k,F)\times\mathrm{O}(V_{m-k})\times\mathrm{Gl}(j-k,F)\times\mathrm{Gl}(k,F)\times\mathrm{Sp}(n-j,F)$ map

$$\xi_k\xi'_k\xi''_k\boldsymbol{\sigma}_k\otimes\omega_{m-k,n-j}\longrightarrow\sigma_1\otimes\sigma_2\otimes\rho_2\otimes\rho_1\otimes\rho.$$

Hence,

$$\mathrm{Hom}_{\mathrm{Gl}(k,F)\times\mathrm{Gl}(k,F)}\big(\boldsymbol{\sigma}_k, (\sigma_1\otimes\xi_k^{-1})\otimes(\rho_1\otimes\xi_k'^{-1})\big)\neq 0$$

and

$$\mathrm{Hom}_{\mathrm{Gl}(j-k,F)}(\xi''_k,\rho_2)\neq 0,\quad \mathrm{Hom}_{\mathrm{O}(V_{m-k})\times\mathrm{Sp}(n-j,F)}(\omega_{m-k,n-j},\sigma_2\otimes\rho)\neq 0.$$

This implies that $\rho_1=\sigma_1^\vee\otimes\chi(\det)$ and $\xi''_k=\rho_2$. Let

$$\lambda = l/2-n+(j-k-1)/2.$$

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Then for $t \in F^\times$,

$$\begin{aligned} \omega_\rho(t) &= \omega_{\rho_1}(t)\omega_{\rho_2}(t) \\ |\omega_{\rho_1}(t)|^{-1} &= |\omega_\rho(t)|^{-1}|\omega_{\rho_2}(t)| \\ |\omega_{\sigma_1}(t)| &= |t|^{(j-k)\lambda - e(\rho)j}. \end{aligned}$$

Assume $k > 0$. Since σ is tempered, by Theorem 2.1, for $|t| < 1$, we have

$$1 \le |\omega_{\sigma_1}(t)| = |t|^{(j-k)\lambda - e(\rho)j}.$$

This implies $(j-k)\lambda \le e(\rho)j = e(\delta_1^\vee)j$. As $n \le l/2$, $(j-k)\lambda \ge 0$, so that $e(\delta_1^\vee) = -e(\delta_1) \ge 0$, a contradiction.

Assume $k = 0$. Then $\rho = \delta_1^\vee = \xi_0''$. Since δ_1 is an essentially tempered representation, we must have $j = 1$ and $\delta_1 = \chi|\ |^{n-l/2}$, so that $n - l/2 = e(\delta_1) > 0$, a contradiction.

Proof of (2). Let $\pi \cong S(\delta_1 \otimes \cdots \otimes \delta_t \otimes \tau)$, as in the proof of (1). Let $e = e(\delta_1)$ and $\delta_1 \otimes |\det|^{-e} \cong \operatorname{Ind}_{P^{\mathrm{Gl}}_{p_1,\ldots,p_s}}^{\mathrm{Gl}(n_1,F)}(\eta_1 \otimes \cdots \otimes \eta_s)$, where $\eta_i \in \operatorname{Irr}\big(\mathrm{Gl}(p_i,F)\big)$, $1 \le i \le s$, are square integrable. Then $\delta_1 \cong \operatorname{Ind}_{P^{\mathrm{Gl}}_{p_1,\ldots,p_s}}^{\mathrm{Gl}(n_1,F)}\big((\eta_1 \otimes |\det|^e) \otimes \cdots \otimes (\eta_s \otimes |\det|^e)\big)$. Let $j = p_1$, $\rho = (\eta_1 \otimes |\det|^e)^\vee$ and

$$\rho' = \operatorname{Ind}_{P_{p_2,\ldots,p_s,n_2,\ldots,n_t}}^{\mathrm{Sp}(n-j,F)}\big((\eta_2 \otimes |\det|^e)^\vee \otimes \cdots \otimes (\eta_s \otimes |\det|^e)^\vee \otimes \delta_2^\vee \otimes \cdots \otimes \delta_t^\vee \otimes \tau\big).$$

Arguing as above, there exist an integer k such that $0 \le k \le \min(m,j)$ and nonzero irreducible subquotients $\sigma_1 \otimes \sigma_2 \in \operatorname{Irr}\big(\mathrm{Gl}(k,F) \times \mathrm{O}(V_{m-k})\big)$ of $\overline{\mathrm{R}}_{N_k}(\sigma)$ and $\rho_2 \otimes \rho_1 \in \operatorname{Irr}\big(\mathrm{Gl}(j-k,F) \times \mathrm{Gl}(k,F)\big)$ of $\overline{\mathrm{R}}_{N^{\mathrm{Gl}}_{j-k,k}}(\rho)$ such that $\xi_k'' = \rho_2$ and $\rho_1 = \sigma_1^\vee \otimes \chi(\det)$.

Assume $k > 0$. As in the proof of (1), we find that $\lambda(j-k) \le je(\rho) = -je$. Now $j - k \ge 0$; as $e > 0$, it follows that $j - k > 0$. Hence, by Corollaries 2.3 and 2.4, $\rho \cong \mathbf{St}_{p_1} \otimes \chi(\det)|\det|^{\lambda + k/2 - a}$ with a an integer such that $0 \le a < k/2$, or equivalently, $\eta \otimes |\det|^e \cong \mathbf{St}_{p_1} \otimes \chi(\det)|\det|^{n-l/2-c/2}$, with $c = j - 1 - 2a$ satisfying $j - 1 \ge c > 0$. This implies that $e = e(\delta_1) = n - l/2 - c/2$. Also, we have that

$$\begin{aligned} \lambda(j-k) &\le -je \\ \big(l/2 - n + (j-k-1)/2\big)(j-k) &\le -j(n - l/2 - c/2) \\ (l/2 - n)(j-k) + (j-k-1)(j-k)/2 &\le -j(n - l/2) + j(c/2) \\ k(n - l/2) + (j-k-1)(j-k)/2 &\le j(j-1)/2 \\ (n - l/2) + (k+1)/2 &\le j, \end{aligned}$$

so that $(n - l/2) + 1 \le j$.

Assume $k = 0$. The we have $\sigma \in R_{n-j}\big(\mathrm{O}(V_m)\big)$ and hence $\sigma \in R_{n-1}\big(\mathrm{O}(V_m)\big)$, a contradiction.

Since $\operatorname{Ind}_{P_{p_1,\ldots,p_s}}^{\mathrm{Gl}(n_1,F)}(\eta_1 \otimes \cdots \otimes \eta_s) \cong \operatorname{Ind}_{P_{p_{z(1)},\ldots,p_{z(s)}}}^{\mathrm{Gl}(n_1,F)}(\eta_{z(1)} \otimes \cdots \otimes \eta_{z(s)})$ for any permutation z of $\{1,\ldots,s\}$, the same results hold for all of the η_i. The claim (2) now follows. ∎

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To prove Theorem 4.4, we recall some results from [K-R1]. For $s \in \mathbb{C}$ and N a positive integer, let $I_N(s,\chi)$ be the degenerate principal series representation induced from the quasi-character $\chi|\ |^s$ of the Siegel parabolic of $\mathrm{Sp}(N,F)$, *i.e.*, the space of smooth functions Φ on $\mathrm{Sp}(N,F)$ such that

$$\Phi\left(\begin{pmatrix} a & b \\ 0 & {}^t a^{-1}\end{pmatrix} g\right) = \chi(\det a)|\det a|^{s+\frac{N+1}{2}}\Phi(g),$$

where $\begin{pmatrix} a & b \\ 0 & {}^t a^{-1}\end{pmatrix} \in P'_N$ and $g \in \mathrm{Sp}(N,F)$. Finally, let $(\omega_{m,N})_{\mathrm{O}(V_m)}$ be the $\mathrm{O}(V_m)$ coinvariants of $\omega_{m,N}$.

PROPOSITION 4.3 (KUDLA-RALLIS). *Assume that* $0 \le l \le N$ *and let* $s_0 = -l/2 + (N+1)/2 = (2N+2-l)/2 - (N+1)/2$. *Then there is a surjective* $\mathrm{Sp}(N,F)$ *map from* $I_N(s_0,\chi)$ *to* $(\omega_{m,N})_{\mathrm{O}(V_m)}$.

PROOF. See Proposition 5.5 of [K-R1]. ■

Finally, we give the proof of Theorem 4.4.

PROOF OF THEOREM 4.4. The argument is similar to the proof of the theorem for supercuspidal π and π' in [K1]. By hypothesis, there exist a nonzero $\mathrm{O}(V_m)\times\mathrm{Sp}(n,F)$ map $\omega_{m,n} \longrightarrow \sigma\otimes\pi$ and a nonzero $\mathrm{O}(V_m)\times\mathrm{Sp}(n',F)$ map $\omega_{m,n'} \longrightarrow \sigma\otimes\pi'$. Since σ and π are pre-unitary, there exists a $\mathbb{C}$-anti-linear isomorphism $\sigma\otimes\pi \longrightarrow \sigma^\vee\otimes\pi^\vee$ which is $\mathrm{O}(V_m)\times\mathrm{Sp}(n,F)$ intertwining. If $\overline{\omega_{m,n}}$ is defined by $\overline{\omega_{m,n}}(g,g')\varphi = \overline{\omega_{m,n}(g,g')\overline{\varphi}}$, where we use the above model for $\omega_{m,n}$, then the map $\overline{\omega_{m,n}} \longrightarrow \omega_{m,n}$ defined by $\varphi \longrightarrow \overline{\varphi}$ is also a $\mathbb{C}$-anti-linear isomorphism which is $\mathrm{O}(V_m)\times\mathrm{Sp}(n,F)$ intertwining. Composing, we obtain a nonzero $\mathrm{O}(V_m)\times\mathrm{Sp}(n,F)$ map $\overline{\omega_{m,n}} \longrightarrow \sigma^\vee\otimes\pi^\vee$. This implies that there is a nonzero $\mathrm{O}(V_m)\times\mathrm{Sp}(n,F)\times\mathrm{Sp}(n',F)$ map

$$\overline{\omega_{m,n}}\otimes\omega_{m,n'} \longrightarrow \sigma^\vee\otimes\sigma\otimes\pi^\vee\otimes\pi',$$

and by composition with the canonical $\mathrm{O}(V_m)$ map $\sigma^\vee\otimes\sigma \longrightarrow \mathbf{1}$, a nonzero $\mathrm{O}(V_m)\times\mathrm{Sp}(n,F)\times\mathrm{Sp}(n',F)$ map

$$\overline{\omega_{m,n}}\otimes\omega_{m,n'} \longrightarrow \pi^\vee\otimes\pi'.$$

It is easy to verify that

$$\overline{\omega_{m,n}} \cong \left(1,\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\right)\cdot\omega_{m,n}.$$

Define a map $T\colon S(V_m^n)\otimes S(V_m^{n'}) \longrightarrow S(V_m^{n+n'})$ by

$$T(\varphi\otimes\varphi')(x\oplus x') = \varphi(x)\varphi'(x').$$

Define an inclusion $\mathrm{Sp}(n,F)\times\mathrm{Sp}(n',F) \hookrightarrow \mathrm{Sp}(n+n',F)$ by

$$\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix},\begin{pmatrix} a' & b' \\ c' & d'\end{pmatrix}\right) \longmapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d'\end{pmatrix}.$$

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Then it is a standard fact that T gives an isomorphism

$$\omega_{m,n} \otimes \omega_{m,n'} \xrightarrow{\sim} \omega_{m,n+n'} |_{\mathrm{O}(V_m)\times(\mathrm{Sp}(n,F)\times\mathrm{Sp}(n',F))},$$

where we regard $\mathrm{Sp}(n,F)\times\mathrm{Sp}(n',F)$ as a subgroup of $\mathrm{Sp}(n+n',F)$ via the above inclusion. It follows that if we use the inclusion $\mathrm{Sp}(n,F)\times\mathrm{Sp}(n',F) \hookrightarrow \mathrm{Sp}(n+n',F)$ given by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right) \mapsto \begin{pmatrix} a & 0 & -b & 0 \\ 0 & a' & 0 & b' \\ -c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{pmatrix}$$

then we have an isomorphism

$$\overline{\omega_{m,n}} \otimes \omega_{m,n'} \xrightarrow{\sim} \omega_{m,n+n'} |_{\mathrm{O}(V_m)\times(\mathrm{Sp}(n,F)\times\mathrm{Sp}(n',F))} .$$

Thus, there is a nonzero $\mathrm{O}(V_m) \times \big(\mathrm{Sp}(n,F)\times\mathrm{Sp}(n',F)\big)$ map

$$\omega_{m,n+n'} \longrightarrow \pi^\vee \otimes \pi'.$$

By Proposition 4.3, since $l \leq 2n \leq N = n+n'$, there is a nonzero $\mathrm{Sp}(n,F)\times\mathrm{Sp}(n',F)$ map

$$I_N(s_0,\chi) \longrightarrow \pi^\vee \otimes \pi'.$$

Now by Proposition 3.4 of [K1] (see also the explicit computations in [G]), $I_N(s_0,\chi)$ admits a filtration of $\mathrm{Sp}(n,F)\times\mathrm{Sp}(n',F)$ representations

$$0 = I^{n+1} \subset I^n \subset \cdots \subset I^1 \subset I^0 = I_N(s_0,\chi)$$

such that

$$I^i/I^{i+1} \cong \mathrm{Ind}_{P'_{n-i}\times P'_{n'-i}}^{\mathrm{Sp}(n,F)\times\mathrm{Sp}(n',F)} \chi|\ |^{s_0+\frac{n'-i}{2}} \otimes \chi|\ |^{s_0+\frac{n-i}{2}} \otimes \rho_i$$

for $0 \leq i \leq n$. Here, ρ_i is the representation of $\mathrm{Sp}(i,F)\times\mathrm{Sp}(i,F)$ on $S\big(\mathrm{Sp}(i,F)\big)$ defined by $\rho_i(g,g')\phi(x) = \phi(g^{-1}xg')$. It follows that for some $0 \leq i \leq n$,

$$\mathrm{Hom}_{\mathrm{Sp}(n,F)\times\mathrm{Sp}(n',F)}(\mathrm{Ind}_{P'_{n-i}\times P'_{n'-i}}^{\mathrm{Sp}(n,F)\times\mathrm{Sp}(n',F)} \chi|\ |^{s_0+\frac{n'-i}{2}} \otimes \chi|\ |^{s_0+\frac{n-i}{2}} \otimes \rho_i, \pi^\vee \otimes \pi') \neq 0.$$

By Frobenius reciprocity, we obtain

$$\mathrm{Hom}_{M'_{n-i}\times M'_{n'-i}}\big(\chi|\ |^{s_0+\frac{n'-i}{2}} \otimes \chi|\ |^{s_0+\frac{n-i}{2}} \otimes \rho_i, \overline{\mathrm{R}}_{N'_{n-i}}(\pi^\vee) \otimes \overline{\mathrm{R}}_{N'_{n'-i}}(\pi')\big) \neq 0.$$

Hence, there exists an irreducible subquotient $\pi_1 \otimes \pi_2 \in \mathrm{Irr}\big(\mathrm{Gl}(n-i,F)\times\mathrm{Sp}(i,F)\big)$ of $\overline{\mathrm{R}}_{N'_{n-i}}(\pi^\vee)$ and an irreducible subquotient $\pi'_1 \otimes \pi'_2 \in \mathrm{Irr}\big(\mathrm{Gl}(n'-i,F)\times\mathrm{Sp}(i,F)\big)$ of $\overline{\mathrm{R}}_{N'_{n'-i}}(\pi')$ such that

$$\mathrm{Hom}_{M'_{n-i}\times M'_{n'-i}}\big(\chi|\ |^{s_0+\frac{n'-i}{2}} \otimes \chi|\ |^{s_0+\frac{n-i}{2}} \otimes \rho_i, (\pi_1\otimes\pi_2)\otimes(\pi'_1\otimes\pi'_2)\big) \neq 0.$$

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In particular, this implies that $\chi|\ |^{s_0+\frac{n'-i}{2}} = \pi_1$.

Now suppose that $i < n$. Then $1 \leq n-i \leq n$. By Theorem 2.1, since π is tempered, $1 \leq |\omega_{\pi_1}(t)|$ for $|t| < 1$. Hence,

$$(n-i)\big(s_0 + (n'-i)/2\big) \leq 0.$$

So, $n-i+2n'+1 \leq l$. But $l \leq 2n$. Therefore, $n-i \leq -2(n'-n)-1$. Since $n' \geq n$, this implies that $i > n$, a contradiction.

Since $i = n$, we obtain

$$\operatorname{Hom}_{\mathrm{Sp}(n,F)\times\mathrm{Sp}(n',F)}\big(\operatorname{Ind}_{\mathrm{Sp}(n,F)\times P'_{n'-n}}^{\mathrm{Sp}(n,F)\times\mathrm{Sp}(n',F)}(\chi|\ |^{s_0}\otimes\rho_n), \pi^\vee\otimes\pi'\big) \neq 0.$$

By Frobenius reciprocity, this implies that

$$\operatorname{Hom}_{\mathrm{Sp}(n,F)\times P'_{n'-n}}\Big(\rho_n, \pi^\vee\otimes\big((\pi'^\vee|_{P'_{n'-n}})^\vee\otimes\chi|\ |^{-s_0}\delta_{P'_{n'-n}}^{1/2}\big)\Big) \neq 0.$$

Let

$$\rho_n \longrightarrow \pi^\vee\otimes\big((\pi'^\vee|_{P'_{n'-n}})^\vee\otimes\chi|\ |^{-s_0}\delta_{P'_{n'-n}}^{1/2}\big)$$

be a nonzero $\mathrm{Sp}(n,F)\times P'_{n'-n}$ map. By the lemma on p. 59 of [MVW], there is an $\mathrm{Sp}(n,F)\times\mathrm{Sp}(n,F)$ isomorphism

$$\pi^\vee\otimes\pi \cong \rho_n \Big/ \bigcap_{\substack{f\in\operatorname{Hom}_{\mathrm{Sp}(n,F)\times 1}(\rho_n,\pi^\vee\otimes U),\\ U \text{ a } \mathbb{C} \text{ vector space}}} \ker(f).$$

Let

$$\rho_n \longrightarrow \pi^\vee\otimes\pi$$

be the quotient map. It follows that there is an $\mathrm{Sp}(n,F)\times 1$ map

$$\pi^\vee\otimes\pi \longrightarrow \pi^\vee\otimes\big((\pi'^\vee|_{P'_{n'-n}})^\vee\otimes\chi|\ |^{-s_0}\delta_{P'_{n'-n}}^{1/2}\big)$$

such that

$$\begin{array}{ccc} \rho_n & \longrightarrow & \pi^\vee\otimes\pi \\ & \searrow & \downarrow \\ & & \pi^\vee\otimes\big((\pi'^\vee|_{P'_{n'-n}})^\vee\otimes\chi|\ |^{-s_0}\delta_{P'_{n'-n}}^{1/2}\big) \end{array}$$

commutes. This map is also an $\mathrm{Sp}(n,F)\times P'_{n'-n}$ map. Hence,

$$\operatorname{Hom}_{P'_{n'-n}}\big(\pi, (\pi'^\vee|_{P'_{n'-n}})^\vee\otimes\chi|\ |^{-s_0}\delta_{P'_{n'-n}}^{1/2}\big) \neq 0.$$

By Frobenius reciprocity,

$$\operatorname{Hom}_{\mathrm{Sp}(n',F)}\big(\operatorname{Ind}_{P'_{n'-n}}^{\mathrm{Sp}(n',F)}(\chi|\ |^{s_0}\otimes\pi), \pi'\big) \neq 0.$$

Now there is a surjective $\mathrm{Gl}(n'-n,F)$ map

$$\operatorname{Ind}_{P^{\mathrm{Gl}}_{1,\dots,1}}^{\mathrm{Gl}(n'-n,F)}(\chi|\ |^{n'-l/2}\otimes\cdots\otimes\chi|\ |^{n+1-l/2}) \longrightarrow \chi|\ |^{s_0}.$$

Hence, there is a surjective $\mathrm{Sp}(n,F')$ map

$$\operatorname{Ind}_{P'_{n'-n}}^{\mathrm{Sp}(n',F)}(\chi|\ |^{n'-l/2}\otimes\cdots\otimes\chi|\ |^{n+1-l/2}\otimes\pi) \longrightarrow \pi'.$$

This implies that $\pi' = L(\chi|\ |^{n'-l/2}\otimes\cdots\otimes\chi|\ |^{n+1-l/2}\otimes\pi)$. ∎

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