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THE GENERALIZED INVERSE $A^{(2)}_{T,S}$ OF A MATRIX OVER AN ASSOCIATIVE RING

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Abstract

In this paper we establish the definition of the generalized inverse $A^{(2)}_{T,S}$ which is a $\{2\}$ inverse of a matrix A with prescribed image T and kernel S over an associative ring, and give necessary and sufficient conditions for the existence of the generalized inverse $A^{(1,2)}_{T,S}$ and some explicit expressions for $A^{(1,2)}_{T,S}$ of a matrix A over an associative ring, which reduce to the group inverse or $\{1\}$ inverses. In addition, we show that for an arbitrary matrix A over an associative ring, the Drazin inverse A_d, the group inverse A_g and the Moore-Penrose inverse $A^{\dagger}$, if they exist, are all the generalized inverse $A^{(2)}_{T,S}$.

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1. Introduction

It is a well known that, over the field of complex numbers, the Moore-Penrose inverse, the Drazin inverse, the group inverse and so on, are all the generalized inverse $A^{(2)}_{T,S}$, which is a $\{2\}$ inverse of a matrix A with prescribed range T and null space S (see [2, 10]). Y. Wei in [11] gave an explicit expression for the generalized inverse $A^{(2)}_{T,S}$ which reduces to the group inverse.

There are some results on generalized inverses of matrices, such as the Drazin inverse, the group inverse and the Moore-Penrose inverse, over an associative ring (see, for example, [3–8]). These results include necessary and sufficient conditions for the existence of these generalized inverses. In [8], Corollary 1 implies that over an associative ring, a von Neumann regular matrix A has a group inverse if and only if $A^2A^{(1)}+I-AA^{(1)}$ is invertible, if and only if $A^{(1)}A^2+I-A^{(1)}A$ is invertible. Recently,

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similar results about the Moore-Penrose inverse and the Drazin inverse appeared in [6, 7]. This is a motivation for our research.

Throughout this paper, R denotes an associative ring with identity 1 and $R^{m\times n}$ denotes the set of $m \times n$ matrices over R. In particular, we write R^m for $R^{m\times 1}$ and $M_n(R)$ for $R^{n\times n}$, the ring of square $n \times n$ matrices over R. By a module we mean a right R-module. If S is an R-submodule of an R-module M then we write $S \subset M$.

Let $A \in R^{m\times n}$. We denote the image of A (that is $\{Ax | x \in R^n\}$) by $R(A)$ and the kernel of A (that is $\{x \in R^n | Ax = 0\}$) by $N(A)$.

An $m \times n$ matrix A over R is said to be *von Neumann regular* if there exists an $n \times m$ matrix X over R such that

(1) $AXA = A$.

In this case X is called a $\{1\}$ *inverse* of A and is denoted by $A^{(1)}$.

An $n \times n$ matrix A over R is said to be *Drazin invertible* if for some positive integer k there exists a matrix X over R such that

(2) $A^kXA = A^k$,

(3) $XAX = X$,

(4) $AX = XA$.

If X exists then it is unique and is called the *Drazin inverse* of A and denoted by A_d. If k is the smallest positive integer such that X and A satisfy (2), (3) and (4), then it is called the *Drazin index* and denoted by k=Ind(A). If $k = 1$ then A_d is denoted by A_g and is called the *group inverse* of A.

Let $*$ be an involution on the matrices over R. Recall that an $m \times n$ matrix A over R is said to be *Moore-Penrose invertible* (with respect to $*$) if there exists an $n \times m$ matrix X such that (1) and (3) hold and

(6) $(AX)^* = AX$,

(7) $(XA)^* = XA$.

If X exists then it is unique and is called the *Moore-Penrose inverse* of A and denoted by $A^\dagger$. If a matrix X satisfies condition (3) then X is called a $\{2\}$ *inverse* of A.

In Section 2 we shall establish the definition of the generalized inverse $A^{(2)}_{T,S}$, which is a $\{2\}$ inverse of a matrix A over an associative ring with prescribed image T and kernel S, and show that for an arbitrary matrix A over an associative ring the Drazin inverse A_d, the group inverse A_g and the Moore-Penrose inverse $A^\dagger$, if they exist, are all the generalized inverse $A^{(2)}_{T,S}$. In Section 3, we give necessary and sufficient conditions for the existence of the generalized inverse $A^{(1,2)}_{T,S}$. In Section 4 we study some explicit expressions for $A^{(1,2)}_{T,S}$ of a matrix A over an associative ring, which reduce to the group inverse or $\{1\}$ inverses, and some equivalent conditions for the existence of $A^{(1,2)}_{T,S}$.

2. The generalized inverse $A^{(2)}_{T,S}$

Suppose that $L, M \subset R^n$ and $L \oplus M = R^n$. Then every $x \in R^n$ can be uniquely written as $x = x_1 + x_2$, where $x_1 \in L, x_2 \in M$. Thus

$$P_{L,M}x = x_1$$

defines a homomorphism $P_{L,M} : R^n \to R^n$ called the *projection* of R^n on L along M. This homomorphism can be represented by a matrix with respect to the standard basis of R^n, since the module R^n is free. The symbol $P_{L,M}$ is used to denote the matrix as well.

About $P_{L,M}$, we have the following results, whose proof is analogous to that over the field of complex numbers.

LEMMA 2.1. *If $L, M \subset R^n$ and $L \oplus M = R^n$ then*

(i) $P_{L,M}A = A$ *if and only if* $R(A) \subset L$,

(ii) $AP_{L,M} = A$ *if and only if* $N(A) \supset M$.

We now characterize the $\{2\}$ inverse of a matrix A over R with prescribed image T and kernel S. The proof of the following theorem is analogous to that of [13, Theorem 1].

THEOREM 2.2. *Let A be an $m \times n$ matrix over an associative ring R with identity and $T \subset R^n$ and $S \subset R^m$. Then the following conditions are equivalent.*

(i) *There exists some $X \in R^{n\times m}$ such that*

$$XAX = X, \quad R(X) = T, \quad N(X) = S. \tag{2.1}$$

(ii) $AT \oplus S = R^m$ *and* $N(A) \cap T = \{0\}$.

If these conditions are satisfied then X is unique.

PROOF. (i)$\Rightarrow$(ii) Since $XAX = X$, AX is an idempotent homomorphism from R^m to R^m. So, by [1, Lemma 5.6],

$$R(AX) \oplus N(AX) = R^m.$$

It is easy to see that $R(AX) = AR(X) = AT$ and $N(AX) = N(X) = S$. Hence

$$AT \oplus S = R^m.$$

Next we will show that $N(A) \cap T = \{0\}$. Let $x \in N(A) \cap T$. Then $Ax = 0$ and there exists a $y \in R^m$ such that $x = Xy$. So $x = Xy = XAXy = XAx = 0$. Therefore we have $N(A) \cap T = \{0\}$.

(ii)$\Rightarrow$(i) Obviously $A|_T$ is an epimorphism from T to AT. Since $N(A|_T) = N(A) \cap T = 0$, $A|_T$ is a monomorphism and so $A|_T$ has an inverse $(A|_T)^{-1} : AT \to T$. From $AT \oplus S = R^m$, we know that any $y \in R^m$, can be uniquely written as $y = y_1 + y_2$, where $y_1 \in AT$, $y_2 \in S$. So we define $X : R^m \to R^n$ by $Xy = (A|_T)^{-1} y_1$. Obviously X is a homomorphism and satisfies

$$\begin{cases} Xy = (A|_T)^{-1} y, & \text{if } y \in AT; \\ Xy = 0, & \text{if } y \in S. \end{cases} \tag{2.2}$$

Because R^m and R^n are both free modules, there exists a matrix of the homomorphism X with respect to the standard bases of R^m and R^n, and we write X for the matrix as well. It is easy to see that $R(X) = T$ and $N(X) = S$ by $AT \oplus S = R^m$.

For every $y \in R^m = AT \oplus S$ we have $y = y_1 + y_2$ where $y_1 \in AT$, $y_2 \in S$. Then

$$XAXy = XAXy_1 = XA(A|_T)^{-1} y_1 = Xy_1 = Xy.$$

This implies that $XAX = X$.

Now we prove the uniqueness. Suppose that X_1 and X_2 both satisfy (2.1). Then $X_1 A$ and AX_2 are idempotent matrices of order m and n respectively, and

$$X_1 A = P_{R(X_1 A), N(X_1 A)} = P_{R(X_1), N(X_1 A)} = P_{T, N(X_1 A)},$$
$$AX_2 = P_{R(AX_2), N(AX_2)} = P_{R(AX_2), N(X_2)} = P_{R(AX_2), S}.$$

By Lemma 2.1, we deduce that

$$X_2 = P_{T, N(X_1 A)} X_2 = (X_1 A) X_2 = X_1 (AX_2) = X_1 P_{R(AX_2), S} = X_1 \qquad \square$$

A matrix $X \in R^{n \times m}$ is called the *generalized inverse which is a $\{2\}$ inverse of a matrix A over R with prescribed image T and kernel S* if it satisfies the equivalent conditions in Theorem 2.2, and is denoted by $A^{(2)}_{T,S}$.

By (2.2), we have that

$$A^{(2)}_{T,S} = (A|_T)^{-1} P_{AT,S}. \tag{2.3}$$

From the proof of uniqueness in the theorem above and Lemma 2.1, we have the following corollary.

COROLLARY 2.3. *Let A and G be matrices over an associative ring R. If the generalized inverse $A^{(2)}_{T,S}$ exists, then*

(i) $A^{(2)}_{T,S} AG = G$ *if and only if* $R(G) \subset T$;
(ii) $GAA^{(2)}_{T,S} = G$ *if and only if* $N(G) \supset S$.

About the generalized inverse, we also have the following property.

THEOREM 2.4. *Let A be a matrix over R. If $A^{(2)}_{T,S}$ exists and there exists a matrix G over R satisfying $R(G) = T$ and $N(G) = S$ then there exists a matrix W over R such that*

$$GAGW = G, \tag{2.4}$$

$$A^{(2)}_{T,S}AGW = A^{(2)}_{T,S}. \tag{2.5}$$

PROOF. Suppose $A^{(2)}_{T,S}$ exists with $R(G) = T$ and $N(G) = S$ for a matrix G. Then $AR(G) \oplus N(G) = R^m$ and so there exists an epimorphism $R^m \to N(G) \to 0$. By [1, Theorem 8.1], $N(G)$ has a finite spanning set whose elements constitute a matrix, denoted by L. Thus $GL = 0$, and the columns of (AG, L) generate R^m, that is, there exists a matrix $(W^T, W_1^T)^T$ such that

$$AGW + LW_1 = I_m.$$

If we multiply the left hand side by G and $A^{(2)}_{T,S}$ respectively, then we obtain (2.4) and (2.5). □

The following theorem shows that for an arbitrary matrix A over an associative ring, $A^{\dagger}$, A_d and A_g, if they exist, are all the generalized inverse $A^{(2)}_{T,S}$.

THEOREM 2.5. (i) *Let A be an $m \times n$ matrix over R and let $*$ be an involution on the matrices over R. If $A^{\dagger}$ exists, then $A^{\dagger} = A^{(2)}_{R(A^*),N(A^*)}$.*

(ii) *Let A be an $n \times n$ matrix over R, and k =Ind(A). If A_d exists, then $A_d = A^{(2)}_{R(A^k),N(A^k)}$.*

(iii) *Let A be an $n \times n$ matrix over R. If A_g exists, then $A_g = A^{(2)}_{R(A),N(A)}$.*

PROOF. (i) Since $A^{\dagger} \in A\{1,2\}$ and $A^{\dagger *} \in A^*\{1,2\}$, we easily see that

$$R(A^{\dagger}) = R(A^{\dagger}A) = R((A^{\dagger}A)^*) = R(A^*A^{\dagger *}) = R(A^*),$$
$$N(A^{\dagger}) = N(AA^{\dagger}) = N((AA^{\dagger})^*) = N(A^{\dagger *}A^*) = N(A^*),$$

and $N(A) = N(A^{\dagger}A)$.

Since $AA^{\dagger}$ and $A^{\dagger}A$ are idempotent, we have

$$R^m = R(AA^{\dagger}) \oplus N(AA^{\dagger}) = AR(A^{\dagger}) \oplus N(AA^{\dagger}) = AR(A^*) \oplus N(A^*)$$

and

$$N(A) \cap R(A^*) = N(A^{\dagger}A) \cap R(A^{\dagger}A) = \{0\}$$

by [1, Lemma 5.6]. So, by Theorem 2.2, $A^{(2)}_{R(A^*),N(A^*)}$ exists and $A^{\dagger} = A^{(2)}_{R(A^*),N(A^*)}$.

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(ii) Firstly, we shall show that

$$R(A_d) = R(AA_d) = R(A^l) \quad \text{and} \quad N(A_d) = N(AA_d) = N(A^l)$$

for any positive integer $l \geq k$. Since

$$R(A_d) = R(AA_d^2) \subset R(AA_d) = R(A_dA) \subset R(A_d),$$

we have $R(A_d) = R(AA_d)$ and so

$$R(AA_d) = AR(A_d) = AR(AA_d) = A^2R(A_d).$$

It is easy to obtain inductively that $R(AA_d) = A^hR(A_d)$ for any positive integer h. This gives us that $R(A_d) = R(AA_d) = R(A^l)$ for any positive integer $l \geq k$. Also, since for any positive integer $l \geq k$,

$$N(A_d) \subset N(A^{l+1}A_d) = N(A^l) \subset N(A_d^lA^l) = N(A_dA) \subset N(A_d^2A) = N(A_d),$$

we get that $N(A_d) = N(AA_d) = N(A^l)$.

Since AA_d is idempotent, by [1, Lemma 5.6], we have

$$R^n = R(AA_d) \oplus N(AA_d) = AR(A^k) \oplus N(A^k) = R^n.$$

Since

$$N(A) \cap R(A^k) \subset N(A^k) \cap R(A^{k+1}) = \{0\},$$

$A^{(2)}_{R(A^k),N(A^k)}$ exists and $A_{R(A^k),N(A^k)} = A_d$ by Theorem 2.2.

(iii) Take $k = 1$ in (ii). □

3. The generalized inverse $A^{(1,2)}_{T,S}$

If the generalized inverse $A^{(2)}_{T,S}$ satisfies $AA^{(2)}_{T,S}A = A$ then it is called the *generalized inverse which is a {1,2} inverse of a matrix A over R with prescribed image T and kernel S*, and is denoted by $A^{(1,2)}_{T,S}$. (Its uniqueness is guaranteed by the following theorem.)

THEOREM 3.1. *Let A be an $m \times n$ matrix over an associative ring R with identity and $T \subset R^n$ and $S \subset R^m$. Then the following conditions are equivalent.*

(i) $AT \oplus S = R^m$, $R(A) \cap S = \{0\}$ *and* $N(A) \cap T = \{0\}$.

(ii) $R(A) \oplus S = R^m$, $N(A) \oplus T = R^n$.

(iii) *There exists some $X \in R^{n\times m}$ such that*

$$AXA = A, \quad XAX = X, \quad R(X) = T, \quad N(X) = S.$$

If these conditions are satisfied then X is unique.

PROOF. (ii) $\Longrightarrow$ (i) It is obvious that $R(A) \cap S = \{0\}$ and $N(A) \cap T = \{0\}$. To obtain $AT \oplus S = R^m$, it suffices to prove $AT = R(A)$.

Obviously, $AT \subset R(A)$. For any $x \in R(A)$, we have $x = Ay$, where $y \in R^n$. Since $N(A) \oplus T = R^n$, we can write $y = y_1 + y_2$, where $y_1 \in N(A)$, $y_2 \in T$. Thus,

$$x = Ay = Ay_1 + Ay_2 = Ay_2 \in AT,$$

and therefore $R(A) \subset AT$. Consequently, $AT = R(A)$.

(i) $\Longrightarrow$ (iii) By Theorem 2.2, from $AT \oplus S = R^m$ and $N(A) \cap T = \{0\}$, we know that $X = A^{(2)}_{T,S}$ exists and that $R(X) = T$ and $N(X) = S$. We shall show $AXA = A$.

Since $XAX = X$, we have $XAXA = XA$ and then $X(AXA - A) = 0$. So

$$R(AXA - A) \subset R(A) \cap N(X) = R(A) \cap S = \{0\}.$$

Hence $AXA = A$.

(iii) $\Longrightarrow$(ii) From (iii), we have $(AX)^2 = AX$, $(XA)^2 = XA$, and

$$\begin{aligned}
N(X) \subset \quad & N(AX) \subset \quad N(XAX) = N(X),\\
N(XA) \subset \quad & N(AXA) = \quad N(A) \subset N(XA),\\
R(XA) \subset \quad & R(X) = \quad R(XAX) \subset R(XA),\\
R(AX) \subset \quad & R(A) = \quad R(AXA) \subset R(AX).
\end{aligned}$$

So

$$\begin{aligned}
N(AX) = N(X) = S, &\quad N(XA) = N(A),\\
R(XA) = R(X) = T, &\quad R(AX) = R(A).
\end{aligned}$$

By [1, Lemma 5.6] and the four equations above, we reach (ii).

By Theorem 2.2, X is unique. □

The next result is concerning the equivalent conditions in Theorem 3.1.

THEOREM 3.2. *Let A be an $m \times n$ matrix over an associative ring R with identity and $T \subset R^n$ and $S \subset R^m$.*

(i) *If $N(A) + T = R^n$ then $AT = R(A)$.*

(ii) *If $AT \oplus S = R^m$ then*

$$AT = R(A) \quad \textit{if and only if} \quad R(A) \cap S = \{0\}.$$

PROOF. (i) From the proof of the theorem above (ii) implies (i).

(ii) Suppose that $R(A) \cap S = \{0\}$. Obviously, $AT \subset R(A)$. Now we will show the inclusion in reverse. For any $x \in R(A)$,

$$x = x_1 + x_2 \in R^m = AT \oplus S,$$

where $x_1 \in AT, x_2 \in S$. By $AT \subset R(A), x_1 \in R(A)$. So

$$x_2 = x - x_1 \in R(A) \cap S = \{0\}.$$

Therefore, $x_2 = 0$ and then $x = x_1 \in AT$. Hence $R(A) \subset AT$.

Conversely, suppose that $AT = R(A)$. Since $AT \oplus S = R^m$ and $AT = R(A)$, we have $R(A) \cap S = AT \cap S = \{0\}$. □

We denote the maximal order of a nonvanishing minor of A over a commutative ring R by $\rho(A)$. This is called the *determinantal rank* of A. Obviously $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$ (see [9, Theorem 2.3]). When R is the complex number field, $\rho(A) = \operatorname{rank}(A)$.

THEOREM 3.3. *Let A be an $m \times n$ matrix over an integral domain R and $T \subset R^n$ and $S \subset R^m$ be free submodules. If $AT \oplus S = R^m$ then the following conditions are equivalent.*

(i) $N(A) \cap T = \{0\}$ *and* $R(A) \cap S = \{0\}$,

(ii) $\dim(T) = \rho(A)$ *and* $\dim(S) = m - \dim(T)$.

PROOF. Suppose that (i) holds and let the columns of U be a basis of T. From the proof of [13, Theorem 2], we have $\dim(T) = \dim(AT) = \rho(AU) \leq \rho(A)$ and $\dim(S) = m - \dim(T)$. By Theorem 3.2, $AT = R(A)$. Thus there exists a matrix X over R such that $A = AUX$. Thus $\rho(A) \leq \rho(AU) = \dim(AT)$. Therefore $\rho(A) = \dim AT = \dim(T)$.

Conversely, suppose that (ii) holds. We have that $\dim(T) = \dim(AT)$ from the proof of [13, Theorem 2]. Thus $\rho(A) = \dim(T) = \dim(AT)$. By [12, Lemma 1], the maximal number of linearly independent columns of A is $\dim(AT)$. Since $AT \subset R(A)$, $R(A) + S = R^m$. Over the quotient field F of R, $AT = R(A)$ because $\rho(A) = \dim(AT)$, and $R(A) \oplus S = R^m$. Therefore x and y are linear independent over F for any $x \in R(A)$, $y \in S$.

On the other hand, over an integral domain R, suppose that $0 \neq z \in R(A) \cap S$. Then there exist $r_i \in R$, $i = 1, \ldots, s$, such that

$$z = \sum_{i=1}^{s} \beta_i r_i, \tag{3.1}$$

where $\{\beta_1, \beta_2, \ldots, \beta_s\}$ is a basis of S and $s = \dim(S)$. But Equation (3.1) is true over F. This is in contradiction to the above reasoning. Hence $R(A) \cap S = \{0\}$.

The remainder of the proof is obtained from [13, Theorem 2]. □

REMARK 1. A module over the field of complex numbers is a vector space. So when R is the field of complex numbers, the above theorem ensures that Theorem 3.1 extends [2, Corollary 2.10].

4. Explicit expressions for $A^{(1,2)}_{T,S}$

We now consider some explicit expressions for $A^{(1,2)}_{T,S}$, which reduce to the group inverse or $\{1\}$ inverses. Firstly we shall prove the following lemma. In the proof, we use the following fact.

PROPOSITION 4.1. *If e is idempotent in a ring R with identity 1 and $x, y \in eRe$ then $xy = e$ if and only if $(x + 1 - e)(y + 1 - e) = 1$.*

LEMMA 4.2. *Let A be an $m \times n$ von Neumann regular matrix over R and G an $n \times m$ matrix over R. Then $U = AGAA^{(1)} + I_m - AA^{(1)}$ is invertible if and only if $V = A^{(1)}AGA + I_n - A^{(1)}A$ is invertible.*

PROOF. If U is invertible then there exists an X such that $UX = XU = I_m$. That is,

$$\left(AGAA^{(1)} + I_m - AA^{(1)}\right) X = I_m \quad \text{and} \quad X\left(AGAA^{(1)} + I_m - AA^{(1)}\right) = I_m.$$

Multiplying on the left by $A^{(1)}AA^{(1)}$ and the right by A and, since $A = AA^{(1)}A$, we have

$$\left(A^{(1)}AGA\right)\left(A^{(1)}AA^{(1)}XA\right) = A^{(1)}A \quad \text{and} \quad \left(A^{(1)}AA^{(1)}XA\right)\left(A^{(1)}AGA\right) = A^{(1)}A.$$

Since $A^{(1)}AGA = A^{(1)}A(GA)A^{(1)}A$ and $A^{(1)}AA^{(1)}XA = A^{(1)}A(A^{(1)}XA)A^{(1)}A$, we know that $A^{(1)}AGA$ has the inverse matrix $A^{(1)}AA^{(1)}XA$ in $A^{(1)}AM_n(R)A^{(1)}A$. Thus $V = A^{(1)}AGA + I_n - A^{(1)}A$ has the inverse matrix

$$A^{(1)}A\left(A^{(1)}AA^{(1)}XA\right)A^{(1)}A + I_n - A^{(1)}A \quad \text{in } M_n(R).$$

The proof of the converse is analogous. □

Next we shall show the main result of this section. The following theorem not only shows some explicit expressions for $A^{(1,2)}_{T,S}$ which reduce to the group inverse or $\{1\}$ inverses, but also gives some equivalent conditions for the existence of $A^{(1,2)}_{T,S}$.

THEOREM 4.3. *Let A be an $m \times n$ matrix over R and G an $n \times m$ matrix over R. Then the following conditions are equivalent.*

(i) *A is von Neumann regular, $U = AGAA^{(1)} + I_m - AA^{(1)}$ is invertible and $N(A) \cap R(G) = \{0\}$.*

(ii) *A is von Neumann regular, $V = A^{(1)}AGA + I_n - A^{(1)}A$ is invertible and $N(A) \cap R(G) = \{0\}$.*

(iii) *$A^{(1,2)}_{R(G),N(G)}$ exists.*

When these conditions are satisfied we have

$$A^{(1,2)}_{R(G),N(G)} = G(AG)_g = (GA)_g G \tag{4.1}$$
$$= G(GAG)^{(1)}G \tag{4.2}$$
$$= G(AG)^{(1)}A(GA)^{(1)}G \tag{4.3}$$
$$= GU^{-2}AG = GU^{-1}AV^{-1}G = GAV^{-2}G. \tag{4.4}$$

PROOF. (i) and (ii) are equivalent by Lemma 4.2.

To show that (ii) implies (iii), set $B = AV^{-2}G$. Using $UA = AGA = AV$, we have $B = (AG)_g$ because

$$\begin{aligned} B(AG) &= AV^{-2}GAG = U^{-2}AGAG = U^{-1}AG = AV^{-1}G = AGAV^{-2}G \\ &= (AG)B, \\ B(AG)B &= U^{-1}AG(AV^{-2}G) = AV^{-2}G = B, \\ (AG)B(AG) &= (AG)AV^{-1}G = AG. \end{aligned}$$

Analogously, we deduce that $(GA)_g$ exists and $(GA)_g = GU^{-2}A$. Let $X = G(AG)_g$. It is obvious that

$$XAX = X. \tag{4.5}$$

Since

$$AG = (AG)^2(AG)_g = AGAX,$$

we have $A(G - GAX) = 0$ and then

$$R(G - GAX) = R(G(I - AX)) \subset N(A) \cap R(G) = \{0\}.$$

Therefore

$$G = GAX \tag{4.6}$$
$$= GA(G(AG)_g) = G(AG)_g AG$$
$$= XAG. \tag{4.7}$$

Using (4.6) and (4.7), we have

$$R(X) = R(G) \quad \text{and} \quad N(X) = N(G). \tag{4.8}$$

Since $AV = AGA$, we get

$$A = AGAV^{-1} = AG(AG)_g AGAV^{-1} = AXA.$$

Using the equation above, together with (4.5) and (4.8), we deduce that $A^{(1,2)}_{R(G),N(G)}$ exists and $A^{(1,2)}_{R(G),N(G)} = X = G(AG)_g$ by Theorem 3.1.

To show that (iii) implies (i), we use Theorem 2.4 to obtain

$$\begin{aligned}(AGAA^{(1)})(AGW^2AA^{(1)}) &= AGAGW^2AA^{(1)} = AGWAA^{(1)}\\ &= AA^{(1,2)}_{R(G),N(G)}AGWAA^{(1)} = AA^{(1,2)}_{R(G),N(G)}AA^{(1)}\\ &= AA^{(1)}. \qquad (4.9)\end{aligned}$$

Therefore,

$$\left(AGAA^{(1)}\right)\left(AGW^2AA^{(1)}\right)\left(AGAA^{(1)}\right) = AA^{(1)}\left(AGAA^{(1)}\right) = AGAA^{(1)}$$

and then

$$AG\Big(\left(AGW^2AA^{(1)}\right)\left(AGAA^{(1)}\right) - AA^{(1)}\Big) = 0.$$

By Theorem 3.1, $R(A) \cap N(G) = \{0\}$ and $N(A) \cap R(G) = \{0\}$ and so

$$R\bigg(G\Big(\left(AGW^2AA^{(1)}\right)\left(AGAA^{(1)}\right) - AA^{(1)}\Big)\bigg) \subset R(G) \cap N(A) = \{0\}.$$

Thus

$$G\Big(\left(AGW^2AA^{(1)}\right)\left(AGAA^{(1)}\right) - AA^{(1)}\Big) = 0.$$

From this, we have

$$R\Big(\left(AGW^2AA^{(1)}\right)\left(AGAA^{(1)}\right) - AA^{(1)}\Big) \subset R(A) \cap N(G) = \{0\},$$

and then

$$\left(AGW^2AA^{(1)}\right)\left(AGAA^{(1)}\right) = AA^{(1)}. \qquad (4.10)$$

By (4.9) and (4.10), $AGAA^{(1)}$ is invertible in $AA^{(1)}M_m(R)AA^{(1)}$ and so is U in $M_m(R)$. Also, obviously, A is von Neumann regular.

Now we shall prove that (4.1) $\sim$ (4.3). Since

$$G(AG)_g = G(AV^{-2}G) = GU^{-1}AV^{-1}G = (GU^{-2}A)G = (GA)_gG,$$

we have $A^{(1,2)}_{R(G),N(G)} = (GA)_gG$ and (4.4).

Next we will prove (4.2). Since

$$GAG = GAG\left((AG)_g\right)^2 AGAG,$$

GAG is von Neumann regular and then

$$\begin{aligned}AG = U^{-1}UAG = U^{-1}AGAG &= (U^{-1}A)GAG(GAG)^{(1)}GAG\\ &= AG(GAG)^{(1)}GAG.\end{aligned}$$

Therefore

$$A\left(G - G(GAG)^{(1)}GAG\right) = 0.$$

Thus

$$R\left(G - G(GAG)^{(1)}GAG\right) \subset N(A) \cap R(G) = \{0\}.$$

So we obtain

$$G = G(GAG)^{(1)}GAG \tag{4.11}$$

Since $A^{(1,2)}_{R(G),N(G)}$ exists, using (2.4) and (4.11), it follows that

$$\begin{aligned} G &= GAGW = GAG(GAG)^{(1)}GAGW \\ &= GAG(GAG)^{(1)}G. \end{aligned} \tag{4.12}$$

Let $Z = G(GAG)^{(1)}G$. Using (4.11) and (4.12), it easily follows that $ZAZ = Z$, $AZA = A$, $R(Z) = R(G)$ and $N(Z) = N(G)$. By Theorem 3.1 we have that $A^{(1,2)}_{R(G),N(G)} = Z = G(GAG)^{(1)}G$.

Finally, we will verify (4.3). It is obvious that AG and GA are von Neumann regular. By Proposition 4.1 and the invertibility of V there exists a matrix $P \in A^{(1)}AM_n(R)A^{(1)}A$ such that $P(A^{(1)}AGA) = A^{(1)}A$. Thus

$$A = A\left(PA^{(1)}AGA\right) = APA^{(1)}A\left(GA(GA)^{(1)}GA\right) = A(GA)^{(1)}GA. \tag{4.13}$$

Using (4.13), we deduce that $(AG)^{(1)}A(GA)^{(1)}$ is a $\{1\}$ inverse of GAG. Therefore, using (4.2), we obtain (4.3). □

REMARK 2. By (4.4), we can compute $A^{(1,2)}_{R(G),N(G)}$ using U or V.

REMARK 3. If $G = A$ where A is such that $V = A^{(1)}A^2 + I_n - A^{(1)}A$ is invertible, then $N(A) \cap R(A) = \{0\}$. Indeed, let $x \in N(A) \cap R(A)$. Then there exists a $y \in R^n$ such that $x = Ay$ and so $A^2y = 0$. Since V is invertible, there exists a matrix P such that $PV = I_n$. Thus $PA^{(1)}A^3 = A^{(1)}A$ and then

$$0 = PA^{(1)}A^3y = A^{(1)}Ay.$$

Hence $Ay = AA^{(1)}Ay = 0$. Consequently, $x = Ay = 0$.

Similarly, if we take $G = A^*$, where $*$ is an involution on the matrices over R such that $U = AA^*AA^{(1)} + I_m - AA^{(1)}$ is invertible, then $N(A) \cap R(A^*) = \{0\}$. Indeed, let $x \in N(A) \cap R(A^*)$. Then there exists a $y \in R^m$ such that $x = A^*y$ and so $AA^*y = 0$. Since U is invertible, there exists a matrix Q such that $AA^*AA^{(1)}Q = AA^{(1)}$ and thus

$$0 = Q^*(A^{(1)})^*A^*AA^*y = (A^{(1)})^*A^*y.$$

So $x = A^*y = A^*(A^{(1)})^*A^*y = 0$.

When G takes the value A (respectively A^*) in the theorem above, we find that $A^{(1,2)}_{R(G),N(G)}$ is A_g (respectively $A^\dagger$).

THEOREM 4.4. *Let A be an $m \times n$ matrix over R. Then*

(i) *$A^{(1,2)}_{R(A),N(A)}$ exists if and only if A_g exists. Moreover, $A^{(1,2)}_{R(A),N(A)} = A_g$.*

(ii) *If $*$ is an involution on the matrices over R then $A^{(1,2)}_{R(A^*),N(A^*)}$ exists if and only if $A^\dagger$ exists. Moreover, $A^{(1,2)}_{R(A^*),N(A^*)} = A^\dagger$.*

PROOF. To show the existence of $A^{(1,2)}_{R(A),N(A)}$ implies existence of A_g in (i), take $G = A$ in (4.1). Then $A^{(1,2)}_{R(A),N(A)} = A(A^2)_g = (A^2)_g A$ and then $AA^{(1,2)}_{R(A),N(A)} = A^{(1,2)}_{R(A),N(A)}A$. Hence $A^{(1,2)}_{R(A^*),N(A^*)}$ is the group inverse of A.

To show that existence of $A^{(1,2)}_{R(A^*),N(A^*)}$ implies existence of $A^\dagger$ in (ii), take $G = A^*$ in (4.1). Then $A^{(1,2)}_{R(A^*),N(A^*)} = A^*(AA^*)_g = (A^*A)_g A^*$ and then

$$\left(AA^{(1,2)}_{R(A^*),N(A^*)}\right)^* = AA^{(1,2)}_{R(A^*),N(A^*)} \quad \text{and} \quad \left(A^{(1,2)}_{R(A^*),N(A^*)}A\right)^* = A^{(1,2)}_{R(A^*),N(A^*)}A.$$

Hence $A^{(1,2)}_{R(A^*),N(A^*)}$ is the Moore-Penrose inverse of A.

The converses follow from Theorem 2.5. □

By Theorems 4.3 and 4.4 and Remark 3, we can obtain the following two corollaries, in which the first is equivalent to [8, Corollary 2] and the second is almost the same as [6, Theorem 1].

COROLLARY 4.5. *Let $A \in R^{n\times n}$. The following conditions are equivalent.*

(i) *A is von Neumann regular and $U = A^3A^{(1)} + I_n - AA^{(1)}$ is invertible.*

(ii) *A is von Neumann regular and $V = A^{(1)}A^3 + I_n - A^{(1)}A$ is invertible.*

(iii) *A_g exists.*

Moreover,

$$A_g = A(A^2)_g = (A^2)_g A \tag{4.14}$$
$$= A(A^3)^{(1)}A \tag{4.15}$$
$$= A(A^2)^{(1)}A(A^2)^{(1)}A. \tag{4.16}$$
$$= AU^{-2}A^2 = AU^{-1}AV^{-1}A = A^2V^{-2}A. \tag{4.17}$$

REMARK 4. The above corollary is unlike [8, Corollary 2], but they are equivalent. This is because V is invertible if and only if $T = A^{(1)}A^2 + I_n - A^{(1)}A$ is invertible. Indeed, if V is invertible, then there exists a matrix $P \in M_n(R)$ such that $PV = VP = I_n$. From this and $V = T^2$, we get $(PT)T = T(TP) = I_n$. Hence T is invertible in $M_n(R)$. The converse is obvious from $V = T^2$.

COROLLARY 4.6. *Let A be an $m \times n$ matrix over R and let $*$ be an involution on the matrices over R. The following conditions are equivalent.*

(i) *A is von Neumann regular and $U = AA^*AA^{(1)} + I_n - AA^{(1)}$ is invertible.*
(ii) *A is von Neumann regular and $V = A^{(1)}AA^*A + I_n - A^{(1)}A$ is invertible.*
(iii) *$A^{\dagger}$ exists.*

Moreover,

$$\begin{aligned} A^{\dagger} &= A^*(AA^*)_g = (A^*A)_g A^* = A^*(A^*AA^*)^{(1)}A^* = A^*(AA^*)^{(1)}A(A^*A)^{(1)}A^* \\ &= A^*U^{-2}AA^* = A^*U^{-1}AV^{-1}A^* = A^*AV^{-2}A^*. \end{aligned}$$

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