4
The action principle

The variational principle is central to covariant field theory. It displays symmetries, field equations and continuity conditions on an equal footing. It can be used as the starting point for every field theoretical analysis. In older books, the method is referred to as Hamilton's principle. In field theory it is referred to more colloquially as the *action principle*. Put plainly, it is a method of *generating functionals*; it compresses all of the kinematics and dynamics of a physical theory into a single integral expression S called the action.

The advantage of the action principle is that it guarantees a well formulated dynamical problem, assuming only the existence of a set of parameters on which the dynamical variables depends. Any theory formulated as, and derived from an action principle, automatically leads to a complete dynamical system of equations with dynamical variables which play the roles of positions and momenta, by analogy with Newtonian mechanics. To formulate a new model in physics, all one does is formulate invariant physical properties in the form of an action, and the principle elucidates the resulting kinematical and dynamical structure in detail.

4.1 The action in Newtonian particle mechanics

Consider a system consisting of a particle with position $q(t)$ and momentum $p(t)$. The kinetic energy of the particle is

$$T = \frac{1}{2} m \dot{q}^2, \tag{4.1}$$

and the potential energy is simply denoted $V(q)$. The 'dot' over the q denotes the time derivative, or

$$\dot{q} = \frac{\mathrm{d}q}{\mathrm{d}t}. \tag{4.2}$$

https://doi.org/10.1017/9781009289887.006 Published online by Cambridge University Press

Classical mechanics holds that the equation of motion for a classical particle is Newton's law:

$$F = m\ddot{q} = -\frac{\mathrm{d}V}{\mathrm{d}q}, \tag{4.3}$$

but it is interesting to be able to derive this equation from a general principle. If many equations of motion could be derived from a common principle, it would represent a significant compression of information in physics. This is accomplished by introducing a generating function L called the Lagrangian. For a conservative system, the Lagrangian is defined by

$$L = T - V, \tag{4.4}$$

which, in this case, becomes

$$L = \frac{1}{2}m\dot{q}^2 - V(q). \tag{4.5}$$

This form, kinetic energy minus potential energy, is a coincidence. It does not apply to all Lagrangians. In relativistic theories, for instance, it is not even clear what one should refer to as the kinetic and potential energies. The Lagrangian is a generating function; it has no unique physical interpretation.

The Lagrangian is formally a function of q and $\dot{q}$. The general rule for obtaining the equations of motion is the well known Euler–Lagrange equations. They are

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0. \tag{4.6}$$

If the physical system is changed, one only has to change the Lagrangian: the general rule will remain true. Evaluating, in this case,

$$\begin{aligned} \frac{\partial L}{\partial q} &= -\frac{\mathrm{d}V}{\mathrm{d}q} \\ \frac{\partial L}{\partial \dot{q}} &= m\dot{q}, \end{aligned} \tag{4.7}$$

one obtains the field equations (4.3), as promised.

Is this approach better than a method in which one simply writes down the field equations? Rather than changing the field equations for each case, one instead changes the Lagrangian. Moreover, eqn. (4.6) was pulled out of a hat, so really there are two unknowns now instead of one! To see why this approach has more to offer, we introduce the *action*.

4.1.1 Variational principle

The fact that one can derive known equations of motion from an arbitrary formula involving a constructed function L is not at all surprising – there are hundreds of possibilities; indeed, the motivation for such an arbitrary procedure is not clear. The fact that one can obtain them from a function involving only the potential and kinetic energies of the system, for any conservative system, is interesting. What is remarkable is the fact that one can derive the Euler–Lagrange equations (i.e. the equations of motion), together with many other important physical properties for any system, from one simple principle: the action principle.

Consider the action S from the Lagrangian by

$$S_{12} = \int_{t_1}^{t_2} L(q, \dot{q}) \mathrm{d}t. \tag{4.8}$$

The action has (naturally) dimensions of *action* or 'energy $\times$ time', and is thought of as being a property of the path $q(t)$ of our particle between the fixed points $q(t_1)$ and $q(t_2)$. The action has no physical significance in itself. Its significance lies instead in the fact that it is a *generating functional* for the dynamical properties of a physical system.

When formulating physics using the action, it is not necessary to consider the fact that q and $\dot{q}$ are independent variables: that is taken care of automatically. In fact, the beauty of the action principle is that all of the useful information about a physical system falls out of the action principle more or less automatically.

To extract information from S, one varies it with respect to its dynamical variables, i.e. one examines how the integral changes when the key variables in the problem are changed. The details one can change are t_1 and t_2, the end-points of integration, and $q(t)$, the path or world-line of the particle between those two points (see figure 4.1). Note however that $Q(t)$ is the path the particle would take from A to B, and that is not arbitrary: it is determined by, or determines, physical law, depending on one's view. So, in order to make the variational principle a useful device, we have to be able to select the correct path by some simple criterion.

Remarkably, the criterion is the same in every case: one chooses the path which minimizes (or more correctly: makes stationary) the action; i.e. we look for paths $q(t)$ satisfying

$$\frac{\delta S}{\delta q(t)} = 0. \tag{4.9}$$

These are the *stable* or *stationary* solutions to the variational problem. This tells us that most physical laws can be thought of as regions of stability in a space of all solutions. The action behaves like a potential, or stability measure, in this space.

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It is an attractive human idea (Occam's razor) that physical systems do the 'least action' possible; however, eqn. (4.9) is clearly no ordinary differentiation. First of all, S is a scalar number – it is integrated over a dummy variable t, so t is certainly not a variable on which S depends. To distinguish this from ordinary differentiation of a function with respect to a variable, it is referred to as *functional differentiation* because it is differentiation with respect to a function.

The functional variation of S with respect to $q(t)$ is defined by

$$\delta S = S[q + \delta q] - S[q], \tag{4.10}$$

where $\delta q(t)$ is an infinitesimal change in the form of the function q at time t. Specifically, for the single-particle example,

$$\delta S = \int \mathrm{d}t \left\{ \frac{1}{2} m(\dot{q} + \delta\dot{q})^2 - V(q + \delta q) \right\} - \int \mathrm{d}t \left\{ \frac{1}{2} m\dot{q}^2 - V(q) \right\}. \tag{4.11}$$

Now, since δq is infinitesimal, we keep only the first-order contributions, so on expanding the potential to first order as a Taylor series about $q(t)$,

$$V(q + \delta q) = V(q) + \frac{\mathrm{d}V}{\mathrm{d}q}\delta q + \cdots, \tag{4.12}$$

one obtains the first-order variation of S,

$$\delta S = \int \mathrm{d}t \left\{ m\dot{q}(\partial_t \delta q) - \frac{\mathrm{d}V}{\mathrm{d}q}\delta q \right\}. \tag{4.13}$$

A 'dot' has been exchanged for an explicit time derivative to emphasize the time derivative of δq. Looking at this expression, one notices that, if the time derivative did not act on δq, we would be able to take out an overall factor of δq, and we would be almost ready to move δq to the left hand side to make something like a derivative. Since we are now operating under the integral sign, it is possible to integrate by parts, using the property:

$$\int \mathrm{d}t\, A(\partial_t B) = \Big[AB \Big]_{t_1}^{t_2} - \int \mathrm{d}t\, (\partial_t A) B, \tag{4.14}$$

so that the time derivative can be removed from δq, giving:

$$\delta S = \int \mathrm{d}t \left\{ -m\ddot{q}(t) - \frac{\mathrm{d}V}{\mathrm{d}q(t)} \right\} \delta q(t) + \Big[m\dot{q} \cdot \delta q(t) \Big]_{t_1}^{t_2}. \tag{4.15}$$

The stationary action criterion tells us that $\delta S = 0$. Assuming that $q(t)$ is not always zero, one obtains a restriction on the allowed values of $q(t)$. This result must now be interpreted.

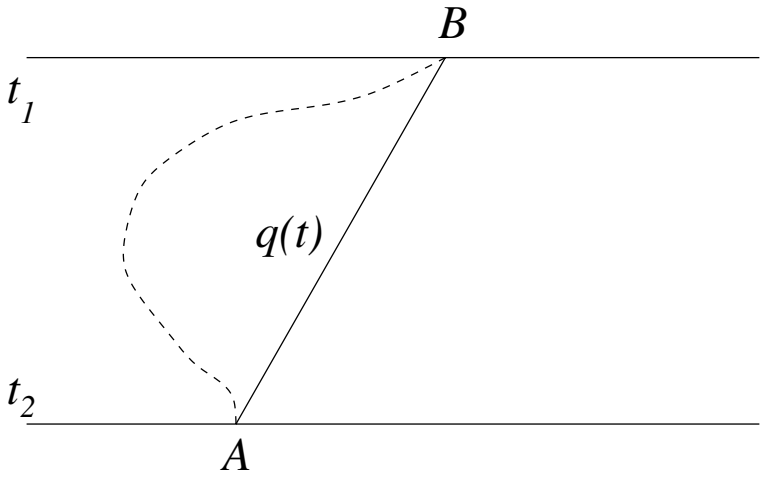

Fig. 4.1. The variational formula selects the path from A to B with a *stationary* value of the action. Stationary or *minimum* means that the solution is stable on the surface of all field solutions. Unless one adds additional perturbations in the action, it will describe the 'steady state' behaviour of the system.

4.1.2 δS: equation of motion

The first thing to notice about eqn. (4.15) is that it is composed of two logically separate parts. The first term is an integral over all times which interpolate between t_1 and t_2, and the second is a term which lives only at the end-points. Now, suppose we ask the question: what path $q(t)$ is picked out by the action principle, if we consider all the possible variations of paths $q(t) + \delta q(t)$, given that the two end-points are always fixed, i.e. $\delta q(t_1) = 0$ and $\delta q(t_2) = 0$?

The requirement of fixed end-points now makes the second term in eqn. (4.15) vanish, so that $\delta S = 0$ implies that the contents of the remaining curly braces must vanish. This gives precisely the equation of motion

$$m\ddot{q} = -\frac{\mathrm{d}V}{\mathrm{d}q}. \tag{4.16}$$

The action principle delivers the required formula as promised. This arises from an equation of *constraint* on the path $q(t)$ – a constraint which forces the path to take a value satisfying the equation of motion. This notion of constraint recurs later, in more advanced uses of the action principle.

4.1.3 The Euler–Lagrange equations

The Euler–Lagrange equations of motion are trivially derived from the action principle for an arbitrary Lagrangian which is a function of q and $\dot{q}$. The action

one requires is simply

$$S = \int \mathrm{d}t\, L(q(t), \dot{q}(t)), \tag{4.17}$$

and its variation can be written, using the functional chain-rule,

$$\delta S = \int \mathrm{d}t \left\{ \frac{\delta L}{\delta q}\delta q + \frac{\delta L}{\delta(\partial_t q)}\delta(\partial_t q) \right\} = 0. \tag{4.18}$$

The variation of the path commutes with the time derivative (trivially), since

$$\delta(\partial_t q) = \partial_t q(\tau + \delta\tau) - \partial_t q(\tau) = \partial_t(\delta q). \tag{4.19}$$

Thus, one may re-write eqn. (4.18) as

$$\delta S = \int \mathrm{d}t \left\{ \frac{\delta L}{\delta q}\delta q + \frac{\delta L}{\delta(\partial_t q)}\partial_t(\delta q) \right\} = 0. \tag{4.20}$$

Integrating the second term by parts, one obtains

$$\delta S = \int \mathrm{d}t \left\{ \frac{\delta L}{\delta q}\delta q - \partial_t \left(\frac{\delta L}{\delta(\partial_t q)} \right)(\delta q) \right\} + \int \mathrm{d}\sigma \left[\frac{\delta L}{\delta(\partial_t q)}\delta q \right] = 0. \tag{4.21}$$

The second term vanishes independently (since its variation is zero at the fixed end-points), and thus one obtains the Euler–Lagrange equations (4.6).

4.1.4 δS: continuity

Before leaving this simple world of classical particles, there is one more thing to remark about eqn. (4.21). Consider the second term; when one asks the question: what is the condition on $q(t)$ for the classical trajectories with stationary action and fixed end-points? – this term drops out. It vanishes by assumption. It contains useful information however. If we consider the example of a single particle, the surface term has the form

$$m\dot{q} \cdot \delta q = p\delta q. \tag{4.22}$$

This term represents the momentum of the particle. For a general Lagrangian, one can use this fact to define a 'generalized momentum'. From eqn. (4.21)

$$p = \frac{\delta L}{\delta(\partial_t q)} \equiv \Pi. \tag{4.23}$$

Traditionally, this quantity is called the canonical momentum, or conjugate momentum, and is denoted generically as Π.

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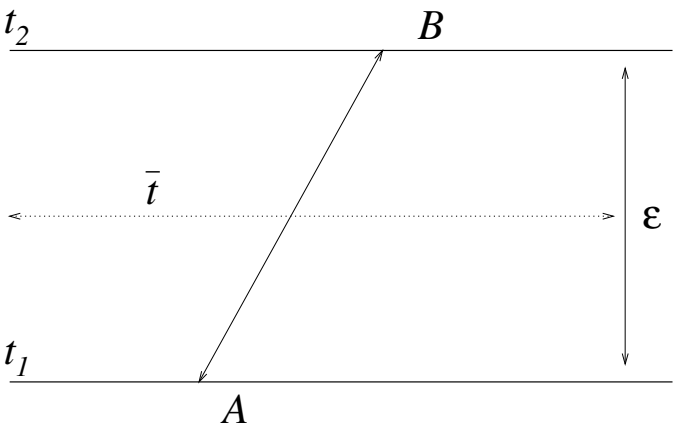

Fig. 4.2. The continuity of paths obeying the equations of motion, over an infinitesimal interval is assured by the null variation of the action over that interval.

Suppose one asks a different question of the variation. Consider only an infinitesimal time period $t_2 - t_1 = \epsilon$, where $\epsilon \to 0$. What happens between the two limits of integration in eqn. (4.21) is now less important. In fact, it becomes decreasingly important as $\epsilon \to 0$, since

$$\delta S_{12} = [\, p\delta q]_{t_1}^{t_2} + \mathrm{O}(\epsilon). \tag{4.24}$$

What infinitesimal property of the action ensures that $\delta S = 0$ for all intermediate points between the limits t_1 and t_2? To find out, we relax the condition that the end-points of variation should vanish. Then, over any infinitesimal interval ϵ, the change in $\delta q(t)$ can itself only be infinitesimal, unless $q(t)$ is singular, but it need not vanish. However, as $\epsilon \to 0$, the change in this quantity must also vanish as long as $q(t)$ is a smooth field, so one must take $\Delta(\delta q) = 0$.[1] This means that

$$\Delta p \equiv p(t_2) - p(t_1) = 0; \tag{4.25}$$

i.e. the change in momentum across any infinitesimal surface is zero, or momentum is conserved at any point. This is a continuity condition on $q(t)$. To see this, ask what would happen if the potential $V(q)$ contained a singular term at the surface:

$$V(q, t) = \delta(t - \bar{t})\Delta V + V(q), \tag{4.26}$$

[1] Note that we are assuming that the field is a continuous function, but the momentum need not be strictly continuous if there are impulsive forces (influences) on the field. This is fully consistent with our new philosophy of treating the 'field' q as a fundamental variable, and p as a derived quantity.

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where $\frac{1}{2}(t_1 + t_2)$ is the mid-point of the infinitesimal interval. Here, the delta function integrates out immediately, leaving an explicit surface contribution from the potential, in addition to the term from the integration by parts:

$$\delta S_{12} = \frac{\mathrm{d}\Delta V}{\mathrm{d}q}\delta q + [\, p\delta q]_{t_1}^{t_2} + \mathrm{O}(\epsilon) = 0, \tag{4.27}$$

Provided ΔV is finite, using the same argument as before, one obtains,

$$\Delta p = -\frac{\mathrm{d}\Delta V}{\mathrm{d}q}, \tag{4.28}$$

i.e. the change in momentum across any surface is a direct consequence of the impulsive force $\mathrm{d}\Delta V/\mathrm{d}q$ at that surface.

We thus have another facet of the action: it evaluates relationships between dynamical variables which satisfy the constraints of stable behaviour. This property of the action is very useful: it generates standard continuity and boundary conditions in field theory, and is the backbone of the canonical formulation of both classical and quantum mechanics. For instance, in the case of the electromagnetic field, we can generate all of the 'electromagnetic boundary conditions' at interfaces using this technique (see section 21.2.2). This issue occurs more generally in connection with the energy–momentum tensor, in chapter 11, where we shall re-visit and formalize this argument.

4.1.5 Relativistic point particles

The relativistically invariant form of the action for a single point particle is

$$S = \int \mathrm{d}t\sqrt{-g_{00}}\left\{-\frac{1}{2}m\frac{\mathrm{d}x^i(t)}{\mathrm{d}t}g_{ij}\frac{\mathrm{d}x^j(t)}{\mathrm{d}t} + V\right\}. \tag{4.29}$$

The particle positions trace out world-lines $q(\tau) = \mathbf{x}(\tau)$. If we re-express this in terms of the proper time τ of the particle, where

$$\begin{aligned} \tau &= t\gamma^{-1} \\ \gamma &= 1/\sqrt{(1-\beta^2)} \\ \beta^2 &= \frac{\mathbf{v}^2}{c^2} = \frac{1}{c^2}\left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right)^2, \end{aligned} \tag{4.30}$$

then the action may now be written in the frame of the particle,

$$\begin{aligned} \mathrm{d}t &\to \gamma\,\mathrm{d}\tau \\ \sqrt{g} &\to \gamma\sqrt{g}, \end{aligned} \tag{4.31}$$

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giving

$$S = \int \mathrm{d}\tau \sqrt{g_{00}} \left\{ -\frac{1}{2} m \left(\frac{\mathrm{d}\mathbf{x}(\tau)}{\mathrm{d}\tau} \right)^2 + V\gamma^{-2} \right\}. \tag{4.32}$$

The field equations are therefore

$$\frac{\delta S}{\delta \mathbf{x}} = m\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}\tau^2} + \frac{\partial V'}{\partial \mathbf{x}} = 0, \tag{4.33}$$

i.e.

$$\mathbf{F} = m\mathbf{a}, \tag{4.34}$$

where

$$\begin{aligned} \mathbf{F} &= -\nabla V' \\ \mathbf{a} &= \frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}\tau^2}. \end{aligned} \tag{4.35}$$

The conjugate momentum from the continuity condition is

$$\mathbf{p} = m\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau}, \tag{4.36}$$

which is simply the relativistic momentum vector $\mathbf{p}$. See section 11.3.1 for the energy of the classical particle system.

In the above derivation, we have treated the metric tensor as a constant, but in curved spacetime $g_{\mu\nu}$ depends on the coordinates. In that case, the variation of the action leads to the field equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \right) - \frac{1}{2}(\partial_\mu g_{\rho\nu})\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} = 0. \tag{4.37}$$

The equation of a free particle on a curved spacetime is called the geodesic equation. After some manipulation, it may be written

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} = 0. \tag{4.38}$$

Interestingly, this equation can be obtained from the absurdly simple variational principle:

$$\delta \int \mathrm{d}s = 0, \tag{4.39}$$

where $\mathrm{d}s$ is the line element, described in section 3.2.5. See also section 25.4.

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4.2 Frictional forces and dissipation

In many branches of physics, phenomenological equations are used for the dissipation of energy. Friction and ohmic resistance are two common examples. Empirical frictional forces cannot be represented by a microscopic action principle, since they arise physically only through time-dependent boundary conditions on the system. No fundamental dynamical system is dissipative at the microscopic level; however, fluctuations in dynamical variables, averaged over time, can lead to a re-distribution of energy within a system, and this is what leads to dissipation of energy from one part of a system to another. More advanced statistical notions are required to discuss dissipation fully, but a few simple observations can be made at the level of the action.

Consider the example of the frictional force represented by Langevin's equation:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t} + \alpha\dot{x} = F(t). \tag{4.40}$$

Initially it appears as though one could write the action in the following way:

$$S = \int \mathrm{d}t \left\{ \frac{1}{2}m\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \frac{1}{2}\alpha x\frac{\mathrm{d}x}{\mathrm{d}t} \right\}. \tag{4.41}$$

However, if one varies this action with respect to x, the term proportional to α gives

$$\int \mathrm{d}t\ \alpha\left(\delta x\,\frac{\mathrm{d}}{\mathrm{d}t}x + x\,\frac{\mathrm{d}}{\mathrm{d}t}\delta x\right). \tag{4.42}$$

But this term is a total derivative. Integrating by parts yields

$$\int \mathrm{d}t_a^b\,\frac{\mathrm{d}}{\mathrm{d}t}(x^2) = x^2\Big|_a^b = 0, \tag{4.43}$$

which may be ignored, since it exists only on the boundary. Because of the reversibility of the action principle, one cannot introduce terms which pick out a special direction in time. The only place where such terms can appear is through boundary conditions. For the same reason, it is impossible to represent Ohm's law

$$J^i = \sigma E^i \tag{4.44}$$

in an action principle. An ohmic resistor has to dissipate heat as current passes through it.

In some cases, the action principle can tricked into giving a non-zero contribution from velocity-dependent terms by multiplying the whole Lagrangian

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with an 'integrating factor' $\exp(\gamma(t))$, but the resulting field equations require $\gamma(t)$ to make the whole action decay exponentially, and often the results are ambiguous and not physically motivated.

We shall return to the issue of dissipation in detail in chapter 6 and show the beginnings of how physical boundary conditions and statistical averages can be incorporated into the action principle, in a consistent manner, employing the principle of causality. It is instructive to show that it is not possible to write down a gauge-invariant action for the equation

$$J^i = \sigma E^i. \tag{4.45}$$

i.e. Ohm's law, in terms of the vector potential A_μ. The equation is only an effective representation of an averaged statistical effect, because it does provide a reversible description of the underlying physics.

(1) By varying with respect to A_μ, one may show that the action

$$S = \int (\mathrm{d}x) \left\{ J^i A_i - \sigma_{ij} A^i E^j \right\} \tag{4.46}$$

with $E_i = -\partial_t A_i - \partial_i A_0$, does not give eqn. (4.45). If one postulates that E^i and J^i may be replaced by their steady state (time-independent) averages $\langle E^i \rangle$ and $\langle J^i \rangle$, then we can show that this does give the correct equation. This is an indication that some averaging procedure might be the key to representing dissipative properties of bulk matter.

(2) Consider the action

$$S = \int (\mathrm{d}x) \left\{ J^\mu A_\mu - \sigma_{ij} A^i E^j \mathrm{e}^{-\gamma^\mu x_\mu} \right\}. \tag{4.47}$$

This may be varied with respect to A_0 and A_i to find the equations of motion; gauge invariance requires the equations to be independent of the vector potential A_μ. On taking $\sigma_{ij} = \sigma \delta_{ij}$, one can show that gauge invariance requires that the vector potential decay exponentially. Readers are encouraged to check whether the resulting equations of motion are a satisfactory representation of Ohm's law.

4.3 Functional differentiation

It is useful to define the concept of functional differentiation, which is to ordinary differentiation what $\delta q(t)$ is to $\mathrm{d}q$. Functional differentiation differs from normal differentiation in some important ways.

The ordinary derivative of a function with respect to its control variable is defined by

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = \lim_{\delta t \to 0} \frac{f(t+\delta t) - f(t)}{\delta t}. \tag{4.48}$$

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It tells us about how a function changes with respect to the value of its control variable at a given point. Functional differentiation, on the other hand, is something one does to an integral expression; it is performed with respect to a function of some variable of integration. The 'point of differentiation' is now a function $f(t)$ evaluated at a special value of its control variable t'. It takes some value from within the limits of the integral. So, whereas we start with a quantity which is not a function of t or t', the result of the functional derivation is a function which is evaluated at the point of differentiation. Consider, as an example, the arbitrary functional

$$F[f] = \int \mathrm{d}t \sum_n a_n (f(t))^n. \tag{4.49}$$

This is clearly not a function of t due to the integral. The variation of such a functional $F[f]$ is given by

$$\delta F[f] = F[f(t) + \delta f(t)] - F[f(t)]. \tag{4.50}$$

We define the functional derivative by

$$\frac{\delta F}{\delta f(t')} = \lim_{\epsilon \to 0} \frac{F[f(t) + \epsilon \delta(t - t')] - F[f(t)]}{\epsilon}. \tag{4.51}$$

This is a function, because an extra variable t' has been introduced. You can check that this has the unusual side effect that

$$\frac{\delta q(t)}{\delta q(t')} = \delta(t - t'), \tag{4.52}$$

which is logical (since we expect the derivative to differ from zero only if the function is evaluated at the same point), but unusual, since the right hand side is not dimensionless – in spite of the fact that the left hand side seems to be. On the other hand, if we define a functional

$$Q = \int \mathrm{d}t q(t) \tag{4.53}$$

then we have

$$\frac{\delta Q}{\delta q(t')} = \int \mathrm{d}t \frac{\delta q(t)}{\delta q(t')} = \int \delta(t - t') = 1. \tag{4.54}$$

Thus, the integral plays a key part in the definition of differentiation for functionals.

4.4 The action in covariant field theory

The action principle can be extended to generally covariant field theories. This generalization is trivial in practice. An important difference is that field theories

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are defined in terms of variables which depend not only on time but also on space; $\phi(\mathbf{x}, t) = \phi(x)$. This means that the action, which must be a scalar, without functional dependence, must also be integrated over space in addition to time. Since the final action should have the dimensions of energy $\times$ time, this means that the Lagrangian is to be replaced by a Lagrangian *density* $\mathcal{L}$

$$S = \int_{\sigma}^{\sigma'} (\mathrm{d}x)\mathcal{L}(\phi(\mathbf{x}, t), \partial_\mu \phi(\mathbf{x}, t), x). \tag{4.55}$$

The integral measure is $(\mathrm{d}x) = \mathrm{d}V_x/c$, where $\mathrm{d}V_x = c\mathrm{d}t\mathrm{d}^n\mathbf{x}\sqrt{g} = \mathrm{d}x^0\mathrm{d}^n\mathbf{x}\sqrt{g}$. Although it would be nice to use $\mathrm{d}V_x$ here (since this is the Minkowski space volume element), this is not possible if $\mathcal{L}$ is an energy density and S is to have the dimensions of action.[2] The non-relativistic action principle has already chosen this convention for us. The special role played by time forces is also manifest in that the volume is taken between an earlier time t and a later time t' – or, more correctly, from one spacelike hyper-surface, σ, to another, σ'.

The classical interpretation of the action as the integral over $T - V$, the kinetic energy minus the potential energy, does not apply in the general case. The Lagrangian density has no direct physical interpretation, it is merely an artefact which gives the correct equations of motion. What is important, however, is how one defines a Hamiltonian, or energy functional, from the action. The Hamiltonian is related to measurable quantities, namely the total energy of the system at a given time, and it is responsible for the time development of the system. One must be careful to use consistent definitions, e.g. by sticking to the notation and conventions used in this book.

Another important difference between field theory and particle mechanics is the role of position. Particle mechanics describes the trajectories of particles, $q(t)$, as a function of time. The position was a function with time as a parameter. In field theory, however, space and time are independent parameters, on a par with one another, and the ambient field is a function which depends on both of them. In particle mechanics, the action principle determines the equation for a constrained path $q(t)$; the field theoretical action principle determines an equation for a field which simultaneously exists at all spacetime points, i.e. it does not single out any trajectory in spacetime, but rather a set of allowed solutions for an omnipresent field space. In spite of this difference, the formal properties of the action principle are identical, but for an extra integration:

[2] One could absorb a factor of c into the definition of the field $\phi(x)$, since its dimensions are not defined, but this would then mean that the Lagrangian and Hamiltonian would not have the dimensions of energy. This blemish on the otherwise beautiful notation is eliminated when one chooses natural units in which $c = 1$.

4.4.1 Field equations and continuity

For illustrative purposes, consider the following action:

$$S = \int (\mathrm{d}x)\left\{\frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) + \frac{1}{2}m^2\phi^2 - J\phi\right\}, \tag{4.56}$$

where $\mathrm{d}V_x = c\mathrm{d}t\,\mathrm{d}\mathbf{x}$. Assuming that the variables $\phi(x)$ commute with one another, the variation of this action is given by

$$\delta S = \int (\mathrm{d}x)\left\{(\partial^\mu\delta\phi)(\partial_\mu\phi) + m^2\phi\delta\phi - J\delta\phi\right\}. \tag{4.57}$$

Integrating this by parts and using the commutativity of the field, one has

$$\delta S = \int (\mathrm{d}x)\left\{-\Box\,\phi + m^2\phi - J\right\} + \int \mathrm{d}\sigma^\mu\ \delta\phi(\partial_\mu\phi). \tag{4.58}$$

From the general arguments given earlier, one recognizes a piece which is purely a surface integral and a piece which applies the field in a general volume of spacetime. These terms vanish separately. This immediately results in the field equations of the system,

$$(-\Box + m^2)\phi(x) = J(x), \tag{4.59}$$

and a continuity condition which we shall return to presently.

The procedure can be reproduced for a general Lagrangian density $\mathcal{L}$ and gives the Euler–Lagrange equations for a field. Taking the general form of the action in eqn. (4.55), one may write the first variation

$$\delta S = \int (\mathrm{d}x)\left\{\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\delta(\partial_\mu\phi)\right\}. \tag{4.60}$$

Now, the variation symbol and the derivative commute with one another since they are defined in the same way:

$$\begin{aligned}\partial_\mu\delta\phi &= \partial_\mu\phi(x+\Delta x) - \partial_\mu\phi(x)\\ &= \delta(\partial_\mu\phi);\end{aligned} \tag{4.61}$$

thus, one may integrate by parts to obtain

$$\delta S = \int (\mathrm{d}x)\left\{\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\right)\right\} + \frac{1}{c}\int \mathrm{d}\sigma^\mu\ \delta\phi\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\right) \tag{4.62}$$

The first of these terms exists for every spacetime point in the volume of integration, whereas the second is restricted only to the bounding hyper-surfaces σ and σ'. These two terms must therefore vanish independently in general.

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The vanishing integrand of the first term gives the Euler–Lagrange equations of motion for the field

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \right) = 0, \tag{4.63}$$

and the vanishing of the second term leads to the boundary continuity condition,

$$\Delta \left(\delta\phi \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \right) = 0. \tag{4.64}$$

If this result is compared with eqns. (4.22) and (4.23), an analogous 'momentum', or conjugate variable to the field $\phi(x)$, can be defined. This conjugate variable is unusually denoted $\Pi(x)$:

$$\Pi(x) = \frac{\delta L}{\partial(\partial^0 \phi)}, \tag{4.65}$$

and is derived by taking the canonical spacelike hyper-surface with $\sigma = 0$. Note the position of indices such that the variable transforms like a covariant vector $p = \partial_0 q$. The covariant generalization of this is

$$\Pi_\sigma(x) = \frac{\delta L}{\partial(\partial^\sigma \phi)}. \tag{4.66}$$

4.4.2 Uniqueness of the action

In deriving everything from the action principle, one could gain the impression that there is a unique prescription at work. This is not the case. The definition of the action itself is not unique. There is always an infinity of actions which generates the correct equations of motion. This infinity is obtained by multiplying the action by an arbitrary complex number. In addition to this trivial change, there may be several actions which give equivalent results depending on (i) what we take the object of variation to be, and (ii) what we wish to deduce from the action principle. For example, we might choose to re-parametrize the action using new variables. The object of variation and its conjugate are then re-defined.

It is clear from eqn. (4.21) that the field equations and boundary conditions would be the same if one were to re-define the Lagrangian by multiplying by a general complex number:

$$S \to (a + \mathrm{i}b)S. \tag{4.67}$$

The complex factor would simply cancel out of the field equations and boundary conditions. Moreover, the Lagrangian itself has no physical meaning, so there

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is no physical impediment to such a re-definition. In spite of this, it is normal to choose the action to be *real*. The main reason for this is that this choice makes for a clean relationship between the Lagrangian and a new object, the Hamiltonian, which is related to the energy of the system and is therefore, by assumption, a real quantity.

Except in the case of the gravitational field, one is also free to add a term on to the action which is independent of the field variables, since this is always zero with respect to variations in the fields:

$$S \to S + \int (\mathrm{d}x)\ \Lambda. \tag{4.68}$$

Such a term is often called a cosmological constant, because it was introduced by Einstein into the theory of relativity in order to create a static (non-expansive) cosmology. Variations of the action with respect to the metric are not invariant under the addition of this term, so the energy–momentum tensor in chapter 11 is not invariant under this change, in general. Since the Lagrangian density is an energy density (up to a factor of c), the addition of this arbitrary term in a flat (gravitation-free) spacetime simply reflects the freedom one has in choosing an origin for the scale of energy density for the field.[3]

Another way in which the action can be re-defined is by the addition of a total derivative,

$$\begin{aligned} S &\to S + \int (\mathrm{d}x)\partial^\mu F_\mu[\phi] \\ &= S + \int \mathrm{d}\sigma^\mu F_\mu[\phi]. \end{aligned} \tag{4.69}$$

The additional term exists only on the boundaries σ of the volume integral. By assumption, the surface term vanishes independently of the rest, thus, since the field equations are defined entirely from the non-surface contributions, they will never be affected by the addition of such a total derivative. However, the boundary conditions or continuity will depend on this addition. This has a physical interpretation: if the boundary of a physical system involves a discontinuous change, it implies the action of an external agent at the boundary. Such a jump is called a *contact potential*. It might signify the connection of a system to an external potential source (a battery attached by leads, for instance). The connection of a battery to a physical system clearly does not change the laws of physics (equations of motion) in the system, but it does change the boundary conditions.

In light of this observation, we must be cautious to write down a 'neutral', or unbiased action for free systems. This places a requirement on the action,

[3] Indeed, the action principle $\delta S = 0$ can be interpreted as saying that only potential differences are physical. The action potential itself has no unique physical interpretation.

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namely that the action must be *Hermitian*, time-reversal-invariant, or symmetrical with respect to the placement of derivatives, so that, if we let $t \to -t$, then nothing is changed. For instance, one writes

$$(\partial^\mu \phi)(\partial_\mu \phi) \qquad \text{instead of} \qquad \phi(-\Box\, \phi), \tag{4.70}$$

for quadratic derivatives, and

$$\frac{1}{2}(\phi^* \overset{\leftrightarrow}{\partial_t} \phi) = \frac{1}{2}(\phi^*(\partial_t \phi) - (\partial_t \phi^*)\phi) \qquad \text{instead of} \qquad \phi^* \partial_t \phi, \tag{4.71}$$

in the case of linear derivatives. These alternatives differ only by an integration by parts, but the symmetry is essential for the correct interpretation of the action principle as presented. This point recurs in more detail in section 10.3.1.

4.4.3 Limitations of the action principle

In 1887, Helmholtz showed that an equation of motion can only be derived from Lagrange's equations of motion (4.6) if the generalized force can be written

$$F_i = -\partial_i V + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial V}{\partial \dot{q}_i}, \tag{4.72}$$

where $V = V(q, \dot{q}, t)$ is the potential $L = T - V$, and the following identities are satisfied:

$$\begin{aligned} \frac{\partial F_i}{\partial \ddot{q}_j} &= \frac{\partial F_j}{\partial \ddot{q}_i} \\ \frac{\partial F_i}{\partial \dot{q}_j} + \frac{\partial F_j}{\partial \dot{q}_i} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F_i}{\partial \ddot{q}_j} + \frac{\partial F_j}{\partial \ddot{q}_i}\right) \\ \partial_j F_i - \partial_i F_j &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F_i}{\partial \dot{q}_j} - \frac{\partial F_j}{\partial \dot{q}_i}\right) \end{aligned} \tag{4.73}$$

For a review and discussion of these conditions, see ref. [67]. These relations lie at the core of Feynman's 'proof' of Maxwell's equations [42, 74]. Although they are couched in a form which derives from the historical approach of varying the action with respect to the coordinate q_i and its associated velocity, $\dot{q}_i$, separately, their covariant generalization effectively summarizes the limits of generalized force which can be derived from a local action principle, even using the approach taken here. Is this a significant limitation of the action principle?

Ohm's law is an example where a Lagrangian formulation does not work convincingly. What characterizes Ohm's law is that it is a substantive relationship between large-scale averages, derived from a deeper theory, whose actual dynamics are hidden and approximated at several levels. The relation summarizes a coarse average result of limited validity. Ohm's law cannot be

derived from symmetry principles, only from a theory with complex hidden variables. The deeper theory from which it derives (classical electrodynamics and linear response theory) does have an action principle formulation however.

Ohm's law is an example of how *irreversibility* enters into physics. The equations of fundamental physics are reversible because they deal only with infinitesimal changes. An infinitesimal interval, by assumption, explores so little of its surrounding phase space that changes are trivially reversed. This is the main reason why a generating functional (action) formulation is so successful at generating equations of motion: it is simply a mechanism for exploring the differential structure of the action potential-surface in a local region; the action is a definition of a conservation book-keeping parameter (essentially energy), parametrized in terms of field variables. The reversible, differential structure ensures conservation and generates all of the familiar quantities such as momentum. Irreversibility arises only when infinitesimal changes are compounded into significant changes; i.e. when one is able to explore the larger part of the phase space and take account of long-term history of a system. The methods of statistical field theory (closed time path [116] and density matrices [49]) may be used to study long-term change, based on sums of differential changes. Only in this way can one relate differential law to macroscopic change.

Another way of expressing the above is that the action principle provides a concise formulation of Markov processes, or processes whose behaviour now is independent of what happened in their past. Non-Markov processes, or processes whose behaviour now depends on what happened to them earlier, require additional long-term information, which can only be described by the combination of many infinitesimal changes.

Clearly, it is possible to write down equations which cannot be easily derived from an action principle. The question is whether such equations are of interest to physics. Some of them are (such as Ohm's law), but these only fail because, employing an action principle formulation of a high-level emergent phenomenon ignores the actual energy accounting taking place in the system. If one jumps in at the level of an effective field theory, one is not guaranteed an effective energy parameter which obeys the reversible accounting rules of the action principle. If an action principle formulation fails to make sense, it is possible to go to a deeper, more microscopic theory and re-gain an action formulation, thereby gaining a more fundamental (though perhaps more involved) understanding of the problem.

So are there any fundamental, elementary processes which cannot be derived from an action principle? The answer is probably not. Indeed, today all formulations of elementary physics assume an action principle formulation at the outset. What one can say in general is that any theory derived from an action principle, based on local fields, will lead to a well defined problem, within a natural, covariant formulation. This does not guarantee any prescription

understanding physical phenomena, but it does faithfully generate differential formulations which satisfy the symmetry principle.

4.4.4 Higher derivatives

Another possibility which is not considered in this book is that of higher derivative terms. The actions used here are at most quadratic in the derivatives. Particularly in speculative gravitational field theories, higher derivative terms do occur in the literature (often through terms quadratic in the curvature, such as Gauss–Bonnet terms or Weyl couplings); these are motivated by geometrical or topological considerations, and are therefore 'natural' to consider. Postulating higher order derivative terms is usually not useful in other contexts.

Higher derivative terms are often problematic, for several reasons. The main reason is that they lead to acausal solutions and 'ghost' excitations, or to field modes which appear to be solutions, but which actually do not correspond to physical propagations. In the quantum field theory, they are non-renormalizable. Although none of these problems is itself sufficient to disregard higher derivatives entirely, it limits their physical significance and usefulness. Some higher derivative theories can be factorized and expressed as coupled local fields with no more than quadratic derivatives; thus, a difficult action may be re-written as a simpler action, in a different formulation. This occurs, for instance, if the theories arise from non-local self-energy terms.

4.5 Dynamical and non-dynamical variations

It is convenient to distinguish between two kinds of variations of tensor quantities. These occur in the derivation of field equations and symmetry generators, such as energy and momentum, from the action.

4.5.1 Scalar fields

The first kind of variation is a *dynamical* variation; it has been used implicitly up to now. A dynamical variation of an object q is defined by

$$\delta q = q'(x) - q(x). \tag{4.74}$$

This represents a change in the *function* $q(x)$ at constant position x. It is like the 'rubber-banding' of a function into a new function: a parabola into a cubic curve, and so on.

The other kind of variation is a coordinate variation, or *kinematical* variation, which we denote

$$\delta_x q(x) = q(x') - q(x). \tag{4.75}$$

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This is the apparent change in the height of the function when making a shift in the coordinates x, or perhaps some other parameter which appears either explicitly or implicitly in the action. More generally, the special symbol δ_ξ is used for a variation with respect to the parameter ξ. By changing the coordinates in successive variations, δ_x, one could explore the entire function $q(x)$ at different points. This variation is clearly related to the partial (directional) derivative of q. For instance, under a shift

$$x^\mu \to x^\mu + \epsilon^\mu, \tag{4.76}$$

i.e. $\delta x^\mu = \epsilon^\mu$, we have

$$\delta_x q(x) = (\partial_\mu q)\epsilon^\mu. \tag{4.77}$$

One writes the total variation in the field q as

$$\delta_{\mathrm{T}} \equiv \delta + \sum_i \delta_{\xi^i}. \tag{4.78}$$

4.5.2 Gauge and vector fields

The coordinate variation of a vector field is simply

$$\begin{aligned} \delta_x V_\mu &= V_\mu(x') - V_\mu(x) \\ &= (\partial_\lambda V_\mu)\epsilon^\lambda. \end{aligned} \tag{4.79}$$

For a gauge field, the variation is more subtle. The field at position x' need only be related to the Taylor expansion of the field at x up to a gauge transformation, so

$$\begin{aligned} \delta_x A_\mu &= A_\mu(x') - A_\mu(x) \\ &= (\partial_\lambda A_\mu)\epsilon^\lambda + \partial_\lambda(\partial_\mu s)\epsilon^\lambda. \end{aligned} \tag{4.80}$$

The gauge transformation s is important because $\delta_x A_\mu(x)$ is a *potential difference*, and we know that potential differences are observable as the electric and magnetic fields, so this variation should be gauge-invariant. To make this so, one identifies the arbitrary gauge function s by $\partial_\lambda s = -A_\lambda$, which is equally arbitrary, owing to the gauge symmetry. Then one has

$$\begin{aligned} \delta_x A_\mu &= (\partial_\lambda A_\mu - \partial_\mu A_\lambda)\epsilon^\lambda \\ &= F_{\lambda\mu}\epsilon^\lambda. \end{aligned} \tag{4.81}$$

Neglect of the gauge freedom has led to confusion over the definition of the energy–momentum tensor for gauge fields; see section 11.5.

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The dynamical variation of a vector field follows from the general tensor transformation

$$V'(x') = \frac{\partial x^\rho}{\partial x'^\mu} V_\rho(x). \tag{4.82}$$

From this we have

$$\begin{aligned}
\delta V_\mu(x) &= V'_\mu(x) - V_\mu(x) \\
&= V'_\mu(x') - (\partial_\lambda V_\mu)\epsilon^\lambda - V_\mu(x) \\
&= \frac{\partial x^\rho}{\partial x'^\mu} V_\rho(x) - (\partial_\lambda V_\mu)\epsilon^\lambda - V_\mu(x) \\
&= -(\partial_\nu \epsilon_\mu) V^\nu - (\partial_\lambda V_\mu)\epsilon^\lambda.
\end{aligned} \tag{4.83}$$

For the gauge field, one should again be wary about the implicit coordinate variation. The analogous derivation gives

$$\begin{aligned}
\delta A_\mu(x) &= A'_\mu(x) - A_\mu(x) \\
&= A'_\mu(x') - F_{\lambda\mu}\epsilon^\lambda - A_\mu(x) \\
&= \frac{\partial x^\rho}{\partial x'^\mu} A_\rho(x) - F_{\lambda\mu}\epsilon^\lambda - A_\mu(x) \\
&= -(\partial_\nu \epsilon_\mu) A^\nu - F_{\lambda\mu}\epsilon^\lambda.
\end{aligned} \tag{4.84}$$

4.5.3 The metric and second-rank tensors

The coordinate variation of the metric is obtained by Taylor-expanding the metric about a point x,

$$\begin{aligned}
\delta_x g_{\mu\nu} &= g_{\mu\nu}(x') - g_{\mu\nu}(x) \\
&= (\partial_\lambda g_{\mu\nu}(x))\epsilon^\lambda.
\end{aligned} \tag{4.85}$$

To obtain the dynamical variation, we must use the tensor transformation rule

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x), \tag{4.86}$$

where

$$\frac{\partial x^\rho}{\partial x'^\mu} = \delta^\rho_\mu - (\partial_\mu \epsilon^\rho) + \cdots + \mathrm{O}(\epsilon^2). \tag{4.87}$$

Thus,

$$\begin{aligned}
\delta g_{\mu\nu} &= g'_{\mu\nu}(x) - g_{\mu\nu}(x) \\
&= \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma(x)} - (\partial_\rho g'_{\mu\nu})\epsilon^\rho - g_{\mu\nu}(x)
\end{aligned}$$

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$$
\begin{aligned}
&= -(\partial_\lambda g_{\mu\nu})\epsilon^\lambda - (\partial_\mu \epsilon^\lambda) g_{\lambda\nu} - (\partial_\nu \epsilon^\lambda) g_{\lambda\mu} \\
&= -(\partial_\lambda g_{\mu\nu})\epsilon^\lambda - \left\{\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\right\}, \qquad (4.88)
\end{aligned}
$$

where one only keeps terms to first order in ϵ^μ.

4.6 The value of the action

There is a frequent temptation to assign a physical meaning to the action, beyond its significance as a generating functional. The differential structure of the action, and the variational principle, give rise to canonical systems obeying conservation laws. This is the limit of the action's physical significance. The impulse to deify the action should be stifled.

Some field theorists have been known to use the value of the action as an argument for the triviality of a theory. For example, if the action has value zero, when evaluated on the constraint shell of the system, one might imagine that this is problematic. In fact, it is not. It is not the numerical value of the action but its differential structure which is relevant.

The vanishing of an action on the constraint shell is a trivial property of any theory which is linear in the derivatives. For instance, the Dirac action and the Chern–Simons [12] action have this property. For example:

$$
\begin{aligned}
S &= \int (\mathrm{d}x)\overline{\psi}(\mathrm{i}\gamma^\mu \partial_\mu + m)\psi \\
\frac{\delta S}{\delta \overline{\psi}} &= (\mathrm{i}\gamma^\mu \partial_\mu + m)\psi = 0 \\
S\Big|_\psi &= 0. \qquad (4.89)
\end{aligned}
$$

The scalar value of the action is irrelevant, even when evaluated on some specified constraint surface. Whether it is zero, or non-zero, it has no meaning. The only exception to this is in the Wick-rotated theory, where a serendipitous link to finite temperature physics relates the Wick-rotated action to the Hamiltonian or energy operator of the non-rotated theory.

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