PROJECTIVE IDEALS OF FINITE TYPE

WILLIAM W. SMITH

1. Introduction. The main results in this paper relate the concepts of flatness and projectiveness for finitely generated ideals in a commutative ring with unity. In this discussion the idea of a multiplicative ideal is used.

Definition. An ideal J is multiplicative if and only if whenever I is an ideal with $I \subset J$ there exists an ideal C such that $I = JC$.

Throughout this paper R will denote a commutative ring with unity. If I and J are ideals of R, then $I: J = \{x | xJ \subset I\}$. By "prime ideal" we will mean "proper prime ideal" and Spec R will denote this set of ideals. R is called a local ring if it has a unique maximal ideal (the ring need not be Noetherian). If P is in Spec R, then R_P is the quotient ring formed using the complement of P. If M is an R-module, then M_P will be the corresponding R_P-module. $J^\perp = (0):J$ is the annihilator of J. We use rad R to denote the prime radical and Rad R to denote the maximal radical of R.

In the following two propositions, the principal results of this paper are given.

PROPOSITION A. *Let J be an ideal of finite type such that $J^\perp \subset$ rad R. The following are equivalent:*

(i) *J is a projective rank one ideal;*

(ii) *J is multiplicative;*

(iii) *J is flat.*

PROPOSITION B. *Let J be an ideal of finite type such that $J^\perp$ is finitely generated. The following are equivalent:*

(i) *J is projective;*

(ii) *J is flat;*

(iii) *J is a direct summand of a projective rank one ideal.*

Several theorems which lead to the proofs of these two propositions are given in § 2. The proofs of both propositions are given in § 3.

2. The proof of the following theorem uses the same basic technique found in (**2**; **3**), where it was shown that a projective ideal containing an element which is not a zero divisor is invertible. It is easily seen that a multiplicative ideal containing an element which is not a zero divisor must be invertible.

THEOREM 1. *If J is a projective ideal of a ring R, then J must be multiplicative.*

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Proof. Let $I \subset J$. By (**2**, p. 132, Proposition 3.1) there exists a family $\{b_\alpha\}$ of elements of J and a family $\{f_\alpha\}$ of R-homomorphisms from J to R such that for each b in J, $b = \sum (f_\alpha(b))b_\alpha$, where $f_\alpha(b)$ is zero for all except a finite number of the indices. Let C be the ideal generated by elements of the form $f_\alpha(x)$, where x is in I. We claim that $I = JC$. If x is in $I \subset J$, then $x = \sum f_\alpha(x)b_\alpha$, and hence x is in JC. Conversely, if x is in I, b is in J, and α is some index, then $bf_\alpha(x) = f_\alpha(bx) = f_\alpha(b)x$ which is in I. As a result, $JC \subset I$ and we obtain equality. This completes the proof.

LEMMA 2. *If J is a finitely generated multiplicative ideal in a local ring, then J is principal.*

Proof. Suppose that $J = \langle b_1, \ldots, b_n\rangle$ with $n \geqq 2$. Let C be such that $\langle b_2, \ldots, b_n\rangle = JC$. There exists $c_1, \ldots, c_n$ in C such that

$$b_n = b_1c_1 + \ldots + b_nc_n,$$

and hence $b_n(1 - c_n) = b_1c_1 + \ldots + b_{n-1}c_{n-1}$. If c_n is not a unit in R, then $1 - c_n$ is a unit and $J = \langle b_1, \ldots, b_{n-1}\rangle$. If c_n is a unit, then $C = R$ and $J = \langle b_2, \ldots, b_n\rangle$. In either case we see that an induction argument would yield J is a principal ideal.

THEOREM 3. *If J is a multiplicative ideal of finite type, then J_P is principal for each P in* Spec R.

Proof. Since every ideal in R_P is the extension of some ideal in R and $(JC)_P = J_PC_P$, it is easily seen that J_P is a multiplicative ideal of R_P. Lemma 2 then yields the desired result.

LEMMA 4. *Let $J = \langle a_1, \ldots, a_n\rangle$ (for $n \geqq 2$) be a flat ideal with $C = \langle a_2, \ldots, a_n\rangle : J$. Then $\langle a_2, \ldots, a_n\rangle = JC$.*

Proof. We define a map from R^n to J by mapping $(x_1, \ldots, x_n)$ onto $\sum a_ix_i$. This yields the exact sequence $0 \to K \to R^n \to J \to 0$. Moreover, $C = \{y|$ there exists $y_2, \ldots, y_n$ such that $(y, y_2, \ldots, y_n)$ is in $K\}$. Since J is flat, we have $(J \times \ldots \times J) \cap K = JK$; see (**1**, p. 33). Now $(a_2, -a_1, 0, \ldots, 0)$ is in both $J \times \ldots \times J$ and K, hence in JK. Therefore there exists $c_1, \ldots, c_m$ in J and $\{(x_{i1}, \ldots, x_{in})\}_{i=1}^m$ in K such that

$$(a_2, -a_1, 0, \ldots, 0) = \sum_{i=1}^m c_i(x_{i1}, \ldots, x_{in}).$$

Thus we obtain $a_2 = \sum c_ix_i$. However, $(x_{i1}, \ldots, x_{in})$ in K implies x_{i1} is in C for each i, hence a_2 is in JC. A similar argument yields a_j is in JC for each $j \geqq 2$. Thus we have $\langle a_2, \ldots, a_n\rangle \subset JC$. The other containment is obvious by the definition of C.

LEMMA 5. *If J is a flat ideal of finite type in a local ring, then J is principal.*

Proof. Suppose that $J = \langle b_1, \ldots, b_n\rangle$ with $n \geqq 2$. By Lemma 4 there is an ideal C such that $\langle b_2, \ldots, b_n \rangle = JC$. The remainder of the proof is exactly the proof of Lemma 2.

LEMMA 6. *If $J = \langle a \rangle$ is a non-zero flat ideal in a local ring R, then a is not a zero divisor in R.*

Proof. Let $K = J^\perp$. Then $J = R/K$ flat implies $IK = I \cap K$ for each ideal I in R. If $B \subset K$ and $I = B{:}K$ we have $B \subset I \cap K = IK \subset B$. Therefore $B = IK$. We also notice that $K^2 = K$ and $K \subset M$ since J is not zero (where M is the maximal ideal of R). If k is in K, then there exists an ideal B such that $\langle k \rangle = KB$. As a result, $K\langle k \rangle = K^2B = KB = \langle k \rangle$. By Nakayama's lemma (**1**, p. 113, Proposition 11) we have $k = 0$. Therefore $K = (0)$ and a is not a zero divisor.

LEMMA 7. *Let J be a flat ideal of finite type and let P be a prime ideal of R. Then either $J_P = (0)$ in R_P or J_P is principal generated by an element which is not a zero divisor in R_P.*

Proof. J_P is a flat ideal of finite type in R_P; see (**1**, p. 115, Proposition 13). The result then follows from Lemmas 5 and 6.

LEMMA 8. *Let $M = \langle a_1, \ldots, a_n \rangle$ be an R-module such that for each P in* Spec R *either $M_P = (0)$ or $(M_P)^\perp = (0)$. Then $(M^\perp)_P = R_P$ whenever $M_P = 0$ and $(M^\perp)_P = (0)$ whenever $(M_P)^\perp = (0)$.*

Proof. The second statement is valid since $(M^\perp)_P \subset (M_P)^\perp$. For the first statement notice that for each i there exists a y_i in R and not in P such that $a_i y = 0$. Let $y = y_1 \ldots y_n$. Then $a_i y = 0$ for each i, hence $M_y = 0$ and $y \in M^\perp$. Since y is not in P, we have $(M^\perp)_P = R_P$.

COROLLARY 9. *Let J be a flat ideal of finite type such that $J^\perp \subset$* rad R. *Then J is a projective rank one ideal.*

Proof. By Lemma 7, either $J_P = (0)$ or J_P is isomorphic to R_P for each P in Spec R. If J_P is isomorphic to R_P, then $(J_P)^\perp = (0)$ in R_P. Furthermore, $J^\perp \subset$ rad R implies $(J^\perp)_P \neq R_P$ for any P in Spec R. By using Lemma 8 we conclude that J_P is isomorphic to R_P for each P in Spec R. Hence J is a projective rank one ideal (**1**, p. 141, Definition 2).

COROLLARY 10. *Let J be a flat ideal of finite type such that $J^\perp \subset$* Rad R. *Then J is a projective rank one ideal.*

Proof. By (**1**, p. 141, Theorem 2) it is sufficient in the proof of Corollary 9 to use only the maximal ideals.

COROLLARY 11. *Let J be a flat ideal of finite type such that $J^\perp$ is also finitely generated. Then $J^\perp \oplus (J^\perp)^\perp = R$.*

Proof. By Lemmas 7 and 8, $(J^\perp)_P$ is either R_P or (0) for each P in Spec R.

As a result, $J^{\perp}$ is a flat ideal. We have assumed that $J^{\perp}$ is of finite type and by Lemma 8, this time applied to $J^{\perp}$, we obtain:

$$(J^{\perp})_P{}^{\perp} = \begin{cases} R_P & \text{when } (J^{\perp})_P = 0, \\ 0 & \text{when } (J^{\perp})_P = R_P. \end{cases}$$

Let $N = J^{\perp} \oplus (J^{\perp})^{\perp}$. We have $N_P = R_P$ for each P in Spec R, hence $N = R$. Let $M = (J^{\perp}) \cap (J^{\perp})^{\perp}$. By the above argument, $M_P = (0)$ for each P in Spec R since $M_P \subset J_P{}^{\perp} \cap (J^{\perp})_P{}^{\perp}$; thus we must have $M = (0)$, and the sum is direct. That is, $R = (J^{\perp}) \oplus (J^{\perp})^{\perp}$.

THEOREM 12. *Let J be a flat ideal of finite type such that $J^{\perp}$ is finitely generated. Then J is projective.*

Proof. By Corollary 11 there exists an idempotent e such that $J^{\perp} = (1 - e)R$ and $J \subset (J^{\perp})^{\perp} = eR$. We assert that eJ is a rank one projective eR-module. To see this, notice that the primes in Spec(eR) are those of the form eP, where P is a prime in R with $J^{\perp} \subset eP$. Using the permutability of residue class ring and quotient ring formation we obtain $(eJ)_{eP} \cong J_P \cong R_P \cong (eR)_{eP}$. This verifies the assertion. To complete the proof of the theorem one observes that we have a ring homomorphism from R onto eR with eJ a projective eR-module and eR a projective R-module. By (**2**, p. 30, Proposition 6.2) we have J is projective as an R-module.

COROLLARY 13. *Suppose that J is a flat ideal of finite type such that J_P is isomorphic to R_P for all except a finite number of P in* Spec R. *Then J is projective.*

Proof. Let $P_1, \ldots, P_n$ be the primes of R for which $J_{P_i} = (0)$ in R_P. Then $(J^{\perp})_{P_i} = R_{P_i}$ for all i (by Lemma 8). Therefore for each i there exists a y_i in $J^{\perp}$ not in P_i. Let $B = \langle y_1, \ldots, y_n \rangle$. For $P \neq P_i$ we have $B_P \subset (J^{\perp})_P = (0)$ in R_P. Moreover, $B_{P_i} = R_{P_i} = (J^{\perp})_{P_i}$ for each i. As a result, $B = J^{\perp}$ and by Theorem 12, J is projective.

3. We are now in a position to prove the main results, Propositions A and B.

Proof of Proposition A. Theorem 1 yields the proof that (i) implies (ii). To see that (ii) implies (iii) notice that Theorem 3 yields J_P is principal for each P in Spec R. J finitely generated and $J^{\perp} \subset \operatorname{rad} R$ tells us that J_P must be generated by an element which is not a zero divisor. As a result, J_P is isomorphic to R_P. Since J_P is then flat for each P in Spec R, J itself must be flat. Corollary 9 is the statement that (iii) implies (i), which completes the proof of Proposition A.

Proof of Proposition B. It is always true that (i) implies (ii). Now suppose that J is flat. By Theorem 12, J is projective and $J \cap J^{\perp} = (0)$. Let $N = J + J^{\perp}$. The sum is direct and N must be projective of finite type. By our previous results, N_P is principal and free for each P in Spec R. Furthermore, $N_P = J_P + (J^{\perp})_P \neq 0$ which yields: N_P is a free rank one module.

Therefore $N = J \oplus J^{\perp}$ is a projective rank one module, proving that (ii) implies (iii). Since it is always true that (iii) implies (i) we have completed the proof of the proposition.

Remark. I received in a communication from Robert L. Pendleton and Sam Cox an example showing that the conditions "$J^{\perp} \subset \operatorname{rad} R$" and "$J^{\perp}$ finitely generated" in Propositions A and B cannot be deleted. They give an example of a principal flat ideal in a commutative ring which is not projective. The problems considered above are special cases of a more general problem. That is, what conditions can be put on a submodule M of a free module to ensure that M will be projective.

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The University of North Carolina,
Chapel Hill, North Carolina