ON REARRANGEMENTS OF INFINITE SERIES

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(Received 22nd January, 1958)

1. Introduction. If a convergent series of real or complex numbers is rearranged, the resulting series may or may not converge. There are therefore two problems which naturally arise.

(i) What is the condition on a given series for every rearrangement to converge?

(ii) What is the condition on a given method of rearrangement for it to leave unaffected the convergence of every convergent series?

The answer to (i) is well known ; by a famous theorem of Riemann, the series must be absolutely convergent. The solution of (ii) is perhaps not so familiar, although it has been given by various authors, including R. Rado [7], F. W. Levi [6] and R. P. Agnew [2]. It is also given as an exercise by N. Bourbaki ([4], Chap. III, § 4, exs. 7 and 8).

It may happen that a rearrangement of a convergent series, though not convergent, is nevertheless summable by some method of summability. Thus, given any method of summability, there are analogues of (i) and (ii).

(iii) What is the condition on a given convergent series for every rearrangement to be summable?

(iv) What is the condition on a given method of rearrangement for it to rearrange every convergent series into a summable series?

Of these problems, (iv) seems to be the more difficult, and the main part of this paper is taken up with an attempt to give a satisfactory solution. The general result obtained in § 3 (Theorem 1) would be difficult to apply, but under certain conditions there is a much simpler reformulation of this solution, given in § 4 (Theorem 2) ; this simpler form is applicable to most methods of summability that occur in practice. Finally, Theorem 3 of § 6 gives the complete solution of the easier problem (iii) ; it turns out that, as for (i), the series must be absolutely convergent.

2. Definitions. We consider a general method A of summability, defined by the matrix (a_{ij}). The series Σu_n, or its sequence of partial sums $x_n = \sum_{r=1}^{n} u_r$, is called A-*bounded* if, for each i, the series $\sum_j a_{ij}x_j$ is convergent and $\sup_i \left| \sum_{j=1}^{\infty} a_{ij}x_j \right| < \infty$. If also $s = \lim_{i\to\infty} \sum_{j=1}^{\infty} a_{ij}x_j$ exists, then the series, or its sequence of partial sums, is called A-*summable* to s. The method A is called *regular* if $\sum u_n$ is A-summable to s whenever $\sum u_n$ is convergent to s. Necessary and sufficient conditions for the method A to be regular are

$$\text{C1} : \sup_i \sum_{j=1}^{\infty} | a_{ij} | < \infty,$$

$$\text{C2} : \lim_{i\to\infty} a_{ij} = 0 \text{ for each } j,$$

$$\text{C3} : \lim_{i\to\infty} \sum_{j=1}^{\infty} a_{ij} = 1.$$

https://doi.org/10.1017/S2040618500033694 Published online by Cambridge University Press

(See, for example, G. H. Hardy [5], p. 43, Theorem 2.) It follows from C1 that, if A is regular, every series with bounded partial sums is A-bounded.

In these definitions and conditions, the variable i may take values and tend to infinity either on the set of positive integers or on the set of positive real numbers. Since we are only interested in the behaviour as $i \to \infty$, it seems reasonable to consider also the weaker definitions of A-summability and A-boundedness which we obtain if we demand that the series $\sum_j a_{ij}x_j$ should converge and that $\left\{ \sum_{j=1}^{\infty} a_{ij}x_j \right\}$ should be bounded only for sufficiently large values of i, say for $i \geqslant m$. It is not obvious that this weaker definition leads to the same class of regular methods of summability, because m may depend on the series $\sum u_n$ considered. But in fact it can be shown (R. P. Agnew [1], Theorem 7.2 and C. A. Rogers [8]) that the method A is regular in this weaker sense if and only if C2 and C3 hold together with

$$\text{C1}' : \text{there is a constant } m \text{ with } \sup_{i \geqslant m} \sum_{j=1}^{\infty} | a_{ij} | < \infty.$$

Since it involves little extra complication, we shall work with these weaker definitions of A-boundedness, A-summability and regularity.

The natural way to specify a method of rearrangement is to take a permutation σ of the set N of positive integers. This rearranges the series $\sum u_n$ into the series $\sum u_{\sigma(n)}$. The partial sums of the rearranged series are of the form

$$\sum_{r=1}^{n} u_{\sigma(r)} = \sum_{r \in P_n} u_r,$$

where

$$P_n = \{\sigma(1), \sigma(2), \ldots, \sigma(n)\}.$$

More generally, if we are given any strictly increasing sequence (P_n) of finite subsets of N with $\bigcup_{n=1}^{\infty} P_n = N$, we shall say that these define a *method* P *of rearrangement*, and that the P-rearrangement of the series Σu_n is the series for which the sequence of partial sums is (y_n), where

$$y_n = \sum_{r \in P_n} u_r.$$

Our main problem is to find the condition that must be satisfied jointly by P and A for the sequence (y_n) to be A-summable to s whenever $\sum u_n$ is convergent to s. We shall then say that *the method* PA *is regular*. Unlike the situation for regular methods of summability, it can happen that the method PA is regular and yet there are (non-convergent) series $\sum u_n$ with bounded partial sums for which the P-rearrangement is not A-bounded. (There is an example in § 5.) If the P-rearrangement of every series with bounded partial sums is A-bounded, we shall call the regular method PA *fully regular*.

3. General conditions for regularity and full regularity.

LEMMA 1. *If the method* PA *is regular, then the conditions* C2 *and* C3 *of* § 2 *hold.*

Proof. For C2, if j is given, there are integers h in P_j but not in P_{j-1} (if $j > 1$) and k in P_{j+1} but not in P_j. Then, if $u_h = 1$, $u_k = -1$ and $u_n = 0$ for all other values of n, the series $\sum u_n$ is convergent to 0 and its P-rearranged partial sums y_n are all zero except for y_j which is 1. Thus

$$0 = \lim_{i \to \infty} \sum_{n=1}^{\infty} a_{in} y_n = \lim_{i \to \infty} a_{ij}.$$

Similarly C3 can be proved by choosing $l \in P_1$ and putting $u_l = 1$, $u_n = 0$ otherwise; then Σu_n converges to 1 and $y_n = 1$ for all n. Hence

$$1 = \lim_{i \to \infty} \sum_{j=0}^{\infty} a_{ij} y_j = \lim_{i \to \infty} \sum_{j=1}^{\infty} a_{ij}.$$

We therefore consider only methods of summability which satisfy C2 and C3. With each convergent series Σu_n we associate its sequence $x = (x_n)$ of partial sums. Then x is a member of the space (c) of all convergent sequences. With the norm $\| x \| = \sup_n | x_n |$, the space (c) becomes a Banach space. Any continuous linear form f on this space is of the form

$$f(x) = f_0 \lim_{n \to \infty} x_n + \sum_{n=1}^{\infty} f_n x_n,$$

where $\| f \| = \sum_{n=0}^{\infty} | f_n | < \infty$. With this norm, the set of all continuous linear forms on (c), that is, the dual of (c), is the Banach space (l^1) of all sequences which form the terms of an absolutely convergent series. Finally, any continuous linear form z on (l^1) is of the form

$$z(f) = \sum_{n=0}^{\infty} z_n f_n,$$

where $\| z \| = \sup_n | z_n | < \infty$. Thus the dual of (l^1) is the Banach space (m) of all bounded sequences with the norm given above. (For these definitions and properties, see Banach [3], pp. 11–12, 65–67.)

In what follows we shall have to consider the convergence of sequences of elements of (l^1) in two natural topologies on (l^1). If, for each $x \in (c)$, $f_k(x) \to f(x)$, the sequence (f_k) is said to converge to f in the weak topology $\sigma((l^1), (c))$. If $\| f_k - f \| \to 0$ the sequence (f_k) is said to converge to f in the norm topology on (l^1).

We associate with a method PA a set of elements of (l^1) in the following way. For each i and each positive integer k, put

$$f_{ik}(x) = \sum_{j=1}^{k} a_{ij} y_j = \sum_{j=1}^{k} a_{ij} \sum_{r \in P_j} u_r = \sum_{j=1}^{k} a_{ij} \sum_{r \in P_j} (x_r - x_{r-1}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

(with the convention $x_0 = 0$). Then $f_{ik}(x)$ is a finite linear combination of the x_n and so f_{ik} is a continuous linear form on (c), *i.e.*, $f_{ik} \in (l^1)$.

LEMMA 2. *Suppose that the method* PA *is regular. Then there is an integer m such that, for each $i \geqslant m$,*

$$\sup_k \| f_{ik} \| < \infty$$

and the linear form f_i defined by

$$f_i(x) = \lim_{k \to \infty} f_{ik}(x)$$

exists and is continuous on (c). If the method PA *is fully regular, then $f_{ik} \to f_i$ in the norm topology on (l^1).*

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Proof. Denote by E_n the vector subspace of (c) consisting of those x for which $\lim_{k\to\infty} f_{ik}(x) = \sum_{j=1}^{\infty} a_{ij}y_j$ exists and is finite for all $i \geqslant n$. Since the method PA is regular, (c) is the union of the E_n and so at least one, E_m say, must be of second category in (c). Then if $i \geqslant m$, $\sup_k |f_{ik}(x)| < \infty$ for each $x \in E_m$, and so, by Banach [3], p. 80, Théorème 4, E_m is the whole of (c). Hence, by the Banach-Steinhaus theorem (*loc. cit.*, Théorème 5), $\sup_k \|f_{ik}\| < \infty$. Thus $f_{ik}(x) \to f_i(x)$ for each $x \in (c)$ and f_i is continuous on (c).

If the method PA is fully regular, the same argument can be applied with (m) instead of (c) to find an integer m such that, for all $i \geqslant m, f_{ik}(x) \to f_i(x)$ for each $x \in (m)$. But (Banach [3], p. 141) this implies that $f_i \in (l^1)$ and (*loc. cit.*, p. 137) that $f_{ik} \to f_i$ in the norm topology on (l^1).

THEOREM 1. *Suppose that the method* A *of summability satisfies the conditions* C2 *and* C3 *of* § 2, *and that the linear forms* f_{ik} *are defined on* (c) *by* (1) *above. Then the method* PA *is regular* [*fully regular*] *if and only if there is an integer* m *such that, for each* $i \geqslant m$, $\lim_{k\to\infty} f_{ik} = f_i$ *exists* (*the limit being taken in the weak topology* $\sigma((l^1), (c))$ [*in the norm topology on* (l^1)]), *and* $\sup_{i\geqslant m} \|f_i\| < \infty$.

Proof. If the method PA is regular or fully regular, the existence of the limits f_i in the appropriate topology follows from Lemma 2. Also, for each $x \in (c)$, $\lim_{i\to\infty} f_i(x) = \lim_{i\to\infty} \sum_{j=1}^{\infty} a_{ij}y_j$ exists, and so, by the Banach-Steinhaus theorem, $(\|f_i\|)$ is bounded for $i \geqslant m$.

Conversely, suppose that for $i \geqslant m$, $f_{ik} \to f_i$ weakly and $\sup_{i\geqslant m} \|f_i\| < \infty$. If $x = (x_n)$ is a sequence for which x_n is constant for sufficiently large n, then

$$y_j = \sum_{r \in P_j} (x_r - x_{r-1}) = \lim_{n\to\infty} x_n$$

for sufficiently large j, say for $j \geqslant l$. Thus

$$\begin{aligned} f_i(x) &= \lim_{k\to\infty} f_{ik}(x) \\ &= \sum_{j=1}^{\infty} a_{ij}y_j \\ &= \sum_{j=1}^{l} a_{ij}(y_j - \lim_{n\to\infty} x_n) + \sum_{j=1}^{\infty} a_{ij} \,.\, \lim_{n\to\infty} x_n \\ &\to 0 + \lim_{n\to\infty} x_n \text{ as } i \to \infty, \text{ by C2 and C3.} \end{aligned}$$

Now, given any sequence $x \in (c)$, put

$$x^{(p)} = (x_1, x_2, x_3, \dots , x_p, x_p, x_p, \dots)$$

Then
$$f_i(x) = f_i(x^{(p)}) + f_i(x - x^{(p)}).$$

Now the first part of the right side of this converges to $\lim_{n\to\infty} x_n^{(p)} = x_p$, by what has been proved, while

$$|f_i(x - x^{(p)})| \leqslant \sup_{i\geqslant m} \|f_i\| \,.\, \|x - x^{(p)}\|,$$

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which converges to zero as $p \to \infty$, uniformly for $i \geqslant m$. Hence $f_i(x) \to \lim_{p\to\infty} x_p$ as $i \to \infty$, which proves that the method PA is regular.

Finally, if $f_{ik} \to f_i$ in norm for $i \geqslant m$, we have, for each $x \in (m)$, $\sum_{j=1}^{\infty} a_{ij}y_j$ exists and

$$\sup_{i \geqslant m} \left| \sum_{j=1}^{\infty} a_{ij}y_j \right| = \sup_{i \geqslant m} |f_i(x)| \leqslant \sup_{i \geqslant m} \| f_i \| \cdot \| x \| .$$

Thus the method PA is fully regular.

4. Special cases. From the point of view of applications, it is clearly desirable to have the conditions of Theorem 1 reformulated in terms of the sequence (P_n) and the matrix (a_{ij}) which specify the method PA. This we have been able to do only with some supplementary hypothesis either on the method P of rearrangement or on the method A of summability. We start by calculating $\| f_{ik} \|$. Denoting by χ_j the characteristic function of the set P_j (so that $\chi_j(r) = 1$ if $r \in P_j$ and $\chi_j(r) = 0$ otherwise), we have

$$f_{ik}(x) = \sum_{j=1}^{k} a_{ij} \sum_r \chi_j(r)\,(x_r - x_{r-1})$$

$$= \sum_{j=1}^{k} a_{ij} \sum_r (\chi_j(r) - \chi_j(r+1))x_r .$$

Hence
$$\| f_{ik} \| = \sum_r \left| \sum_{j=1}^{k} a_{ij}(\chi_j(r) - \chi_j(r+1)) \right| .$$

An exactly similar argument shows that if $l < k$,

$$\| f_{il} - f_{ik} \| = \sum_r \left| \sum_{j=l+1}^{k} a_{ij}(\chi_j(r) - \chi_j(r+1)) \right| .$$

Now, for fixed r, both $\chi_j(r)$ and $\chi_j(r+1)$ are increasing with j and so all the terms $(\chi_j(r) - \chi_j(r+1))$ have the same sign. Thus the last two formulae can be written in the form

$$\left. \begin{aligned} \| f_{ik} \| &= \sum_r | \Sigma_1 a_{ij} | , \\ \| f_{il} - f_{ik} \| &= \sum_r | \Sigma_2 a_{ij} | , \end{aligned} \right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

where Σ_1 denotes summation over those values of $j \leqslant k$ for which just one of r, $r+1$ belongs to P_j, and Σ_2 denotes summation over the same values of j with the further restriction $j \geqslant l+1$.

The position of the modulus signs in the formulae (2) makes further simplification difficult, but if they can be moved inside the inner summation, we can then change the order of summation and obtain much simpler expressions. Clearly this will be possible if either all the a_{ij} in the inner sum are real and have the same sign, or if the inner sum contains only one term. The first of these conditions is satisfied if the method A of summability is *positive* (*i.e.* if $a_{ij} \geqslant 0$ for all i and all j) and the second if the method P of rearrangement satisfies the following condition : for each n, if $r \in P_n$, then $r+1 \in P_{n+1}$ and (if $r > 1$) $r-1 \in P_{n+1}$. When this condition is satisfied we shall call the method P of rearrangement *rapid*.

If either of these conditions is satisfied we shall have

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$$\|f_{ik}\| = \sum_r \Sigma_1 |a_{ij}| = \Sigma_1 \sum_r |a_{ij}| = \sum_{j=1}^{k} |a_{ij}| \phi_j,$$

where ϕ_j is the number of integers r for which just one of r, $r+1$ belongs to P_j. Similarly

$$\|f_{il} - f_{ik}\| = \sum_{j=l+1}^{k} |a_{ij}| \phi_j.$$

Even when the modulus signs cannot be moved in (2), we have inequalities ; for example

$$\|f_{ik}\| = \sum_r |\Sigma_1 a_{ij}| \leqslant \sum_r \Sigma_1 |a_{ij}|.$$

These results are summarised in the next lemma.

LEMMA 3. *If the method* A *of summability is positive or the method* P *of rearrangement is rapid,*

$$\|f_{ik}\| = \sum_{j=1}^{k} |a_{ij}| \phi_j, \quad \|f_{il} - f_{ik}\| = \sum_{j=l+1}^{k} |a_{ij}| \phi_j.$$

For ***any*** *method* PA,

$$\|f_{ik}\| \leqslant \sum_{j=1}^{k} |a_{ij}| \phi_j, \quad \|f_{il} - f_{ik}\| \leqslant \sum_{j=l+1}^{k} |a_{ij}| \phi_j.$$

THEOREM 2. *Consider the three statements*

(i) *the method* PA *is regular,*

(ii) *the method* PA *is fully regular,*

(iii) *conditions* C2 *and* C3 *of* § 2 *are satisfied, together with*

$$\text{C1}'' : \textit{there is a constant } m \textit{ with } \sup_{i \geqslant m} \sum_{j=1}^{\infty} |a_{ij}| \phi_j < \infty.$$

Then (iii) ⇒ (ii) ⇒ (i). *If either the method* A *of summability is positive or the method* P *of rearrangement is rapid, then* (i) ⇔ (ii) ⇔ (iii).

Proof. Suppose first that (iii) holds. Then by Lemma 3, if $i \geqslant m$, $\|f_{il} - f_{ik}\| \to 0$ as $l, k \to \infty$ and so $f_{ik} \to f_i$ in the norm topology on (l^1). Also

$$\sup_{i \geqslant m} \|f_i\| = \sup_{i \geqslant m} \lim_{k \to \infty} \|f_{ik}\| \leqslant \sup_{i \geqslant m} \sum_{j=1}^{\infty} |a_{ij}| \phi_j < \infty.$$

Hence, by Theorem 1, (ii) holds (and therefore (i) holds).

Next let A be positive or P rapid and let (i) hold. Then by Lemma 2, there is a constant m with $\sup_k \|f_{ik}\| < \infty$ for all $i \geqslant m$. By Lemma 3 this shows that the series $\sum_{j=1}^{\infty} |a_{ij}| \phi_j$ is convergent. Hence $\|f_{il} - f_{ik}\| \to 0$ as $l, k \to \infty$, and so, as $k \to \infty$, the sequence (f_{ik}) converges in norm to some element of (l^1). But this element must be its $\sigma((l^1), (c))$-limit f_i, whose existence is guaranteed by Theorem 1. Thus, by Theorem 1, (ii) holds. Also

$$\sup_{i \geqslant m} \sum_{j=1}^{\infty} |a_{ij}| \phi_j = \sup_{i \geqslant m} \lim_{k \to \infty} \|f_{ik}\| = \sup_{i \geqslant m} \|f_i\| < \infty,$$

by Theorem 1, and this is C1″. Since C2 and C3 are consequences of (i), by Lemma 1, (iii) has been proved.

Corollary 1. *Suppose that* A *is a* **positive** *method of summability; if there is some method* P *of rearrangement for which the method* PA *is regular, then the method* A *is also regular. If* A *is* **any** *method of summability and if there is some* **rapid** *method* P *of rearrangement for which the method* PA *is regular, then the method* A *is also regular.*

Proof. Each $\phi_j \geqslant 1$ and so C1″ implies C1′.

Corollary 2. *If* A *is a positive regular method of summability, then the following three conditions are equivalent:*

(i) *the method* PA *is regular,*

(ii) *the method* PA *is fully regular,*

(iii) *the sequence* (ϕ_n) *is* A*-bounded.*

On putting $a_{ij} = 1$ if $i = j$ and $a_{ij} = 0$ if $i \neq j$ we obtain the answer to the question (ii) of § 1:

Corollary 3. *The method* P *of rearrangement leaves unaffected the convergence of every convergent series if and only if the sequence* (ϕ_n) *is bounded.*

We recall that ϕ_n was defined to be the number of integers r for which just one of $r, r+1$ belongs to P_n. In Theorem 2, ϕ_n can be replaced by a more natural quantity. Any finite set P of positive integers can be written as a disjoint union of intervals (*i.e.* runs of consecutive positive integers), separated by integers not in the set. The number of intervals in this (unique) partition of P is the smallest number of intervals whose union is P. Let ψ_n be the smallest number of intervals whose union is P_n. It is easy to see that just one of $r, r+1$ belongs to P_n only if either r is a right-hand end point or $r+1$ a left-hand end point of one of the intervals of P_n; hence $\phi_n = 2\psi_n$ if $1 \notin P_n$ and $\phi_n = 2\psi_n - 1$ if $1 \in P_n$. Thus in Theorem 2 and its corollaries, we may replace ϕ_n throughout by ψ_n.

5. Counterexamples. In Theorem 2, the equivalence of (i), (ii), and (iii) fails in the absence of any restriction on either the method A of summability or the method P of rearrangement. Also Corollaries 1 and 2 are no longer valid if A and P are unrestricted. In this section we justify these assertions by exhibiting the following examples:

(1) a regular method PA which is not fully regular,

(2) a fully regular method PA for which $\sum_{j=1}^{\infty} | a_{ij} | \phi_j = \infty$ for all i,

(3) a fully regular method PA for which A is not regular,

(4) a method PA which is not regular, but for which the sequence (ϕ_n) is A-bounded.

Example (1) shows that in Theorem 2, (i) does not imply (ii) and example (2) shows that (ii) does not imply (iii). Examples (3) and (4) refer to Corollaries 1 and 2 respectively.

The same method P of rearrangement will suffice for examples (1), (2) and (4). Any positive integer j can be written in the form $j = n^3 + r$, where $1 \leqslant r \leqslant 3n^2 + 3n + 1$. Put

$$P_{n^3+1} = \{1, 2, 3, \ldots, 2n^3, 2n^3+2, 2n^3+4, \ldots, 2n^3+2n^2\},$$

$$P_{n^3+2} = \{1, 2, 3, \ldots, 2n^3+2, 2n^3+4, 2n^3+6, \ldots, 2n^3+2n^2+2\},$$

$$P_{n^3+r} = \{1, 2, 3, \ldots, 2n^3+2n^2+r\}, \text{ for } 3 \leqslant r \leqslant 3n^2+3n+1.$$

Then clearly $\phi_{n^3+1} = \phi_{n^3+2} = 2n^2+1$ and $\phi_{n^3+r} = 1$ for $3 \leqslant r \leqslant 3n^2+3n+1$. Also

$$y_{n^3+1} = \sum_{r=2n^3}^{2n^3+2n^2} (-1)^r x_r, \quad y_{n^3+2} = \sum_{r=2n^3+2}^{2n^3+2n^2+2} (-1)^r x_r$$

and

$$y_{n^3+r} = x_{2n^3+2n^2+r} \text{ for } 3 \leqslant r \leqslant 3n^2+3n+1.$$

Now if we put

$$b_{ij}=1/i^3 \text{ for } 1 \leqslant j \leqslant i^3,$$
$$=0 \quad \text{otherwise},$$

the matrix (b_{ij}) defines a positive regular method B of summability. Since

$$\sum_{j=1}^{\infty} b_{ij}\phi_j = \frac{1}{i^3}\left(\sum_{j=1}^{i^3} 1 + 2\sum_{n=1}^{i-1} 2n^2\right) \leqslant 1+\frac{4}{3},$$

the method PB is fully regular (Theorem 2, Corollary 2).

Next suppose that $\alpha > 1$ and put

$$a_{i,\, n^3+1} = +1/n^\alpha, \quad a_{i,\, n^3+2} = -1/n^\alpha,$$

for each positive integer $n \geqslant i$, and let $a_{ij}=b_{ij}$ otherwise. Then the conditions C1, C2 and C3 are easily verified and so the method A of summability defined by the matrix (a_{ij}) is regular. Also, since

$$\sum_{j=1}^{\infty} a_{ij}\phi_j = \sum_{j=1}^{\infty} b_{ij}\phi_j$$

the sequence (ϕ_n) is A-bounded. But

$$\sum_{j=1}^{\infty} |a_{ij}|\, \phi_j = \sum_{j=1}^{\infty} b_{ij}\phi_j + \sum_{n=i}^{\infty} \frac{2(2n^2+1)}{n^\alpha}.$$

Since the last series converges only for $\alpha > 3$, the condition C1″ holds only if $\alpha > 3$.

We now investigate the values of α for which the method PA is regular and the values for which it is fully regular.

$$\sum_{j=1}^{k} a_{ij}y_j = \sum_{j=1}^{k} b_{ij}y_j + h_{ik}(x), \quad \text{say},$$

where

$$h_{ik}(x) = \sum_{n \geqslant i,\, n^3+1 \leqslant k} n^{-\alpha} y_{n^3+1} - \sum_{n \geqslant i,\, n^3+2 \leqslant k} n^{-\alpha} y_{n^3+2}.$$

Since the method PB is fully regular, the method PA is regular if and only if, for each $x \in (c)$, $\lim_{k\to\infty} h_{ik}(x)$ exists and tends to zero as $i \to \infty$. Also the method PA is fully regular if (and only if) also, for each $x \in (m)$, $\lim_{k\to\infty} h_{ik}(x)$ exists and is bounded with respect to i.

Now if $n \geqslant i$ and $2 \leqslant r \leqslant 3n^2+3n+1$,

$$|h_{i,\, n^3+r}(x)| = \left|\sum_{m=i}^{n} m^{-\alpha}(y_{m^3+1} - y_{m^3+2})\right|$$
$$= \left|\sum_{m=i}^{n} m^{-\alpha}(x_{2m^3} - x_{2m^3+1} + x_{2m^3+2m^2+1} - x_{2m^3+2m^2+2})\right|$$
$$\leqslant \sum_{m=i}^{\infty} m^{-\alpha}\, 4\, \|x\| \to 0 \text{ as } i \to \infty,$$

for all $x \in (m)$, since $\alpha > 1$. However

$$h_{i,\, n^3+1}(x) = h_{i,\, n^3}(x) + n^{-\alpha} y_{n^3+1},$$

and, while the first term is of the type already considered, the second one is

$$n^{-\alpha}\sum_{r=2n^3}^{2n^3+2n^2}(-1)^r x_r = n^{-\alpha}\left(x_{2n^3}-\sum_{s=n^3+1}^{n^3+n^2}(x_{2s-1}-x_{2s})\right).$$

If $x \in (c)$ this term is $o(n^{2-\alpha})=o(1)$ as $i \to \infty$, if $\alpha \geqslant 2$. Hence the method PA is regular for $\alpha \geqslant 2$. Also if $x \in (m)$ this term is $O(n^{2-\alpha})=o(1)$ as $i \to \infty$ when $\alpha > 2$, and so the method PA is fully regular for $\alpha > 2$. But if $\alpha=2$ the method PA is not fully regular, for then $h_{i,\,n^3+1}(x)-h_{i,\,n^3}(x)=n^{-2}(2n^2+1)\geqslant 2$ for the bounded sequence (x_n) for which $x_n=(-1)^n$. Thus $\lim_{k\to\infty} h_{ik}(x)$ does not exist for all $x \in (m)$. Also if $1<\alpha<2$ the method PA is not regular; in fact if $x_n=(-1)^n/\log n$,

$$y_{n^3+1}=\sum_{r=2n^3}^{2n^3+2n^2}\frac{1}{\log r}>\frac{2n^2+1}{\log(2n^3+2n^2)}$$

and so

$$h_{i,\,n^3+1}(x)-h_{i,\,n^3}(x)>2n^{2-\alpha}/\log(2n^3+2n^2).$$

Thus $\lim_{k\to\infty} h_{ik}(x)$ does not exist for all $x \in (c)$.

To obtain counterexample (1) take $\alpha=2$; for example (2) take any α in the range $2<\alpha\leqslant 3$ and, for example (4), take any α with $1<\alpha<2$.

Example (3) is simpler. For each positive integer n, and each r with $1\leqslant r\leqslant n$ put

$$\begin{aligned}
P_{n^2} &=\{1, 2, 3, \dots, n^2\},\\
P_{n^2+2r-1} &=\{1, 2, 3, \dots, n^2+r-1, n^2+n+1, n^2+n+2, \dots, n^2+n+r\},\\
P_{n^2+2r} &=\{1, 2, 3, \dots, n^2+r, n^2+n+1, n^2+n+2, \dots, n^2+n+r\}.
\end{aligned}$$

Then $y_{n^2}=x_{n^2}$ and $y_{n^2+2r}-y_{n^2+2r-1}=x_{n^2+r}-x_{n^2+r-1}$. Also take $a_{i,\,i^2+r}=(-1)^r$ for $0\leqslant r\leqslant 2i$ and put $a_{ij}=0$ otherwise. Then

$$\begin{aligned}
\sum_{j=1}^{\infty} a_{ij}y_j &= \sum_{r=0}^{2i}(-1)^r y_{i^2+r}\\
&= x_{i^2}+\sum_{r=1}^{i}(x_{i^2+r}-x_{i^2+r-1})\\
&= x_{i^2+i}.
\end{aligned}$$

Hence, for each $x \in (c)$, $\sum_{j=1}^{\infty} a_{ij}y_j$ converges to $\lim_{n\to\infty} x_n$ as $i\to\infty$, and so the method PA thus defined is regular. In fact, it is easily seen to be fully regular. On the other hand, the method A does not satisfy the condition C1 (nor C1′) and so it is not regular.

It is perhaps worth pointing out that the method P of rearrangement used for this example arises from a permutation σ defined by

$$\sigma(n^2)=n^2,\ \sigma(n^2+2r-1)=n^2+n+r,\ \sigma(n^2+2r)=n^2+r,$$

for $1\leqslant r\leqslant n$.

6. An extension of Riemann's theorem. If A is any regular method of summability, there is always a series with bounded partial sums which is not summed by the method A. (See, for example, Hardy [5], Theorems 2 and 3; the conditions that A shall be regular and that A shall sum every series with bounded partial sums are inconsistent.) The theorem proved below represents a simultaneous generalisation of this result and of Riemann's theorem referred to in § 1. It also answers question (iii) of § 1.

THEOREM 3. *Given any conditionally convergent series* $\sum_{n=1}^{\infty} u_n$ *and any regular method* A *of summability, there is always a rearrangement* $\sum_{n=1}^{\infty} u_{\sigma(n)}$ *which has bounded partial sums but is not* A-*summable.*

Proof. To construct σ we divide the positive integers into blocks by means of a strictly increasing sequence $(j(n))$ of positive integers; for each n we define a permutation of the set of r satisfying $j(2n-1) < r \leqslant j(2n+1)$ and piece these together to form σ. The purpose of the even terms $j(2n)$ will appear later.

Write as before $x_n = \sum_{r=1}^{n} u_r$ and $y_n = \sum_{r=1}^{n} u_{\sigma(r)}$ (when $\sigma(r)$ has been defined for $r \leqslant n$), and let $s = \lim_{n\to\infty} x_n$. Also put $k = 4 + \sup_n |x_n|$. We shall so construct σ that

$$|y_r| \leqslant k \text{ for all } r. \qquad \text{......................................(3)}$$

First, (x_n) is A-summable to s and so there is a positive integer $i(1)$ such that

$$\left|\sum_{j=1}^{\infty} a_{i(1),j}\, x_j - s\right| < \frac{1}{5}.$$

Also, by C1 (or C1′), there is a positive integer $j(1)$ such that

$$\sum_{j=j(1)+1}^{\infty} |a_{i(1),j}| < \frac{1}{10k},$$

and we may suppose $j(1)$ chosen so that also

$$|x_p - x_q| < 1 \text{ for all } p, q \geqslant j(1). \qquad \text{..............................(4)}$$

Put $\sigma(r) = r$ for $1 \leqslant r \leqslant j(1)$; then, for these values of r, $|y_r| = |x_r| \leqslant k$ and so (3) is satisfied. Then, however $\sigma(r)$ is defined for $r > j(1)$, provided only that (3) still holds, we shall have $|x_r - y_r| \leqslant |x_r| + |y_r| \leqslant 2k$, and so

$$\left|\sum_{j=1}^{\infty} a_{i(1),j}\, y_j - s\right| < \frac{1}{5} + 2k \cdot \frac{1}{10k} = \frac{2}{5}. \qquad \text{.....................(5)}$$

Now if I_n denotes the set $\{j(1)+1, j(1)+2, \dots, n\}$, write $c_n = \sup_{F \subseteq I_n} \left| y_{j(1)} + \sum_{r \in F} u_r - s \right|$.

Then, by (4), $c_{n+1} < c_n + 1$ and $c_{j(1)} \leqslant 1$. Also $c_n \to \infty$ as $n \to \infty$. (For a series Σu_n with real terms this is familiar. If the terms are complex, either the series of real parts or the series of imaginary parts must converge conditionally; suppose the former. Then, by picking out terms u_r for $r > j(1)$ with positive real parts we can make $\left|\sum_{r \in F} u_r\right|$ arbitrarily large.) Hence there is an n' with $2 < c_{n'} < 3$; let the upper bound $c_{n'}$ be attained when $F = F'$. Then if $\sigma(j(1)+1), \sigma(j(1)+2), \dots, \sigma(m)$, say, are defined to be the elements of F' in order, we shall have, for $j(1) < r \leqslant m$,

$$|y_r - s| \leqslant c_{n'} < 3 \quad \text{and} \quad |y_m - s| = c_{n'} > 2.$$

Also, for these values of r, $|y_r| < |s|+3 \leqslant k$ as required by (3). Now the series

$$u_{\sigma(1)}+u_{\sigma(2)}+\ldots+u_{\sigma(m)}+u_{\sigma(m)+1}+u_{\sigma(m)+2}+\ldots$$

converges to a sum s', say, which, by (4), satisfies $|y_m - s'| \leqslant 1$, and therefore $|s-s'| \geqslant 1$. If z_n denotes the sum to n terms of this series, we can find, as before, $i(2) > i(1)$ such that

$$\left|\sum_{j=1}^{\infty} a_{i(2),j}\, z_j - s'\right| < \frac{1}{5},$$

and $j(2) \geqslant m$ such that

$$\sum_{j=j(2)+1}^{\infty} |a_{i(2),j}| < \frac{1}{10k}.$$

Put $\sigma(r)=\sigma(m)+r-m$ for $m < r \leqslant j(2)$; then for these values of r,

$$|y_r - y_m| = \left|\sum_{n=\sigma(m)+1}^{\sigma(m)+r-m} u_n\right| < 1,$$

by (4). Hence $|y_r| \leqslant |y_m|+1 < |s|+3+1 \leqslant k$. Thus $|y_r| \leqslant k$ for $m < r \leqslant j(2)$, as required by (3). Similarly, $|z_r| \leqslant k$ for all r. Also, however $\sigma(r)$ is defined for $r > j(2)$, provided only that (3) still holds, we shall have $|z_r - y_r| \leqslant 2k$ for all r, and so, as for (5),

$$\left|\sum_{j=1}^{\infty} a_{i(2),j}\, y_j - s'\right| < \frac{2}{5}.$$

Since $|s-s'| \geqslant 1$, this with (5) gives

$$\left|\sum_{j=1}^{\infty} a_{i(1),j}\, y_j - \sum_{j=1}^{\infty} a_{i(2),j}\, y_j\right| > \frac{1}{5}.$$

We can now define $\sigma(r)$ for $j(2) < r \leqslant m'$, say, so as to fill in the gaps left in $I_{n'}$. For these values of r we shall have

$$|y_r - s| = \left|y_{j(1)} + \sum_{r\in F} u_r + \sum_{r=\sigma(m)+1}^{\sigma(j(2))} u_r - s\right|,$$

where F is some finite subset of $I_{n'}$. Hence by the method of construction of F' and (4),

$$|y_r - s| \leqslant c_{n'}+1 < 3+1=4,$$

and so $|y_r| \leqslant k$, as required by (3). Then the series

$$u_{\sigma(1)}+u_{\sigma(2)}+\ldots+u_{\sigma(m')}+u_{\sigma(m')+1}+u_{\sigma(m')+2}+\ldots$$

has x_n for its nth partial sum for $n \geqslant m'$, and therefore it converges to s. Thus we can find $i(3) > i(2)$ and $j(3) \geqslant m'$, as before, such that, if we put $\sigma(r)=\sigma(m')+r-m'$ for $m' < r \leqslant j(3)$, then, for these values of r, $|y_r|=|x_r| \leqslant k$. Then, however $\sigma(r)$ is defined for $r > j(3)$, provided only that (3) still holds, we shall have

$$\left|\sum_{j=1}^{\infty} a_{i(3),j}\, y_j - s\right| < \frac{2}{5},$$

and therefore

$$\left| \sum_{j=1}^{\infty} a_{i(2),j}\, y_j - \sum_{j=1}^{\infty} a_{i(3),j}\, y_j \right| > \frac{1}{5}.$$

This process can clearly be continued ; we next rearrange the terms of Σu_n for $n > j(3)$ to make it converge to a new sum s'' with $|s - s''| \geqslant 1$, and so on. The permutation σ is thus built up step by step ; at each stage (3) is satisfied and we have, for all n,

$$\left| \sum_{j=1}^{\infty} a_{i(n),j}\, y_j - \sum_{j=1}^{\infty} a_{i(n+1),j}\, y_j \right| > \frac{1}{5}.$$

Since $i(n) \to \infty$ as $n \to \infty$, this shows that $\Sigma u_{\sigma(n)}$ is not A-summable, though its sequence (y_n) of partial sums is bounded, which completes the proof of the theorem.

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https://doi.org/10.1017/S2040618500033694 Published online by Cambridge University Press