CORRESPONDING POLYHEDRA IN THE THREE SPACES OF CONSTANT CURVATURE

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1. Introduction. The five Platonic solids can be drawn in elliptic or hyperbolic space just as well as in Euclidean space. Their numerical properties are, of course, the same in all three. So are the various angles subtended at the centre. But the face-angles and dihedral angles are greater in elliptic space, smaller in hyperbolic. It is a special feature of the non-Euclidean spaces that we cannot change the size of a solid without changing its shape. The edge-length $2l$, circum-radius R, and in-radius r are conveniently expressed as functions of the *mid-radius* ρ: the distance from the centre to the mid-point of an edge. With each non-Euclidean polyhedron we shall associate a corresponding Euclidean polyhedron of a definite size, whose mid-radius serves as a parameter for expressing all the metrical properties of the non-Euclidean polyhedron. When an elliptic polyhedron and a hyperbolic polyhedron have the same parameter, the edge-length of the former is the complement of the angle of parallelism corresponding to the edge-length of the latter.

For these purposes it is immaterial whether we take the space of positive curvature to be elliptic (single-elliptic) or spherical (double-elliptic). To simplify the formulae we shall employ the natural unit of length. In hyperbolic space this means that a horocyclic arc of length 1 has the tangent at either end parallel to the diameter through the other end.

2. Corresponding polygons. Let us begin by recapitulating the results of an earlier paper [**5**].

In Euclidean space, spherical triangles are usually drawn on a sphere of radius 1, because then the geodesic lengths of their sides are equal to the angles subtended at the centre of the sphere. In hyperbolic space the same result is achieved by using a sphere of radius

$$\arg\sinh 1 = \log(\sqrt{2} + 1).$$

On such a sphere, consider a circle of angular radius $\sigma_s \leqslant \frac{1}{2}\pi$ or straight radius σ_h, so that

$$\sinh \sigma_h = \sin \sigma_s \leqslant 1. \tag{1}$$

Through this circle draw a plane and a horosphere (say, for definiteness, the horosphere curved in the same direction as the sphere). Draw p tangents to the circle in its plane, so as to form a regular p-gon, $\{p\}$, of in-radius σ_h and side

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$2l_h$, say. With the same points of contact, draw p great-circular tangents on the sphere, and p horocyclic tangents on the horosphere, so as to form a spherical $\{p\}$ of in-radius σ_s and side $2l_s$, and a horospherical $\{p\}$ of in-radius σ_e and side

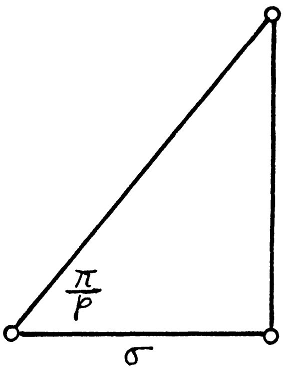

FIG. 1

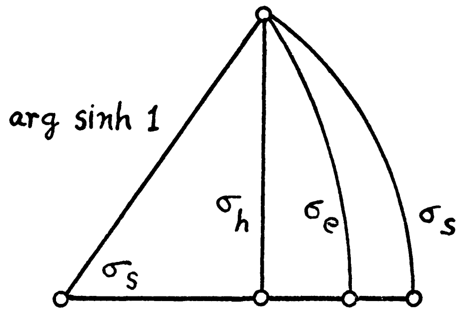

FIG. 2

$2l_e$. The relation between each in-radius and side can be computed from a right-angled triangle (Fig. 1). Using the appropriate trigonometry (hyperbolic, spherical, or Euclidean), we find

$$(2) \qquad \tanh l_h = \sinh \sigma_h \tan \frac{\pi}{p}, \qquad \tan l_s = \sin \sigma_s \tan \frac{\pi}{p}, \qquad l_e = \sigma_e \tan \frac{\pi}{p}.$$

Since σ_h is the semi-chord of a circular arc σ_s and of a horocyclic arc σ_e (Fig. 2), we have

$$(3) \qquad \sinh \sigma_h = \sin \sigma_s = \sigma_e$$

[**7**, p. 62, with $S = 1$], and therefore

$$(4) \qquad \tanh l_h = \tan l_s = l_e.$$

This last result can also be seen directly, since l_s and l_h are a circular arc and its tangent in a diametral plane of the sphere, while l_e and l_h are a horocyclic arc and its tangent in a diametral plane of the horosphere. In particular, when the hyperbolic $\{p\}$ is "asymptotic" (with its vertices at infinity),

$$l_h = \infty, \qquad l_e = 1.$$

By (1) and (2), $\tanh l_h \leqslant \tan \pi/p$; therefore a hyperbolic $\{p\}$ has no corresponding spherical $\{p\}$ if

$$\tanh l_h > \tan \frac{\pi}{p}.$$

(This failure cannot occur when $p = 3$ or 4, but may occur for a pentagon or higher polygon.) By (4),

$$\tan l_s = \tanh l_h \leqslant 1;$$

therefore a spherical $\{p\}$ has no corresponding hyperbolic $\{p\}$ if $l_s > \frac{1}{4}\pi$, that is, if the $\{p\}$ is a spherical triangle whose sides are obtuse. Thus the only regular polygon for which the correspondence holds both ways, for every possible length of side, is the *square.*

The equation $\tanh l_h = \tan l_s$ may be expressed geometrically [**4**, p. 3] by saying that $2l_s$ is the complement of Lobatschefsky's angle of parallelism for the distance $2l_h$.

3. Regular polyhedra. Analogous considerations in hyperbolic 4-space yield a definition for "corresponding" polyhedra. Instead of a common in-circle, these have a common *mid-sphere* (sphere touching all the edges) with its radius measured three different ways.

Let $\{p, q\}$, where $(p - 2)(q - 2) < 4$, denote a regular polyhedron whose faces are p-gons, q at each vertex, in Euclidean or non-Euclidean space, so that the number of edges is [**2**, p. 13]

$$E = 2pq/ [4 - (p - 2)(q - 2)].$$

Let ϕ and ψ denote the angles subtended at the centre by a half-edge and by the in-radius of a face. Then

$$(5)\qquad \begin{aligned} \cos\phi &= \frac{\cos \pi/p}{\sin \pi/q}, & \sin\phi &= \frac{\sin \pi/h}{\sin \pi/q}, & \tan\phi &= \frac{\sin \pi/h}{\cos \pi/p}, \\ \cos\psi &= \frac{\cos \pi/q}{\sin \pi/p}, & \sin\psi &= \frac{\sin \pi/h}{\sin \pi/p}, & \tan\psi &= \frac{\sin \pi/h}{\cos \pi/q}, \end{aligned}$$

where $h = \sqrt{(4E + 1)} - 1$ [**2**, pp. 19, 21].

Let R denote the circum-radius, ρ the mid-radius, r the in-radius, σ the in-radius of a face, 2α the dihedral angle, and 2θ the face-angle. These can be

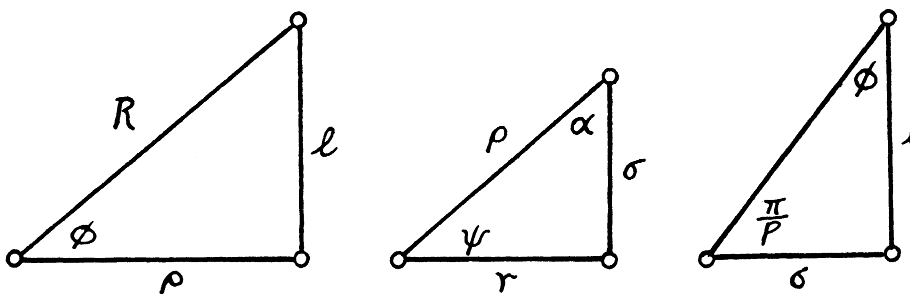

FIG. 3

computed by solving the three right-angled triangles shown in Figure 3. The results are collected in Table I.

The ρ and r of [**4**] have been interchanged for the sake of agreement with Sommerville [**7**, p. 123]. Note, however, that his

are our	a,	α,	θ,	δ,	n,	p,	r_a
	$2l$,	2ϕ,	2θ,	2α,	p,	q,	σ.

For the extension to n dimensions see [**8**, p. 22].

In terms of Schläfli functions [**1**, pp. 15, 23, 28], the volume of the spherical $\{p, q\}$ is $ES(\frac{1}{2}\pi - \pi/p, \pi/q, \frac{1}{2}\pi - \alpha)$, while that of the hyperbolic $\{p, q\}$ is $EiS(\frac{1}{2}\pi - \pi/p, \pi/q, \frac{1}{2}\pi - \alpha)$.

TABLE I

THE GENERAL REGULAR POLYHEDRON

Elliptic or spherical space	Euclidean space	Hyperbolic space
$\tan l = \sin \rho \tan \phi$	$l = \rho \tan \phi$	$\tanh l = \sinh \rho \tan \phi$
$\tan R = \tan \rho \sec \phi$	$R = \rho \sec \phi$	$\tanh R = \tanh \rho \sec \phi$
$\tan r = \tan \rho \cos \psi$	$r = \rho \cos \psi$	$\tanh r = \tanh \rho \cos \psi$
$\sin \sigma = \sin \rho \sin \psi$	$\sigma = \rho \sin \psi$	$\sinh \sigma = \sinh \rho \sin \psi$
$\cot \alpha = \cos \rho \tan \psi$	$\alpha = \frac{1}{2}\pi - \psi$	$\cot \alpha = \cosh \rho \tan \psi$
$\sin \theta = \sec l \cos \pi/p$	$\theta = \frac{1}{2}\pi - \pi/p$	$\sin \theta = \operatorname{sech} l \cos \pi/p$

4. A digression on rectangular polyhedra. It is interesting to see which non-Euclidean polyhedra $\{p, q\}$ share with the Euclidean cube $\{4, 3\}$ the property that the edges at each vertex form an orthogonal trihedron. This requires

$$q = 3, \qquad \alpha = \theta = \tfrac{1}{4}\pi.$$

Since θ is greater or less than $\frac{1}{2}\pi - \pi/p$ according as the space is elliptic or hyperbolic, we have $p < 4$ in the former case and $p > 4$ in the latter. Thus we find an elliptic rectangular $\{3, 3\}$ with

$$\cos \rho = \cot \psi = \sqrt{\tfrac{1}{2}},$$

a hyperbolic rectangular $\{5, 3\}$ with

$$\cosh \rho = \cot \psi = \tau, \qquad \tau = \tfrac{1}{2}(\sqrt{5} + 1),$$

and also, if we allow ρ to be infinite, a hyperbolic rectangular $\{6, 3\}$ inscribed in a horosphere. In each case the edge $2l$ is given by

$$\cos l \quad \text{or} \quad \cosh l = \sqrt{2} \cos \pi/p.$$

Table II lists the chief properties of these polyhedra, with the cube of mid-radius 1 for comparison. (Note that the mid-radius of the rectangular dodecahedron is equal to its edge: $\rho = 2l$.)

By repeatedly reflecting such a solid $\{p, 3\}$ in its faces, we obtain a regular honeycomb $\{p, 3, 4\}$, filling the whole space.

$\{3, 3, 4\}$, when drawn on a hypersphere in Euclidean 4-space, has the same

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vertices as the regular cross polytope β_4, analogous to the octahedron β_3 [**2**, p. 121]. There are sixteen cells $\{3, 3\}$. By identifying antipodes we derive a honeycomb of eight rectangular tetrahedra filling elliptic space [**7**, p. 124].

$\{4, 3, 4\}$ is the ordinary space-filling of cubes.

$\{5, 3, 4\}$ was described by Schlegel and Sommerville [**6**, p. 444; **8**, p. 17].

$\{6, 3, 4\}$ was described by Coxeter and Whitrow [**3**, pp. 426, 427].

TABLE II

THE RECTANGULAR POLYHEDRA

Elliptic or spherical $\{3,3\}$		Euclidean $\{4,3\}$	Hyperbolic $\{5,3\}$	Hyperbolic $\{6,3\}$
$R = \frac{1}{3}\pi$,	$\sin^2 R = \frac{3}{4}$	$R^2 = \frac{3}{2}$	$\sinh^2 R = \frac{3}{2}\tau$	$R = \infty$
$\rho = \frac{1}{4}\pi$,	$\sin^2 \rho = \frac{1}{2}$	$\rho^2 = 1$	$\sinh^2 \rho = \tau$	$\rho = \infty$
$r = \frac{1}{6}\pi$,	$\sin^2 r = \frac{1}{4}$	$r^2 = \frac{1}{2}$	$\sinh^2 r = \frac{1}{2}\tau$	$r = \infty$
$l = \frac{1}{4}\pi$,	$\cos 2l = 0$	$2l = \sqrt{2}$	$\cosh 2l = \tau$	$\cosh 2l = 2$

5. Corresponding polyhedra. The formulae in Table I apply to polyhedra $\{p, q\}$ in the three spaces separately. In each case we can assign a value to ρ and deduce all the other properties. *Corresponding* polyhedra are given by identifying the Euclidean ρ with the elliptic $\sin \rho$ and the hyperbolic $\sinh \rho$, say

$$\sin \rho_s = \rho_e = \sinh \rho_h,$$

whence

$$\tan l_s = l_e = \tanh l_h, \qquad 2l_s = \tfrac{1}{2}\pi - \Pi(2l_h),$$

and

$$\sin \sigma_s = \sigma_e = \sinh \sigma_h.$$

The remaining relations are not quite so simple, but we can express all the properties in terms of ρ_e and l_e ($= \rho_e \tan \phi$), as follows:

$$\tan R_s = \frac{\rho_e}{\sqrt{(1-\rho_e^2)}} \sec \phi, \quad R_e = \rho_e \sec \phi, \quad \tanh R_h = \frac{\rho_e}{\sqrt{(1+\rho_e^2)}} \sec \phi;$$

$$\tan r_s = \frac{\rho_e}{\sqrt{(1-\rho_e^2)}} \cos \psi, \quad r_e = \rho_e \cos \psi, \quad \tanh r_h = \frac{\rho_e}{\sqrt{(1+\rho_e^2)}} \cos \psi;$$

$$\cot \alpha_s = \sqrt{1-\rho_e^2} \tan \psi, \quad \alpha_e = \tfrac{1}{2}\pi - \psi, \quad \cot \alpha_h = \sqrt{1+\rho_e^2} \tan \psi;$$

$$\sin \theta_s = \sqrt{1+l_e^2} \cos \frac{\pi}{p}, \quad \theta_e = \tfrac{1}{2}\pi - \frac{\pi}{p}, \quad \sin \theta_h = \sqrt{1-l_e^2} \cos \frac{\pi}{p}.$$

It is interesting to observe that

$$\cot^2 \alpha_s + \cot^2 \alpha_h = 2 \tan^2 \psi = 2 \cot^2 \alpha_e$$

and

$$\sin^2 \theta_s + \sin^2 \theta_h = 2 \cos^2 \frac{\pi}{p} = 2 \sin^2 \theta_e,$$

whence

$$\cos 2\theta_s + \cos 2\theta_h = 2 \cos 2\theta_e.$$

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6. Limitations of the correspondence. In elliptic or spherical space, the largest possible $\{p, q\}$ has

$$R_s = \rho_s = r_s = \tfrac{1}{2}\pi, \qquad l_s = \phi, \qquad \sigma_s = \psi, \qquad \alpha_s = \tfrac{1}{2}\pi, \qquad \theta_s = \pi/q.$$

Referring to a table of the five Platonic solids [**2**, p. 293], we see that $\phi > \frac{1}{4}\pi$ for the tetrahedron $\{3, 3\}$, $\phi = \frac{1}{4}\pi$ for the octahedron $\{3, 4\}$, and $\phi < \frac{1}{4}\pi$ for the remaining three. Since

$$\tan l_s = \tanh l_h \leqslant 1,$$

an elliptic $\{p, q\}$ has no corresponding hyperbolic $\{p, q\}$ if $l_s > \frac{1}{4}\pi$, that is, if the elliptic $\{p, q\}$ is a tetrahedron of edge greater than $\frac{1}{2}\pi$.

In hyperbolic space, the largest possible $\{p, q\}$ has

$$R_h = l_h = \infty, \quad \tanh \rho_h = \cos \phi, \quad \sinh \rho_h = \cot \phi, \quad \theta_h = 0,$$

$$\tanh r_h = \cos\phi \cos\psi = \cot\frac{\pi}{p}\cot\frac{\pi}{q},$$

$$\sinh \sigma_h = \cot\phi \sin\psi = \cot\frac{\pi}{p},$$

$$\cot \alpha_h = \csc\phi \tan\psi = \tan\frac{\pi}{q},$$

[**1**, p. 28]. Since $\tanh l_h = \sinh \rho_h \tan\phi = \sin \rho_s \tan\phi \leqslant \tan\phi$, a hyperbolic $\{p, q\}$ has no corresponding elliptic $\{p, q\}$ if

$$\tanh l_h > \tan \phi$$

(which implies $\phi < \frac{1}{4}\pi$). Thus, for the correspondence to hold, a hyperbolic hexahedron or dodecahedron or icosahedron must not be too large.

The only regular polyhedron for which the correspondence holds both ways, for every possible size, is the *octahedron*.

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