ON THE FORCED LIENARD EQUATION

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We consider the second order differential equation

$$(1) \qquad \ddot{x} + f(x)\dot{x} + x = p(t) \qquad \left(\cdot = \frac{d}{dt}\right)$$

with the assumptions that

(2) $f(x)$ is continuous $(-\infty < x < \infty)$ and $p(t)$ is continuous and bounded: $|p(t)| \leq E$, $-\infty < t < \infty$.

Also, throughout this paper, $F(x)$ denotes an antiderivative of $f(x)$.

Many results are known concerning periodicity and boundedness of solutions of (1) under various conditions, all of which include the assumptions in (2). For example, a special case of a result of W.S. Loud [4] concerning bounded solutions of (1) is the following which we state in a form suitable for our purposes here.

THEOREM 1. *If there is a constant* $c > 0$ *such that* $f(x) \geq c$ *for all real* x *and* $x(t)$ *is any solution of* (1), *then there exists* t_0 *such that for all* $t \geq t_0$, $|x(t)| + |\dot{x}(t)| \leq K$, *the constant* K *depending only on* E *and* c.

Remark 1. We note that the statement of Theorem 1 above, as given in [3], with the hypothesis "$f(x) \geq c$" replaced by "$|f(x)| \geq c$" is false, as easily constructed examples show. (See Example 1 below.)

More recently, Frederickson and Lazer [2] have proved the following result concerning periodic solutions of (1).

THEOREM 2. *If* $f(x) > 0$ *for all real* x *and* $p(t)$ *is* 2π-*periodic, then (1) has a* 2π-*periodic solution if and only if*

$$(3) \qquad \left| \int_0^{2\pi} p(t)\, e^{it}\, dt \right| < 2 \int_{-\infty}^{\infty} f(x)\, dx .$$

It is our purpose here to investigate solutions $x(t)$ of (1) which satisfy

$$\text{(I)} \qquad x(t) = o(t) \text{ and } \dot{x}(t) = o(t) \text{ as } t \to +\infty ,$$

or

$$\text{(II)} \qquad x(t) = O(t) \text{ and } \dot{x}(t) = O(t) \text{ as } t \to +\infty .$$

Clearly, conditions (I) and (II) are weaker than those of boundedness and periodicity of Theorems 1 and 2 and, accordingly, our main result (Theorem 4) gives a necessary condition that solutions of (1) satisfy (I). This condition is similar to (3) above. Sufficient conditions are stated in Theorem 5.

The following example shows that conditions more stringent than those in (2) must be placed on equation (1) in order that its solution satisfy condition (II). Such conditions are given in Theorem 3.

<u>Example 1.</u> Let $p(t) \equiv 0$ and $f(x) = -2x - \frac{1}{x}$ for all $x \geq 1$ and $f(x) = -3$ for $x < 1$. Then the solution of the resulting equation (1) satisfying $x(0) = 1$, $\dot{x}(0) = 1$ is

$$x(t) = \frac{1}{1-t} \qquad \text{for} \qquad 0 \leq t < 1 .$$

In this case, not all solutions of (1) exist in the future.

THEOREM 3. <u>If (2) holds and</u>

$$(\operatorname{sgn} x)F(x) \geq -M \quad \underline{\text{and}} \quad |F(x)| \leq C|x|$$

<u>holds for all real</u> x, <u>where</u> M <u>and</u> C <u>are positive constants</u>, <u>then all solutions</u> $x(t)$ <u>of (1) exist in the future and satisfy (II).</u>

<u>Proof.</u> Equation (1) is equivalent to the system

$$\text{(1')} \qquad \begin{aligned} \dot{x} &= y - F(x) \\ \dot{y} &= p(t) - x . \end{aligned}$$

Let $V(x, y) = x^2 + y^2$. Then along a solution $x(t)$, $y(t)$ of (1):

$$\dot{V} = -2xF(x) + 2yp(t) \le 2\,M|x| + 2E|y| \le 4(M+E)\,V^{\frac{1}{2}}$$

and thus, for all t for which $V(t) \neq 0$, we have letting $B = 4(M+E)$

$$\frac{d}{dt}\left(2\,V^{\frac{1}{2}} - Bt\right) \le 0 ,$$

$2V^{\frac{1}{2}} - Bt$ is decreasing and

$$-Bt \le 2V^{\frac{1}{2}} - Bt \le 2\,V^{\frac{1}{2}}(t_0) - Bt_0 ,$$

$$0 \le 2V^{\frac{1}{2}}(t) \le 2V^{\frac{1}{2}}(0) - Bt_0 + Bt ,$$

which shows that $V^{\frac{1}{2}}(t) = O(t)$ as $t \to +\infty$. Thus $x = O(t)$ and $y = O(t)$ and since $\dot{x} = y - F(x)$, $\dot{x} = O(t)$ also.

THEOREM 4. *If (2) holds and* $p(t)$ *is* 2π-*periodic then there exists a solution* $x(t)$ *of (1) satisfying condition (I) only if*

$$(4) \qquad \left| \int_0^{2\pi} p(t)\, e^{it}\, dt \right| \le 2 \operatorname{Rng} F .$$

Here $\operatorname{Rng} F \equiv \sup |F(a) - F(b)|$ where the supremum is taken over all a, b in the domain of the function $F(x)$ (all real a, b).

For the proof of Theorem 4, we require the following inequality.

LEMMA 1. *If* $h(t)$ *is a bounded Lebesgue measurable function for* $0 \le t \le 2\pi$, *then*

$$(5) \qquad \left| \int_0^{2\pi} h(t)\, e^{it}\, dt \right| \le 2 \operatorname{Rng} h .$$

Proof. It is clearly no restriction to assume that

$$0 \le h(t) \le 1, \quad \operatorname{Rng} h = 1$$

and that $h(t)$ is a step function:

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$$h(t) = \alpha_k \text{ for } a_{k-1} \leq t < a_k \ , \qquad \text{where}$$

$$0 = a_0 < a_1 < \ldots < a_n = 2\pi \quad \text{and} \quad 0 \leq \alpha_k \leq 1 \ .$$

Then

$$\left| \int_0^{2\pi} h(t)\, e^{it}\, dt \right| = \left| \sum_{k=1}^{n} \alpha_k \left(e^{ia_k} - e^{ia_{k-1}}\right) \right| . \tag{6}$$

The indicated sum in (6) lies inside the set

$$\left\{ \sum_{k=1}^{n} \beta_k \left(e^{ia_k} - e^{ia_{k-1}}\right) ; \ 0 \leq \beta_k \leq 1 \right\} ,$$

which is a convex polygon whose vertices are a subset of

$$\left\{ \sum_{k=1}^{n} \beta_k \left(e^{ia_k} - e^{ia_{k-1}}\right) ; \ \beta_k = 0 \text{ or } 1 \right\} , \tag{7}$$

and each element of (7) is not greater in modulus than the diameter of the unit circle.

$$\{ z \ ; \ |z| = 1 \} \ .$$

Hence, $\left| \int_0^{2\pi} h(t)\, e^{it}\, dt \right| \leq 2 \leq 2 \text{ Rng } h,$ as desired.

Remark 2. Lemma 1 represents a generalization of an old result of de La Vallée Poussin [1; p. 16] who proved, using a modulus of continuity argument, that under the hypothesis of Lemma 1,

$$\left| \int_0^{2\pi} h(t)\, e^{it}\, dt \right| \leq \sqrt{2\pi} \text{ Rng } h.$$

Proof of Theorem 4. Suppose that $x(t)$ is a solution of (1) satisfying condition (I). Multiplying (1) by e^{it} and integrating,

$$\int_0^{2n\pi} [\ddot{x}\ e^{it} + f(x)\dot{x}\ e^{it} + x\ e^{it}]\ dt = \int_0^{2n\pi} p(t)\ e^{it}\ dt\ .$$

An integration by parts yields

$$(8) \qquad \dot{x} - ix + F(x) \Big|_0^{2n\pi} - i \int_0^{2n\pi} F(x)e^{it}\ dt = n\int_0^{2n} p(t)\, e^{it}\ dt\ ,$$

using the 2π - periodicity of $p(t)$.

We may assume, of course, in proving (4) that $F(x)$ is bounded for all real x (Rng $F < \infty$). Thus,

$$(9) \qquad o(n) - i \int_0^{2n} F(x)\ e^{it}\ dt = n \int_0^{2n} p(t)\ e^{it}\ dt\ .$$

Using Lemma 1, we have

$$\begin{aligned} \left| \int_0^{2n\pi} F(x)e^{it}\ dt \right| &= \left| \sum_{k=1}^{n} \int_{2(k-1)\pi}^{2k\pi} F(x)e^{it}\ dt \right| \\ &\le \sum_{k=1}^{n} \left| \int_{2(k-1)\pi}^{2k\pi} F(x)e^{it}\ dt \right| \\ &= \sum_{k=1}^{n} \left| \int_0^{2\pi} F(\,x(t+2(k-1)\pi\,)\ e^{it}\ dt \right| \\ &\le n\ 2\ \mathrm{Rng}\ F(x(t)) \le n\ 2\ \mathrm{Rng}\ F \end{aligned}$$

and, from (9), it follows that

$$(10) \qquad \left| \int_0^{2\pi} p(t)e^{it}\ dt \right| \le \frac{o(n)}{n} + 2\ \mathrm{Rng}\ F\ ,$$

which implies the inequality (4).

THEOREM 5. _If (2) holds and_

$$(11) \qquad xF(x) \ge -M \quad \text{and} \quad |F(x)| \le B \quad \text{for all real } x, \text{ with}$$

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M *and* B *positive constants, and if* $x(t)$ *is a solution of* (1) *such that* $p(t)x(t)$ *is bounded above,* $p(t)x(t) \leq C$ *for all* x, *then* $x(t)$ *satisfies condition* (I).

Proof. We note first that the stated conditions imply, by Theorem 3, that all solutions of (1) exist in the future and satisfy (II). Let $x(t)$, $y(t)$ be the corresponding solution of (1') above and let $V = x^2 + y^2$. Then

$$V = 2yp - 2xF(x) \leq 2yp + M = 2(x + F(x))p + M \leq 2(C + BE) + M$$

and thus $V(t) = o(t^2)$, which implies that $x = o(t)$ and $y = o(t)$. Since $x = y - F(x)$, $x = o(t)$ and the proof is complete.

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