Glasgow Math. J. **50** (2008) 531–537. © 2008 Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089508004394. Printed in the United Kingdom

BOUNDARY VALUE PROBLEMS VIA AN INTERMEDIATE VALUE THEOREM

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(Received 24 September 2007; revised 20 March 2008; accepted 19 February 2008)

Abstract. We use an intermediate value theorem for quasi-monotone increasing functions to prove the existence of the smallest and the greatest solution of the Dirichlet problem $u'' + f(t, u) = 0, u(0) = \alpha, u(1) = \beta$ between lower and upper solutions, where $f : [0, 1] \times E \to E$ is quasi-monotone increasing in its second variable with respect to a regular cone.

2000 *Mathematics Subject Classification.* 34B15, 34C12, 34G20.

1. Introduction. Let E be a real Banach space ordered by a cone K. A cone K is a non-empty closed convex subset of E such that $\lambda K \subseteq K$ $(\lambda \geq 0)$, and $K \cap (-K) = \{0\}$. As usual $x \leq y :\iff y - x \in K$. For $x \leq y$ let $[x, y]$ denote the order interval of all z with $x \leq z \leq y$. Let K^* denote the dual wedge of K, that is the set of all $\varphi \in E^*$ with $\varphi(x) \geq 0$ $(x \geq 0)$.

For $D \subseteq E$ a function $g : D \to E$ is called quasi-monotone increasing (qmi for short), in the sense of Volkmann [**16**], if

$$x, y \in D,\ x \leq y,\ \varphi \in K^*, \quad \varphi(x) = \varphi(y) \Longrightarrow \varphi(g(x)) \leq \varphi(g(y)).$$

For $I \subseteq \mathbb{R}$ (an interval) a function $g : I \times D \to E$ is called qmi if $x \mapsto g(t, x)$ is qmi for each $t \in I$.

Let $f : [0, 1] \times E \to E$ be continuous and qmi. We consider the Dirichlet boundary value problem (BVP)

$$u''(t) + f(t, u(t)) = 0 \ \ (t \in [0, 1]), \quad u(0) = \alpha, \ \ u(1) = \beta. \tag{1}$$

As usual, functions $v, w \in C^2([0, 1], E)$ are called lower and upper solutions for problem (1) in case

$$v''(t) + f(t, v(t)) \geq 0 \ \ (t \in [0, 1]), \quad v(0) \leq \alpha, \ \ v(1) \leq \beta, \tag{2}$$

$$w''(t) + f(t, w(t)) \leq 0 \ \ (t \in [0, 1]), \quad w(0) \geq \alpha, \ \ w(1) \geq \beta, \tag{3}$$

respectively.

The use of lower and upper solutions to obtain existence of solutions of boundary value problems dates back to Perron's method for the Dirichlet problem for elliptic equations, and since that time, hundreds of papers have used lower and upper solutions for all kind of equations and boundary conditions. For a survey on the history of this subject, we refer to [**3**, Chapter 4.3], and the references given there.

If $v(t) \leq w(t)$ ($t \in [0, 1]$), the question whether (1) has a solution $u \in C^2([0, 1], E)$ between v and w is answered positively under different assumptions on E and K. The typical methods of proofs are variants of Nagumo's cutting off method and the method of monotone iteration, see [**1**, **2**, **4**, **7**, **9**, **11**, **13**], and the references given there. In this paper, we will study this problem by a construction that allows the application of intermediate value theorems for qmi functions, in a quite natural way.

2. Preliminaries. We will make use of the following existence and comparison theorem:

THEOREM 1. *Let $h : [a, b] \times E \to E$ be continuous, qmi and Lipschitz continuous in its second variable with constant $L < \pi^2/(b-a)^2$. Then*

(1) *The BVP $u''(t) + h(t, u(t)) = 0$, $u(a) = \alpha$, $u(b) = \beta$ has a unique solution $u(\cdot, \alpha, \beta) \in C^2([a, b], E)$ for each choice of $\alpha, \beta \in E$, and the solution operator $S : E^2 \to C^1([a, b], E)$,*

$$S(\alpha, \beta) = u(\cdot, \alpha, \beta)$$

is Lipschitz continuous.

(2) *If $v, w \in C^2([a, b], E)$ satisfy*

$$v''(t) + h(t, v(t)) \geq w''(t) + h(t, w(t)) \ \ (t \in [a, b]),$$

and

$$v(a) \leq w(a), \quad v(b) \leq w(b),$$

then $v(t) \leq w(t)$ ($t \in [a, b]$).

REMARKS. Part 1 of Theorem 1 is Lettenmeyer's existence theorem for BVPs [**10**]. The Lipschitz continuity of S follows from the corresponding integral equation. Part 2 of Theorem 1 is a special case of [**6**, Theorem 3]. In particular, part 2 of Theorem 1 proves that the solution operator S in part 1 is monotone increasing, if E^2 is ordered by the cone K^2, and if $C^1([a, b], E)$ is ordered by the cone

$$\{z \in C^1([a, b], E) : z(t) \geq 0 \ (t \in [a, b])\}.$$

3. Main result. Let $f : [0, 1] \times E \to E$ be continuous, qmi and Lipschitz continuous in its second variable with constant $L > 0$. We consider problem (1). Let $Z = \{t_0, t_1, \ldots, t_n, t_{n+1}\}$ be a partition of $[0, 1]$, that is, $0 = t_0 < t_1 \cdots < t_n < t_{n+1} = 1$, such that

$$\max\{t_{k+1} - t_k : k = 0, \ldots, n\} < \frac{\pi}{\sqrt{L}}.$$

Under these assumptions, Theorem 1 applies to each interval $[a, b] = [t_k, t_{k+1}]$. For convenience of the notation, we set $x_0 = \alpha$ and $x_{n+1} = \beta$.

In particular, we can define a function $G = (G_1, \ldots, G_n) : E^n \to E^n$ by setting

$$G_k(x) = u'_{k+1}(t_k) - u'_k(t_k) \ \ (k = 1, \ldots, n),$$

for each $x = (x_1, \dots, x_n) \in E^n$, where $u_1, \dots, u_{n+1}$ denote the solutions of the boundary value problems

$$u_k''(t) + f(t, u_k(t)) = 0, \quad u_k(t_{k-1}) = x_{k-1}, \; u_k(t_k) = x_k \quad (k = 1, \dots, n+1).$$

Now let E^n be ordered by the cone K^n, let $v, w \in C^2([0, 1], E)$ be lower and upper solutions for problem (1) with $v \leq w$ on $[0, 1]$, and let

$$V = (v(t_1), \dots, v(t_n)), \qquad W = (w(t_1), \dots, w(t_n)).$$

THEOREM 2. *Under the assumptions given above we have*

(1) *The function $G = (G_1, \dots, G_n) : E^n \to E^n$ is Lipschitz continuous and qmi on E^n. Moreover $G(V) \geq 0$ and $G(W) \leq 0$.*

(2) *The function $u : [0, 1] \to E$ defined by*

$$u(t) = u_k(t) \quad (t \in [t_{k-1}, t_k], \;\; k = 1, \dots, n+1)$$

is a solution of (1) *if and only if $G(x) = 0$.*

Proof: (1) Lipschitz continuity of G follows immediately from part 1 of Theorem 1. Next, note that $\psi \in (K^n)^*$ if and only if there exist $\varphi_1, \dots, \varphi_n \in K^*$ such that

$$\psi(x) = \varphi_1(x_1) + \cdots + \varphi_n(x_n) \quad (x = (x_1, \dots, x_n) \in E^n).$$

For this reason, G is qmi on E^n if and only if for each $x = (x_1, \dots, x_n) \in E^n$ and each $k \in \{1, \dots, n\}$ the following condition holds:

$$\xi \mapsto G_k(x_1, \dots, x_{j-1}, \xi, x_{j+1}, \dots, x_n)$$

is monotone increasing on E for $j \in \{1, \dots, n\} \backslash \{k\}$, and

$$\xi \mapsto G_k(x_1, \dots, x_{k-1}, \xi, x_{k+1}, \dots, x_n)$$

is qmi on E. To verify this property fix $k \in \{1, \dots, n\}$. First, let $j \neq k$ and $\xi, \eta \in E$ with $\xi \leq \eta$. It is clear that G_k does not depend on the jth coordinate if $|j - k| > 1$.

We consider $j = k + 1$. Thus, we have to deal with the solutions of the following BVPs:

$$q''(t) + f(t, q(t)) = 0, \quad q(t_k) = x_k, \; q(t_{k+1}) = \xi,$$
$$r''(t) + f(t, r(t)) = 0, \quad r(t_k) = x_k, \; r(t_{k+1}) = \eta.$$

According to part 2 of Theorem 1 we have $q \leq r$ on $[t_k, t_{k+1}]$ and $q(t_k) = r(t_k)$. Therefore,

$$\begin{aligned} G_k(x_1, \dots, x_{j-1}, \eta, x_{j+1}, \dots, x_n) &- G_k(x_1, \dots, x_{j-1}, \xi, x_{j+1}, \dots, x_n) \\ &= r'(t_k) - q'(t_k) \geq 0. \end{aligned}$$

For $j = k - 1$ we consider the BVPs

$$q''(t) + f(t, q(t)) = 0, \quad q(t_{k-1}) = \xi, \; q(t_k) = x_k,$$
$$r''(t) + f(t, r(t)) = 0, \quad r(t_{k-1}) = \eta, \; r(t_k) = x_k.$$

Now, $q \leq r$ on $[t_{k-1}, t_k]$ and $q(t_k) = r(t_k)$. Therefore,

$$\begin{aligned} &G_k(x_1, \ldots, x_{j-1}, \eta, x_{j+1}, \ldots, x_n) - G_k(x_1, \ldots, x_{j-1}, \xi, x_{j+1}, \ldots, x_n) \\ &\quad = -(r'(t_k) - q'(t_k)) \geq 0. \end{aligned}$$

Next, let

$$\xi, \eta \in E, \quad \xi \leq \eta, \quad \varphi \in K^*, \quad \varphi(\xi) = \varphi(\eta).$$

We consider the BVPs

$$q''(t) + f(t, q(t)) = 0, \quad q(t_k) = \xi, \ q(t_{k+1}) = x_{k+1},$$
$$r''(t) + f(t, r(t)) = 0, \quad r(t_k) = \eta, \ r(t_{k+1}) = x_{k+1}.$$

We have $q \leq r$ on $[t_k, t_{k+1}]$. Therefore,

$$\varphi(q'(t_k)) = \lim_{t \to t_k+} \frac{\varphi(q(t)) - \varphi(\xi)}{t - t_k} \leq \lim_{t \to t_k+} \frac{\varphi(r(t)) - \varphi(\eta)}{t - t_k} = \varphi(r'(t_k)).$$

Analogously, if we consider the BVPs

$$q''(t) + f(t, q(t)) = 0, \quad q(t_{k-1}) = x_{k-1}, \ q(t_k) = \xi,$$
$$r''(t) + f(t, r(t)) = 0, \quad r(t_{k-1}) = x_{k-1}, \ r(t_k) = \eta,$$

we obtain $q \leq r$ on $[t_{k-1}, t_k]$, and now

$$\varphi(q'(t_k)) = \lim_{t \to t_k-} \frac{\varphi(q(t)) - \varphi(\xi)}{t - t_k} \geq \lim_{t \to t_k-} \frac{\varphi(r(t)) - \varphi(\eta)}{t - t_k} = \varphi(r'(t_k)).$$

Thus, we have

$$\varphi(G_k(x_1, \ldots, x_{k-1}, \xi, x_{k+1}, \ldots, x_n)) \leq \varphi(G_k(x_1, \ldots, x_{k-1}, \eta, x_{k+1}, \ldots, x_n)).$$

Summing up, we have proved that G is qmi on E^n.

Now, consider $V = (v(t_1), \ldots, v(t_n))$. For each $k \in \{1, \ldots, n+1\}$ let $u_k : [t_{k-1}, t_k] \to E$ be the corresponding solution of the BVPs in the definition of G with $x = V$. From the properties of v in (2) and part 2 of Theorem 1, we obtain

$$v(t) \leq u_k(t) \ \ (t \in [t_{k-1}, t_k], \ k \in \{1, \ldots, n+1\})$$

and

$$v(t_{k-1}) = u_k(t_{k-1}), \quad v(t_k) = u_k(t_k) \ \ (k = 2, \ldots, n).$$

Thus, $u'_{k+1}(t_k) \geq v'(t_k)$ and $u'_k(t_k) \leq v'(t_k)$ $(k = 1, \ldots, n)$. Therefore

$$u'_{k+1}(t_k) - u'_k(t_k) \geq v'(t_k) - v'(t_k) = 0 \ \ (k = 1, \ldots, n).$$

This means $G(V) \geq 0$. Analogously, from the properties of w in (3) and part 2 of Theorem 1, we obtain $G(W) \leq 0$.

(2) It is trivial that $G(x) = 0$ if u is a solution of (1). Now let $G(x) = 0$, that is

$$u'_{k+1}(t_k) = u'_k(t_k) \;\; (k = 1, \dots, n).$$

Thus, u is C^1 on $[0, 1]$ and u is C^2 on each interval $[t_{k-1}, t_k]$ $(k = 1, \dots, n+1)$. From the differential equation, we obtain

$$(u')'_-(t_k) = (u')'_+(t_k) \;\; (k = 1, \dots, n).$$

Therefore, $u \in C^2([0, 1], E)$ and solves $u''(t) + f(t, u(t)) = 0$. Since $u(0) = \alpha$, $u(1) = \beta$ anyway, u is a solution of (1). □

4. Application of an intermediate value theorem. For qmi mappings, several intermediate value (or equivalently fixed point) theorems are known; see [**5**, **8**, **12**, **15**], and the references given there. Note, that a cone $K \subseteq E$ is called regular if each increasing and order bounded sequence is convergent (if $\dim E < \infty$ each cone is regular). A suitable intermediate value theorem for our purposes is the following:

THEOREM 3. *Let D be an open subset of a Banach space E ordered by a regular cone K, let $\xi, \eta \in E$ be such that $\xi \leq \eta$ and $[\xi, \eta] \subseteq D$. Let $g : D \to E$ be Lipschitz continuous and qmi, and let $g(\eta) \leq 0 \leq g(\xi)$. Then the equation $g(x) = 0$ has the smallest solution $\underline{x}$ and the greatest solution $\overline{x}$ in $[\xi, \eta]$.*

Theorem 3 follows from the results and methods in [**5**] and [**15**].

Now, let E be ordered by a regular cone K (then K^n is a regular cone in E^n) and let f, v, w and G be as assumed in Section 3. Under these assumptions, Theorem 2 proves that Theorem 3 applies to G, V and W. Thus, the equation $G(x) = 0$ has the smallest solution $\underline{x}$ and the greatest solution $\overline{x}$ in $[V, W]$. The corresponding functions $\underline{u}, \overline{u} : [0, 1] \to E$ are solutions of (1). Moreover if $u : [0, 1] \to E$ is a solution of (1) with

$$v(t) \leq u(t) \leq w(t) \;\; (t \in [0, 1]),$$

then for $x = (u(t_1), \dots, u(t_n))$ we find $V \leq x \leq W$ and $G(x) = 0$. Thus $\underline{x} \leq x \leq \overline{x}$ and part 2 of Theorem 1 once more prove

$$\underline{u}(t) \leq u(t) \leq \overline{u}(t) \;\; (t \in [0, 1]).$$

Summing up we have proved.

THEOREM 4. *Let E be ordered by a regular cone, and let f, v and w be as assumed in Section 3. Then the BVP (1) has a smallest solution $\underline{u} : [0, 1] \to E$ and a greatest solution $\overline{u} : [0, 1] \to E$ between v and w.*

REMARKS: (1) Theorem 4 applies if $\dim E < \infty$. In this case, we have no restriction to the cone, and, as it is to our knowledge, this general case was unknown. In the infinite-dimensional case, further intermediate value theorems can be applied to G under special assumptions on f. For example, [**15**, Theorem 4.1] can be applied whenever $\lambda G + \mathrm{id}$ restricted to $[V, W]$ is γ-condensing for some $\lambda > 0$, with respect to Hausdorff's or Kuratowski's measure of non-compactness (compare also [**5**, Theorem 1]).

(2) Of course, we have the restriction that f is Lipschitz continuous in its second variable. We have chosen this global assumption to avoid technical notations occluding

https://doi.org/10.1017/S0017089508004394 Published online by Cambridge University Press

the idea of the method. Let $B_r(0)$ denote the open ball with center 0 and radius r in E. For example our method works in the same way in the following case:

Let $f : [0, 1] \times B_r(0) \to E$ be continuous, qmi and Lipschitz continuous in its second variable. Let v and w be lower and upper solutions of problem (1) with $v \leq w$ on $[0, 1]$, and

$$[v(t), w(t)] \subseteq B_r(0)\,(t \in [0, 1]).$$

Then, by choosing Z sufficiently fine, we obtain a Lipschitz continuous and qmi function $G : D \to E^n$ on an open set $D \subseteq E^n$ containing $[V, W]$. Thus, we have the following generalization of Theorem 4 which, if $\dim E < \infty$, is applicable to locally Lipschitz continuous functions.

THEOREM 5. *Let E be ordered by a regular cone, and let $f : [0, 1] \times E \to E$ be continuous, qmi and Lipschitz continuous in its second variable on each set* $[0, 1] \times B_r(0)$ $(r > 0)$. *Let v and w be lower and upper solutions of problem* (1) *with $v \leq w$ on* $[0, 1]$. *Then the BVP* (1) *has the smallest solution* $\underline{u} : [0, 1] \to E$ *and the greatest solution* $\overline{u} : [0, 1] \to E$ *between v and w.*

5. Example.

Let $E = \mathbb{R}^3$ be ordered by the ice-cream cone

$$K = \{(x, y, z) \in \mathbb{R}^3 : z \geq \sqrt{x^2 + y^2}\}.$$

The linear qmi mappings in this case were characterized in [**14**]. It follows by linearization that the following mappings are qmi:

$$(x, y, z) \mapsto \pm(2zx + y, 2zy - x, x^2 + y^2 + z^2).$$

Hence, if $c : [0, 1] \to \mathbb{R}$ is continuous, then $f : [0, 1] \times \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$f(t, (x, y, z)) = c(t)(2zx + y, 2zy - x, x^2 + y^2 + z^2)$$

is continuous, qmi and Lipschitz continuous in its second variable on each set $[0, 1] \times B_r(0)$ $(r > 0)$.

Now, assume that $\lambda, \mu : [0, 1] \to \mathbb{R}$ satisfy $\lambda \leq \mu$ on $[0, 1]$ and

$$\lambda''(t) + c(t)\lambda^2(t) \geq 0 \geq \mu''(t) + c(t)\mu^2(t) \quad (t \in [0, 1]).$$

Then $v(t) = (0, 0, \lambda(t))$ and $w(t) = (0, 0, \mu(t))$ are lower and upper solutions for problem (1) with $v \leq w$ on $[0, 1]$ whenever

$$\alpha \in [(0, 0, \lambda(0)), (0, 0, \mu(0))], \quad \beta \in [(0, 0, \lambda(1)), (0, 0, \mu(1))],$$

and according to Theorem 5, problem (1) has the smallest and the greatest solution between v and w in this case.

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