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VANISHING l^1-SUMS OF THE POISSON KERNEL, AND SUMS WITH POSITIVE COEFFICIENTS

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1. Introduction

For z in D and ζ in ∂D, we denote by $p_z(\zeta)$ the Poisson kernel $(1-|z|^2)|1-\bar{z}\zeta|^{-2}$ for the open unit disc D. We ask for what countable sets $\{a_n : n \in \mathbb{N}\}$ of points of D there exist complex numbers λ_n with

$$0 < \sum_{n=1}^{\infty} |\lambda_n| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \lambda_n p_{a_n} = 0,$$

by which we mean that the series converges to zero in the norm of $L^1(\partial D)$.

For each a in D, we have $\|p_a\|_1 = 1$. Therefore, given $\lambda = \{\lambda_n\}$ in l^1, the series $\sum_{n=1}^{\infty} \lambda_n p_{a_n}$ converges in the norm of $L^1(\partial \mathrm{D})$ to an element of $L^1(\partial D)$, which we denote by $T\lambda$. Plainly, $\|T\lambda\|_1 \leqq \|\lambda\|_1 = \sum_{n=1}^{\infty} |\lambda_n|$, and so T is a bounded linear mapping of l^1 into $L^1(\partial D)$. In this notation, we are asking for what $\{a_n\}$ the kernel of T is non-zero.

It is known [1] that $Tl^1 = L^1(\partial D)$ if and only if the sequence $\{a_n\}$ is *non-tangentially dense* for ∂D, that is almost every point ζ of ∂D is the limit of some subsequence of $\{a_n\}$ that converges to ζ non-tangentially. As pointed out in [1], it follows at once that $\ker T \neq \{0\}$ if $\{a_n\}$ is non-tangentially dense for ∂D.

We show (Theorem 2) that if $\lim_{n\to\infty} |a_n| = 1$, then $\ker T \neq \{0\}$ if and only if $\{a_n\}$ is non-tangentially dense for ∂D. After drawing a few corollaries, we obtain partial results for the more complicated case in which $\{a_n\}$ is allowed to have limit points in D. Our proofs make substantial use of ideas from Brown, Shields and Zeller [2].

In Section 3, we are concerned with sums of Poisson kernels with positive coefficients. We say that a subset A of D is a *positive Poisson basic set* (P.P.B. set) if, for every positive continuous function f on ∂D, there exist sequences $\{\lambda_n\}$ and $\{a_n\}$ with λ_n positive and a_n in A, such that

$$f(\zeta) = \sum_{n=1}^{\infty} \lambda_n p_{a_n}(\zeta)$$

for all ζ in ∂D. We are indebted to W. R. Rudin for asking one of us (in effect) what subsets A of D are P.P.B. sets.

Our work on Rudin's problem has benefitted greatly from frequent discussion with W.K. Hayman. We are also indebted to him and to T. J. Lyons for access to their forthcoming paper [5] in which they obtain a remarkable solution to the problem. This solution can be stated as follows.

Let $z_{m,n}=r_n\exp(i\psi_{m,n})$ with $r_n=1-2^{-n}$, $\psi_{m,n}=2\pi m2^{-n}$, and let E denote the set of pairs (m,n) such that there exists a point a of A with $r_n\leqq|a|\leqq r_{n+1}$ and $\psi_{m,n}\leqq\arg(a)\leqq\psi_{m+1,n}$.

Theorem (Hayman and Lyons, [5]). *A is a positive Poisson basic set if and only if*

$$\sum_{(m,n)\in E} 2^{-n}p_{z_{m,n}}(\zeta)=+\infty$$

for all ζ in ∂D.

It follows at once from this theorem that A is a P.P.B. set if every point of ∂D is a non-tangential limit of points of A.

In the present paper, we give an independent approach to the problem, based on duality. Let h^1 denote the space of differences of positive harmonic functions on D. With $0<\delta<1$, let $K(a,\delta)=\{z\in D:d(z,a)\leqq\delta\}$, where $d(\ ,\ )$ is the pseudo-hyperbolic distance function on D, that is $d(z,a)=|z-a|\,|1-a\bar{z}|^{-1}$, and let $A(\delta)=\bigcup\{K(a,\delta):a\in A\}$.

We prove (Theorem 10) that A is a P.P.B. set if and only if

$$\sup_{z\in D} h(z)=\sup_{z\in A} h(z)$$

for every function h in h^1; and also prove that A is a P.P.B. set if and only if $A(\delta)$ is a P.P.B. set for some δ with $0<\delta<1$. As a corollary, we show that A is a P.P.B. set if it is uniformly non-tangential for ∂D.

Since the proofs of these results are quite transparent, it would be very interesting if they could be extended to yield the Hayman–Lyons theorem [5].

In Theorem 15, we give an approximation theorem for continuous functions, in which the coefficients of the Poisson kernels are determined.

2. Vanishing l^1-sums

For f in $L^1(\partial D)$ and z in $D, f(z)$ will denote the Poisson integral of f at z; and $[\ ,\ ]$ will denote the bilinear form

$$[f,g]=(2\pi)^{-1}\int_0^{2\pi} f(e^{i\theta})g(e^{i\theta})\,d\theta$$

on $L^1(\partial D)\times L^\infty(\partial D)$ and also the bilinear form

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$$[\{\alpha_n\},\{\beta_n\}]=\sum_{n=1}^{\infty}\alpha_n\beta_n$$

on $l^1\times l^\infty$. Throughout this section $\{a_n\}=\{a_n: n\in\mathbb{N}\}$ is a countable subset of D.

The key to the characterization of those countable sets $\{a_n\}$ for which $\ker T\neq\{0\}$ is the following elementary lemma.

Lemma 1. *Let $\{\lambda_n\}\in l^1$. If $\sum_{n=1}^{\infty}\lambda_n p_{a_n}=0$, then $\sum_{n=1}^{\infty}\lambda_n(1-a_n z)^{-1}=0$ for all z in D. For real sequences $\{\lambda_n\}$ the reverse implication holds.*

Proof. We write $u_z(\zeta)=(1-z\zeta)^{-1}$ for z in D and ζ in ∂D. If $T\lambda=0$, then, since $u_z\in L^\infty(\partial D)$, we have

$$0=[T\lambda,u_z]=\sum_{n=1}^{\infty}\lambda_n[p_{a_n},u_z]=\sum_{n=1}^{\infty}\lambda_n u_z(a_n),$$

and so $\sum_{n=1}^{\infty}\lambda_n(1-a_n z)^{-1}=0$ for all z in D.

Suppose on the other hand that $\{\lambda_n\}$ is a real sequence. Since $p_a(\zeta)=1+\sum_{n=1}^{\infty}((\bar{a}\zeta)^n+(a\bar{\zeta})^n)$ for ζ in ∂D, we have

$$p_a(z)=(1-\bar{a}z)^{-1}+(1-a\bar{z})^{-1}-1,$$

for all z in D. If $\sum_{n=1}^{\infty}\lambda_n(1-a_n z)^{-1}=0$ for all z in D, then we also have, for all z in D, $\sum_{n=1}^{\infty}\lambda_n(1-\bar{a}_n z)^{-1}=0$ and $\sum_{n=1}^{\infty}\lambda_n(1-a_n\bar{z})^{-1}=0$. Then $\sum_{n=1}^{\infty}\lambda_n p_{a_n}(z)=0$ for all z in D, and it follows that $\sum_{n=1}^{\infty}\lambda_n p_{a_n}=0$.

Theorem 2. *Let $\lim_{n\to\infty}|a_n|=1$. Then $\ker T\neq\{0\}$ if and only if $\{a_n\}$ is non-tangentially dense for ∂D.*

Proof. Suppose that $\lambda=\{\lambda_n\}\in l^1\backslash\{0\}$ and that $T\lambda=\sum_{n=1}^{\infty}\lambda_n p_{a_n}=0$. By Lemma 1, we have, for all z in D,

$$\sum_{n=1}^{\infty}\lambda_n(1-a_n z)^{-1}=0.$$

Geometric series expansion and change of order of summation gives

$$\sum_{n=1}^{\infty}\lambda_n a_n^k=0\quad(k=0,1,2,\ldots),\tag{1}$$

and therefore

$$\sum_{n=1}^{\infty} \lambda_n \exp(a_n z) = 0$$

for all z in $\mathbb{C}$. It now follows at once from Brown, Shields and Zeller [2, Theorem 3] that $\{a_n\}$ is non-tangentially dense for ∂D. Alternatively, we can complete the proof by using Theorem 7 below.

Corollary 3. *Let* $\lim_{n\to\infty}|a_n|=1$. *Then* $Tl^1 = L^1(\partial D)$ *if and only if* $\ker T \neq \{0\}$.

Proof. Theorem 2 and [1].

Corollary 4. *Let* $\lim_{n\to\infty}|a_n|=1$. *Then* Tl^1 *is closed in* $L_1(\partial D)$ *if and only if either* $Tl^1 = L^1(\partial D)$ *or* $T^*L^\infty(\partial D) = l^\infty$. *The second alternative occurs if and only if* $\{a_n\}$ *is an interpolating sequence for* H^∞.

Proof. Let Tl^1 be closed in, but not equal to, $L^1(\partial D)$. By Banach's closed range theorem [3, p. 488], $T^*L^\infty(\partial D)$ is closed in l^∞ and is the annihilator of $\ker T$ in l^∞. Since $\ker T = \{0\}$, we therefore have $T^*L^\infty(\partial D) = l^\infty$. Conversely, if $T^*L^\infty(\partial D) = l^\infty$, then, again by Banach's closed range theorem, Tl^1 is closed in $L^1(\partial D)$.

We note next that, for g in $L^\infty(\partial \mathrm{D})$, we have

$$T^* g = \{g(a_n)\}. \tag{2}$$

For, with $\lambda = \{\lambda_n\}$ in l^1, and g in $L^\infty(\partial D)$, we have

$$[\lambda, T^* g] = [T\lambda, g] = \sum_{n=1}^{\infty} \lambda_n [p_{a_n}, g] = \sum_{n=1}^{\infty} \lambda_n g(a_n) = [\lambda, \{g(a_n)\}].$$

Thus $T^*L^\infty(\partial D) = l^\infty$ if and only if every bounded sequence is of the form $\{h(a_n)\}$ with h a bounded harmonic function on D; and, by Garnett [4], this holds if and only if $\{a_n\}$ is an interpolating sequence for H^∞.

Corollary 5. *Let* $\lim_{n\to\infty}|a_n|=1$. *The set of sequences* $\{g(a_n)\}$ *with* g *in* $L^\infty(\partial D)$ *is closed in* l^∞ *if and only if either* $\{a_n\}$ *is an interpolating sequence for* H^∞ *or* $\{a_n\}$ *is non-tangentially dense for* ∂D.

Proof. By (2), the set of sequences $\{g(a_n)\}$ with g in $L^\infty(\partial D)$ is $T^*L^\infty(\partial D)$. By Banach's closed range theorem, this is closed in l^∞ if and only if Tl^1 is closed in $L^1(\partial D)$. Thus Corollary 4 applies.

Corollary 6. *Let* $\{a_n\}$ *be an interpolating sequence. Then each element* f *of the closed linear span of* $\{p_{a_n} : n \in \mathbb{N}\}$ *in* $L^1(\partial D)$ *is of the form*

$$f = \sum_{n=1}^{\infty} \lambda_n p_{a_n}, \tag{3}$$

with $\sum_{n=1}^{\infty} |\lambda_n| < \infty$, *and*

$$\lambda_n = [f, B_n](B_n(a_n))^{-1} \quad (n \in \mathbb{N}),$$

where B_n *is the Blaschke product with its zeros at the points* a_k *with* $k \neq n$.

Proof. Let X be the closed linear span of $\{p_{a_n}: n \in \mathbb{N}\}$. Since Tl^1 is closed in $L^1(\partial D)$, we have $Tl^1 = X$. Thus each f in X is of the form (3), and we have

$$[f, B_n] = \sum_{k=1}^{\infty} \lambda_k [p_{a_k}, B_n] = \sum_{k=1}^{\infty} \lambda_k B_n(a_k) = \lambda_n B_n(a_n).$$

Notation. With ζ in ∂D, $b > 0$, $0 < \alpha < \pi/2$, we define

$$D(b, \alpha) = \{x + yi \in D : b > 1 - x > |y| \cot \alpha\},$$

and

$$D(\zeta, b, \alpha) = \zeta D(b, \alpha).$$

We define the *firm boundary* of an open subset G of D to be the set of ζ in ∂D such that, for every α with $0 < \alpha < \pi/2$, there exists $b > 0$ with $D(\zeta, b, \alpha) \subset G$.

We note that the firm boundary F of an open subset G of D is a Borel subset of ∂D. For, let $F(b, \alpha) = \{\zeta \in \partial D : D(\zeta, b, \alpha) \subset G\}$ and $F(\alpha) = \bigcup \{F(b, \alpha): b > 0\}$, so that $F = \bigcap \{F(\alpha): 0 < \alpha < \pi/2\}$. The set $F(b, \alpha)$ is closed, for if $z \in D(\zeta, b, \alpha)$, then $z \in D(\zeta', b, \alpha)$ for all ζ' sufficiently near ζ. Since $F(b, \alpha)$ is a decreasing function of each of its variables, $F(\alpha)$ and F are of the forms $\bigcup \{F(b_n, \alpha): n \in \mathbb{N}\}$ and $\bigcap \{F(\alpha_n): n \in \mathbb{N}\}$ respectively.

The idea of the following theorem derives from Lemma 4 in Brown, Shields and Zeller [2], which serves a somewhat similar purpose.

Theorem 7. (i) *If there exists an open subset* G *of* $D \setminus \{a_n\}$ *such that the firm boundary of* G *has positive Lebesgue measure, then* $\{a_n\}$ *is not non-tangentially dense for* ∂D.

(ii) *Let* $\{a_n\}$ *be not non-tangentially dense for* ∂D, *but let there exist* λ *in* $l^1 \setminus \{0\}$ *such that* $\sum_{n=1}^{\infty} \lambda_n (1 - \bar{a}_n z)^{-1} = 0$ *for all* z *in* D. *Then there exists an open subset* G *of* $D \setminus \{a_n\}$ *such that* $\sum_{n=1}^{\infty} \lambda_n (z - a_n)^{-1} = 0$ *for all* z *in* G *and the firm boundary of* G *has positive Lebesgue measure.*

Proof. (i) Let G be an open subset of $D \setminus \{a_n\}$ such that the firm boundary F of G has positive Lebesgue measure. If $\{a_n\}$ is non-tangentially dense for ∂D, there exist ζ in F

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and a subsequence $\{a_{n_k}\}$ that converges non-tangentially to ζ. Thus there exists α with $0<\alpha<\pi/2$ and $D(\zeta,b,\alpha)\cap\{a_{n_k}:k\in\mathbb{N}\}$ non-void for every $b>0$. But, since $\zeta\in F$, there exists $b>0$ with $D(\zeta,b,\alpha)\subset G\subset D\backslash\{a_n\}$.

(ii) By [1] or [2], there exists g in H^∞ with

$$\sup\{|g(a_n)|:n\in\mathbb{N}\}<1<\|g\|_\infty. \tag{4}$$

Let $G=\{z\in D:|g(z)|>1\}$. Then G is an open subset of $D\backslash\{a_n\}$. Since $\|g\|_\infty>1$, $E=\{\zeta\in\partial D:|g(\zeta)|>1\}$ has positive Lebesgue measure, and, for almost all ζ in E, $g(z)\to g(\zeta)$ as $z\to\zeta$ non-tangentially. Thus for such ζ and α in $(0,\pi/2)$, there exists $b>0$ such that $|g(z)|>1$ for all z in $D(\zeta,b,\alpha)$, that is $D(\zeta,b,\alpha)\subset G$. This shows that the firm boundary of G has positive Lebesgue measure, and so the theorem is proved unless there is a point a in G with

$$\mu=\sum_{n=1}^{\infty}\lambda_n(a-a_n)^{-1}\neq 0. \tag{5}$$

By (1), we have

$$\sum_{n=1}^{\infty}\lambda_n a_n^j(1-a_nz)^{-k}=0 \quad (j,k=0,1,2,\ldots;z\in D). \tag{6}$$

For $|z|<1/4$, we have $(|a|+|a_n|)|z|(1-|a_n||z|)^{-1}<2/3$, and so

$$\begin{aligned}(1-az)^{-1}&=(1-a_nz)^{-1}\{1-(a-a_n)z(1-a_nz)^{-1}\}^{-1}\\&=(1-a_nz)^{-1}+\sum_{k=1}^{\infty}(a-a_n)^kz^k(1-a_nz)^{-(k+1)}.\end{aligned}$$

(5) and (6) now give

$$\begin{aligned}\mu(1-az)^{-1}&=\sum_{n=1}^{\infty}\lambda_n(a-a_n)^{-1}(1-a_nz)^{-1}+\sum_{k=1}^{\infty}z^k\sum_{n=1}^{\infty}\lambda_n(a-a_n)^{k-1}(1-a_nz)^{-(k+1)}\\&=\sum_{n=1}^{\infty}\lambda_n(a-a_n)^{-1}(1-a_nz)^{-1}.\end{aligned}$$

Assuming (5), we can define $\mu_n=\mu^{-1}\lambda_n(a-a_n)^{-1}$; and, since a is at positive distance from the complement of G, we have $\sum_{n=1}^{\infty}|\mu_n|<\infty$. We have proved that

$$(1-az)^{-1}=\sum_{n=1}^{\infty}\mu_n(1-a_nz)^{-1}$$

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when $|z|<1/4$. and therefore for all z in D. It follows that

$$a^k=\sum_{n=1}^{\infty}\mu_n a_n^k \quad (k=0.1,2,\ldots),$$

and therefore

$$p(a)=\sum_{n=1}^{\infty}\mu_n p(a_n)$$

for all polynomials p. By dominated convergence, it follows that for all h in H^∞,

$$h(a)=\sum_{n=1}^{\infty}\mu_n h(a_n),$$

and hence

$$|h(a)|\leqq\left(\sum_{n=1}^{\infty}|\mu_n|\right)\sup\{|h(a_n)|:n\in\mathbb{N}\}.$$

By taking $h=g^m$ with m sufficiently large, we have contradicted (4).

Remarks. If $\sum_{n=1}^{\infty}\lambda_n p_{a_n}=0$, then Lemma 1 allows us to apply Theorem 7(ii).

If $\lim_{n\to\infty}|a_n|=1$, then $D\backslash\{a_n\}$ is a connected open set. Thus if $\sum_{n=1}^{\infty}\lambda_n(z-a_n)^{-1}$ vanishes on a non-void open subset of $D\backslash\{a_n\}$, it vanishes on all of $D\backslash\{a_n\}$. Since each a_n is at positive distance from $\{a_k:k\neq n\}$, it follows that $\lambda_n=0$ for all n. Thus Theorem 7(ii) shows that, if $\lim_{n\to\infty}|a_n|=1$ and $\ker T\neq 0$, then $\{a_n\}$ is non-tangentially dense for ∂D, completing the proof of Theorem 2, where we quoted [2].

Corollary 8. *Let Ω be the set of limit points in D of the set $A=\{a_n:n\in\mathbb{N}\}$, let $\Omega\cap A$ be void or finite, and let $D\backslash\Omega$ be connected. Then* $\ker T\neq\{0\}$ *if and only if A is non-tangentially dense for ∂D.*

Proof. Let λ belong to $l^1\backslash\{0\}$ with $\sum_{n=1}^{\infty}\lambda_n(1-a_n z)^{-1}=0$ for all z in D, and suppose that A is not non-tangentially dense for ∂D. By Theorem 7(ii), there exists a non-void open set G in $D\backslash A$ with $\sum_{n=1}^{\infty}\lambda_n(z-a_n)^{-1}=0$ for all z in G. Let F be the closure of A. Then $D\backslash F$ is connected and $G\subset D\backslash F$. Thus $\sum_{n=1}^{\infty}\lambda_n(z-a_n)^{-1}=0$ for all z in $D\backslash F$. If $a_n\notin\Omega$, then sufficiently small neighbourhoods of a_n contain no other a_k, but contain points of $D\backslash F$, and so $\lambda_n=0$. Thus there exists a positive integer N such that $\lambda_n=0$ when $n>N$. Therefore $\sum_{n=1}^{N}\lambda_n(z-a_n)^{-1}=0$ for all z in $D\backslash F$. But this is impossible, since the points a_n are distinct.

Corollary 9. *Let at most countably many limit points of the sequence $\{a_n\}$ be in D and, for some λ in $l^1\backslash\{0\}$, let*

$$\sum_{n=1}^{\infty} \lambda_n(1-a_nz)^{-1}=0 \ (z \in D).$$

Then $\{a_n\}$ is non-tangentially dense for ∂D.

Proof. Let $A=\{a_n : n \in \mathbb{N}\}$, and suppose that A is not non-tangentially dense for ∂D. Then, by Theorem 7(ii), there exists a non-void open set G in $D\backslash A$ with $\sum_{n=1}^{\infty} \lambda_n(z-a_n)^{-1}=0$ for all z in G. Let F be the closure of A and let $E=F \cap D$. By hypothesis, E is a countable relatively closed subset of D. Therefore $D\backslash E$ is connected, for each pair of distinct points of $D\backslash E$ are joined by uncountably many circular arcs in D and each point of E is on only one of these arcs. Since $G \subset D\backslash E$, it follows that $\sum_{n=1}^{\infty} \lambda_n(z-a_n)^{-1}=0$ for all z in $D\backslash E$. If a_n is an isolated point of A, then $\lambda_n=0$. Therefore we may assume that, for every n, a_n is a limit point of A.

Since $F=E \cup (\partial D \cap F)$, F is a countable union of closed sets, namely $\partial D \cap F$ and the one-point subsets of E. Since also F is a perfect set, these closed subsets of F are nowhere dense in F, contradicting Baire's category theorem. This contradiction shows that $\{a_n\}$ is non-tangentially dense for ∂D.

Remark. The following example shows that Corollary 8 can fail if $D\backslash\Omega$ is not connected. Let $\{2a_n\}$ be a countable subset of D that is non-tangentially dense for ∂D. For a, z in D, we have

$$p_a(z)=(1-|a|^2|z|^2)|1-\bar{a}z|^{-2},$$

and therefore, for ζ in ∂D,

$$p_{2a_n}(\zeta/2)=p_{a_n}(\zeta).$$

Since $\{2a_n\}$ is non-tangentially dense for ∂D, there exists λ in $l^1\backslash\{0\}$ with $\sum_{n=1}^{\infty} \lambda_n p_{2a_n}=0$. Therefore, for all ζ in ∂D,

$$\sum_{n=1}^{\infty} \lambda_n p_{a_n}(\zeta)=\sum_{n=1}^{\infty} \lambda_n p_{2a_n}(\zeta/2)=0.$$

Thus $\sum_{n=1}^{\infty} \lambda_n p_{a_n}=0$, but obviously $\{a_n\}$ is not non-tangentially dense for ∂D.

3. Sums with positive coefficients

Notation. We denote by h^1 the set of real harmonic functions f on D such that

$$\sup_{0>r>1} \int_0^{2\pi} |f(re^{i\theta})|\, d\theta < \infty,$$

or, equivalently, the set of differences of positive harmonic functions on D. As is well known, h^1 is the set of harmonic functions $\tilde{\mu}$,

$$\tilde{\mu}(z) = \frac{1}{2\pi} \int_0^{2\pi} P_z(t)\, d\mu(t),$$

for real Borel measures μ on $[0, 2\pi]$.

The pseudo-hyperbolic distance function on D is denoted by $d(\ ,\ )$, that is $d(a,b) = |a-b|\,|1-\bar{a}b|^{-1}$. With a in D and $0<\delta<1$, $K(a,\delta)$ denotes the closed ball $\{z \in D : d(z,a) \leqq \delta\}$, and, given a subset A of D,

$$A(\delta) = \bigcup \{K(a,\delta) : a \in A\}.$$

We recall, from the introduction, that a subset A of D is a P.P.B. set if every positive continuous function f on ∂D is of the form

$$f(\zeta) = \sum_{n=1}^{\infty} \lambda_n p_{a_n}(\zeta) \ (\zeta \in \partial D)$$

with λ_n positive and a_n in A for every n.

Our main result is the following theorem.

Theorem 10. *The following statements are equivalent to each other.*

(i) *A is a P.P.B. set.*

(ii) *For all h in h^1, $\sup_{z \in D} h(z) = \sup_{z \in A} h(z)$.*

(iii) *For all h in h^1 and some δ in $(0,1)$, $\sup_{z \in D} h(z) = \sup_{z \in A(\delta)} h(z)$.*

(iv) *For some δ in $(0,1)$, $A(\delta)$ is a P.P.B. set.*

The proof depends on the following three elementary lemmas, of which the second and third are probably well known.

Lemma 11. *Let X be a compact Hausdorff space, F a set of non-negative continuous functions on X, $0 \leqq \kappa < 1$. If, for every positive continuous function f on X and $\varepsilon > 0$, there exists v in F with*

$$0 < f(x) - v(x) < \varepsilon + \kappa \|f\|_\infty \ (x \in X),$$

then every such function f is of the form

$$f(x)=\sum_{n=1}^{\infty} v_n(x)\ (x\in X),$$

with $v_1, v_2, \ldots$ in F.

Proof. Let f be a positive continuous function on X. Take $f_1=f$, and, having chosen the positive continuous function f_n, take v_n in F with

$$0<f_n(x)-v_n(x)<\kappa^{2n}+\kappa\|f_n\|_\infty\ (x\in X),$$

and take $f_{n+1}=f_n-v_n$. We have $f_{n+1}=f-\sum_{k=1}^{n} v_k$, and

$$\begin{aligned}0<f_{n+1}(x)&<\kappa^{2n}+\kappa\|f_n\|_\infty\\ &<\kappa^{2n}+\kappa^{2n-1}+\kappa^2\|f_{n-1}\|_\infty\\ &<\ldots\\ &<\kappa^{2n}+\kappa^{2n-1}+\cdots+\kappa^{n+1}+\kappa^n\|f_1\|_\infty\\ &<\kappa^{n+1}(1-\kappa)^{-1}+\kappa^n\|f\|_\infty.\end{aligned}$$

Lemma 12. *Let $a,b\in D$ with $d(a,b)=\delta<1$, and let h be a positive harmonic function on D. Then*

(i) $h(a)\leqq(1+\delta)(1-\delta)^{-1}h(b)$,

(ii) $|h(a)-h(b)|\leqq 2\delta(1-\delta)^{-1}h(b)$.

Proof. This is an easy consequence of Harnack's inequality and the invariance of the distance function $d(\ ,\ )$. Let τ be a Möbius transformation of D onto D with $\tau(0)=a$, and let $c=\tau^{-1}(b)$. Then

$$|c|=d(0,c)=d(a,b)=\delta.$$

Take $g(z)=h(\tau(z))$. By Harnack's inequality,

$$(1-\delta)(1+\delta)^{-1}g(0)\leqq g(c)\leqq(1+\delta)(1-\delta)^{-1}g(0),$$

that is

$$(1-\delta)(1+\delta)^{-1}h(a)\leqq h(b)\leqq(1+\delta)(1-\delta)^{-1}h(a).$$

The first of these inequalities gives (i) and

$$h(a)-h(b)\leqq 2\delta(1-\delta)^{-1}h(b),$$

while the second gives

$$h(b)-h(a)\leqq 2\delta(1+\delta)^{-1}h(b).$$

Given a subset A of D and $0<\delta<1$, define $A(\delta,n)$ by $A(\delta,0)=A$ and $A(\delta,n+1)=(A(\delta,n))(\delta)$.

Lemma 13. *Let $0<\delta<\Delta<1$. Then there exists a positive integer m such that, for every subset A of D,*

$$A(\Delta)\subset A(\delta,m).$$

Proof. Define T on $[0,1]$ by $Tx=(x+\delta)(1+\delta x)^{-1}$. For all x in $[0,1)$, we have

$$x<Tx<1,\ d(x,Tx)=\delta.$$

Take $\Delta_0=0$, $\Delta_n=T\Delta_{n-1}$. Then $\{\Delta_n\}$ is an increasing sequence in $[0,1)$ with limit t, say. Continuity of T gives $Tt=t$, and so $t=1$. Take m to be the least positive integer with $\Delta_m\geqq\Delta$.

Let b belong to $A(\Delta)\backslash A$, so that there exists a in A with $0<d(a,b)\leqq\Delta$. Let τ be the Möbius transformation of D onto D with $\tau(a)=0$ and $\tau(b)>0$. Then $\tau(b)=d(0,\tau(b))=d(a,b)\leqq\Delta$. Choose $n\leqq m$ with $\Delta_{n-1}<\tau(b)\leqq\Delta_n$. Then we have $d(\Delta_{k-1},\Delta_k)=\delta$, $d(\Delta_{n-1},\tau(b))\leqq\delta$. Let $z_k=\tau^{-1}(\Delta_k)$ $(0\leqq k<n)$. Then $z_0=a$, $d(z_{k-1},z_k)=\delta$ $(1\leqq k\leqq n-1)$, and $d(z_{n-1},b)\leqq\delta$. Thus $b\in A(\delta,n)\subset A(\delta,m)$.

Proof of Theorem 10. (i)$\Rightarrow$(ii). Let h belong to h^1 and $M=\sup_{z\in A}h(z)$. If $M=+\infty$, then $\sup_{z\in D}h(z)=+\infty$. Assume that $M<+\infty$ and take $g=h-M$. Let μ be the real Borel measure on $[0,2\pi]$ with $\tilde{\mu}=g$, and let $w\in D$. By (i), there exist positive λ_k and a_k in A with

$$p_w(\zeta)=\sum_{k=1}^{\infty}\lambda_k p_{a_k}(\zeta)\ (\zeta\in\partial D).$$

Therefore

$$g(w)=\sum_{k=1}^{\infty}\lambda_k g(a_k)\leqq 0.$$

(ii)$\Rightarrow$(i). Let K be the uniform closure in $C_{\mathbf{R}}(\partial D)$ of the set of finite sums with

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non-negative coefficients of the functions p_a with a in A. By Lemma 11, with $\kappa=0$, it suffices to prove that $K=C_{\mathbf{R}}^{+}(\partial D)$, the set of non-negative continuous functions on ∂D. Let $g\in C_{\mathbf{R}}^{+}(\partial D)\backslash K$. By the Hahn–Banach theorem, using the subadditive positive-homogeneous functional p on $C_{\mathbf{R}}(\partial D)$ defined by

$$p(f)=\inf\{\|f-u\|_{\infty}:u\in K\},$$

there exists a real Borel measure μ on $[0,2\pi]$ with

$$\int_{0}^{2\pi} f(e^{it})\,d\mu(t)\leqq 0\ (f\in K),$$

and

$$\int_{0}^{2\pi} g(e^{it})\,d\mu(t)>0.$$

Thus, with $h=\tilde{\mu}$, we have $h\in h^1$, $\sup_{z\in A} h(z)\leqq 0$. Therefore, by (ii), $\sup_{z\in D} h(z)\leqq 0$, $-\mu$ is a positive measure, $\int_0^{2\pi} g(e^{it})\,d\mu(t)\leqq 0$. This contradiction gives $K=C_{\mathbf{R}}^{+}(\partial D)$.

(ii)$\Rightarrow$(iii). Obvious (for all δ).

(iii)$\Rightarrow$(iv). Replace A by $A(\delta)$ in (ii)$\Rightarrow$(i).

(iv)$\Rightarrow$(i). Suppose that $0<\Delta<1$ and that $A(\Delta)$ is a P.P.B. set. Take $\delta=1/4$. By Lemma 13, there exists a positive integer m with $A(\Delta)\subset A(\delta,m)$. Then $A(\delta,m)$ is a P.P.B. set, and it is now sufficient to prove that A is a P.P.B. set whenever $A(1/4)$ is.

Assume then that $A(1/4)$ is a P.P.B. set, and let F denote the set of linear combinations with positive coefficients of the functions p_a with a in A. Let f be a positive continuous function on ∂D and let $\varepsilon>0$. Then there exist positive λ_k and points b_k in $A(1/4)$ with

$$\sum_{k=1}^{\infty} \lambda_k p_{b_k}(\zeta)=f(\zeta)\ (\zeta\in\partial D).$$

Since, by Dini's theorem, the convergence is uniform on ∂D, there exists n with

$$0<f(\zeta)-\sum_{k=1}^{n} \lambda_k p_{b_k}(\zeta)<\varepsilon\ (\zeta\in\partial D).$$

Choose $a_1, a_2, \ldots, a_n$ in A with $d(a_k, b_k) \leqq 1/4$. By Lemma 12,

$$|p_{a_k}(\zeta) - p_{b_k}(\zeta)| \leqq \tfrac{2}{3} p_{b_k}(\zeta), \quad p_{a_k}(\zeta) \leqq \tfrac{5}{3} p_{b_k}(\zeta).$$

Take $v(\zeta) = \frac{3}{5} \sum_{k=1}^{n} \lambda_k p_{a_k}(\zeta)$. Then

$$\begin{aligned} 0 < f(\zeta) - v(\zeta) &= \tfrac{2}{5} f(\zeta) + \tfrac{3}{5}\left(f(\zeta) - \sum_{k=1}^{n} \lambda_k p_{a_k}(\zeta)\right) \\ &< \tfrac{2}{5} f(\zeta) + \tfrac{3}{5}\varepsilon + \tfrac{3}{5} \sum_{k=1}^{n} \lambda_k (p_{b_k}(\zeta) - p_{a_k}(\zeta)) \\ &< \tfrac{2}{5} f(\zeta) + \varepsilon + \tfrac{2}{5} \sum_{k=1}^{n} \lambda_k p_{b_k}(\zeta) < \varepsilon + \tfrac{4}{5} f(\zeta). \end{aligned}$$

We have now proved that, for every positive continuous function f on ∂D and every positive ε, there exists v in F with

$$0 < f(\zeta) - v(\zeta) < \varepsilon + \tfrac{4}{5} f(\zeta) \quad (\zeta \in \partial D).$$

By Lemma 11, it follows that every such f is of the form

$$f(\zeta) = \sum_{k=1}^{\infty} v_k(\zeta) \quad (\zeta \in \partial D),$$

with v_k in F. Thus A is a P.P.B. set.

Definitions. Let A be a subset of D and E a subset of ∂D. We say that A is *uniformly non-tangential* for E if there exists a fixed α with $0 < \alpha < \pi/2$ such that, for every ζ in E and every $b > 0$, $D(\zeta, b, \alpha) \cap A$ is not void. [$D(\zeta, b, \alpha)$ is defined in Section 2]. We say that A is *radial at all points of* E if, for each ζ in E, there exists a sequence of points of A that converges radially to ζ.

Corollary 14. *Each subset of D that is uniformly non-tangential for ∂D is a P.P.B. set.*

Proof. Let A be a subset of D that is uniformly non-tangential for ∂D with the fixed angle α. Then there exists δ with $0 < \delta < 1$ such that $A(\delta)$ is radial at all points of ∂D. In fact, it is enough to take any δ with $\sin \alpha < \delta < 1$. For, given $b > 0$, we have $z = x + yi$ in $D(b, \alpha) \cap A$, that is with $b > 1 - x > |y| \cot \alpha$. For such z,

$$d = d(z, x) = |y|\,|1 - xz|^{-1},$$

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$$d^2 = y^2\{(1-x^2)^2 + x^2y^2\}^{-1} \leqq \{\cot^2\alpha + x^2\}^{-1},$$

and the last term tends to $\sin^2\alpha$ as $x \to 1$.

Let $h \in h^1$ with $\sup_{z \in A(\delta)} h(z) = 0$, and let μ, F be the corresponding real Borel measure and function of bounded variation on $[0, 2\pi]$. Thus

$$h(z) = \frac{1}{2\pi}\int_0^{2\pi} P_z(t)\, d\mu(t) = \frac{1}{2\pi}\int_0^{2\pi} P_z(t)\, dF(t).$$

We prove first that $\mu(\{t_0\}) \leqq 0$ for each point t_0 of $[0, 2\pi]$. There exist points $b_n = r_n e^{it_0}$ in $A(\delta)$ with $0 < r_n < 1$ and $\lim_{n\to\infty} r_n = 1$. Let $g_n(t) = (1-r_n)(1+r_n)^{-1} P_{b_n}(t)$. Then $\|g_n\|_\infty = 1$, and

$$\lim_{n\to\infty} g_n(t) = \varepsilon_{t_0}(t) = \begin{cases} 1 & (t = t_0) \\ 0 & (t \neq t_0). \end{cases}$$

By Lebesgue's theorem of dominated convergence, it follows that

$$\begin{aligned} \mu(\{t_0\}) &= \int_0^{2\pi} \varepsilon_{t_0}(t)\, d\mu(t) \\ &= \lim_{n\to\infty} \int_0^{2\pi} g_n(t)\, d\mu(t) \\ &= \lim_{n\to\infty} 2\pi(1-r_n)(1+r_n)^{-1} h(b_n) \leqq 0. \end{aligned}$$

We prove next that $F'(t) \leqq 0$ whenever the derivative exists, finite or infinite. Suppose that $F'(t_0)$ exists, and let b_n be as above. Since Fatou's theorem is valid for radial convergence even when the derivative is infinite, we have

$$F'(t_0) = \lim_{n\to\infty} h(r_n e^{it_0}) \leqq 0.$$

Let X be the countable set of discontinuities of F, N the set of points of $[0, 2\pi] \backslash X$ at which $F'(t)$ does not exist (in $[-\infty, \infty]$), and let Y be the complement of $X \cup N$. It is known (see Saks [**6**, Theorem 9.1, p. 125]) that $|\mu|(N) = 0$. Let E be a Borel subset of $[0, 2\pi]$. Then $\mu(E \cap X) \leqq 0$ and $\mu(E \cap N) = 0$. Also, since $F'(t) \leqq 0$ at all points of $E \cap Y$, it is known that $\mu(E \cap Y) \leqq 0$ (see Saks [**6**, Lemma 9.4 p. 126]). Thus $\mu(E) \leqq 0$, $-\mu$ is a positive measure, $h(z) \leqq 0$ for all z in D. It follows that, for all functions h in h^1, we have

$$\sup_{z \in D} h(z) = \sup_{z \in A(\delta)} h(z).$$

Therefore, by Theorem 10, A is a P.P.B. set.

Remarks. For ζ in ∂D and $0<\rho<1$, let $G(\zeta,\rho)$ denote the open disc in D of radius ρ that is tangent to ∂D *at* ζ. We are indebted to W. K. Hayman for the observation that if A is a P.P.B. set, then every open tangent disc $G(\zeta,\rho)$ contains points of A. To see this, note that $z\in G(\zeta_0,\rho)$ if and only if $p_z(\zeta_0)>(1-\rho)/\rho$. Suppose that $G(\zeta_0,\rho)\cap A=\varnothing$. Then

$$p_a(\zeta_0)\leqq(1-\rho)/\rho \ (a\in A).$$

If, for some w in D, p_w is of the form

$$p_w(\zeta)=\sum_{n=1}^{\infty}\lambda_n p_{a_n}(\zeta) \ (\zeta\in\partial D),$$

with $\lambda_n>0$ and a_n in A, then integration gives $\sum_{n=1}^{\infty}\lambda_n=1$, and so

$$p_w(\zeta_0)\leqq(1-\rho)/\rho.$$

This shows that w is not in $G(\zeta_0,\rho)$, and so A is not a P.P.B. set.

Taking $A=D\backslash G(\zeta_0,\rho)$, we have an example of a set A, radial at all points of $\partial D\backslash\{\zeta_0\}$, that is not a P.P.B. set.

On the other hand, a slight modification of the proof of Corollary 14 shows that, if A is non-tangential at every point of ∂D and uniformly non-tangential for Z with $\partial D\backslash Z$ countable, then A is a P.P.B. set. For, given t_0 in $[0,2\pi]$, we have a_n in A such that $\{a_n\}$ converges to e^{it_0} non-tangentially. Then $[p_{a_n}(t_0)]^{-1}p_{a_n}(t)$ converges boundedly to $\varepsilon_{t_0}(t)$, and so $\mu(\{t_0\})\leqq 0$ as before. Therefore, given a Borel set E in ∂D, we have $\mu(E\backslash Z)\leqq 0$. That $\mu(E\cap Z)\leqq 0$ is proved as before.

In the proof of Corollary 14, we need to use radial convergence because Fatou's theorem can fail for non-tangential convergence to e^{it_0} if $F'(t_0)=+\infty$.

Every real or complex continuous function on ∂D can be expressed as a uniformly convergent series $\sum_{k=1}^{\infty}\lambda_k p_{a_k}(\zeta)$ with the a_k in a given P.P.B. set. In fact, let A be a P.P.B. set, f a real continuous function on ∂D and $\varepsilon>0$. Then there exist a_k in A and real λ_k with $\sum_{k=1}^{\infty}|\lambda_k|<\|f\|_1+\varepsilon$, such that

$$f(\zeta)=\sum_{k=1}^{\infty}\lambda_k p_{a_k}(\zeta) \ (\zeta\in\partial D),$$

with the series uniformly convergent on ∂D. For, we have non-negative continuous functions f_+, f_- with $f=f_+-f_-$, $|f|=f_++f_-$. There exist a_k in A and non-negative α_k,β_k with

$$f_+(\zeta)+\varepsilon/2=\sum_{k=1}^{\infty}\alpha_k p_{a_k}(\zeta),\ f_-(\zeta)+\varepsilon/2=\sum_{k=1}^{\infty}\beta_k p_{a_k}(\zeta) \ (\zeta\in\partial D).$$

The series are uniformly convergent, and, taking $\lambda_k=\alpha_k-\beta_k$, we have

$$\sum_{k=1}^{\infty}|\lambda_k|\leqq\sum_{k=1}^{\infty}(\alpha_k+\beta_k)=\|f_+\|_1+\|f_-\|_1+\varepsilon=\|f\|_1+\varepsilon$$

and f is of the required form.

For applications, it may be valuable to have uniform approximations for a given function with known coefficients λ_k. By imposing stronger conditions on the set A, we obtain an approximation theorem of this kind.

Theorem 15. *Let $\{r_n\}$, $\{R_n\}$ satisfy $0\leqq r_n\leqq R_n<1$, $\lim_{n\to\infty}r_n=1$, $\lim_{n\to\infty}n^{-1}(1-r_n)^{-1}=0$, $\lim_{n\to\infty}(R_n-r_n)(1-r_n)^{-1}=0$. Let A be a subset of D such that, for some arbitrarily large n and for each k in $(1,2,\dots,2n)$, there exists $a_k=\rho_k e^{i\psi_k}$ in A with $r_n\leqq\rho_k\leqq R_n$ and $(k-1)\pi/n\leqq\psi_k\leqq k\pi/n$.*

Then, given a complex valued function f continuous on ∂D and a positive constant ε, there exists a positive integer n such that, with $a_1,a_2,\dots,a_{2n}$ as stated,

$$\left\|f-\frac{1}{2n}\sum_{k=1}^{2n}f(e^{i\psi_k})p_{a_k}\right\|_\infty<\varepsilon,$$

Proof. Let f be a complex continuous function on ∂D, $M=\|f\|_\infty$, $\varepsilon>0$, and, for $0\leqq r<1$, let

$$f(re^{i\theta})=\frac{1}{2\pi}\int_0^{2\pi}f(e^{it})P_{re^{i\theta}}(t)\,dt.$$

Choose n_1 such that

$$|f(e^{i\theta})-f(r_ne^{i\theta})|<\varepsilon/4\ (n\geqq n_1,\theta\in\mathbb{R}).$$

Then choose $n_2\geqq n_1$ such that, for $n\geqq n_2$, $|f(e^{i\phi})-f(e^{i\psi})|\leqq\varepsilon/4$ whenever $|\phi-\psi|\leqq\pi/n$. Finally, choose $n\geqq n_2$ such that there exist $a_1,a_2,\dots,a_{2n}$ in A of the stated form, and such that $(R_n-r_n)(1-r_n)^{-1}+\pi n^{-1}(1-r_n)^{-1}<\delta$ with $2\delta(1-\delta)^{-1}<\varepsilon/4M$. Note that, with $(k-1)\pi/n\leqq t\leqq k\pi/n$, we have

$$\begin{aligned}d(a_k,r_ne^{it})&\leqq d(a_k,r_ne^{i\psi_k})+d(r_ne^{i\psi_k},r_ne^{it})\\&\leqq(\rho_k-r_n)(1-\rho_kr_n)^{-1}+r_n|\psi_k-t|\,|1-r_n^2e^{i(\psi_k-t)}|^{-1}\\&\leqq(R_n-r_n)(1-r_n)^{-1}+\pi n^{-1}(1-r_n)^{-1}<\delta,\end{aligned}$$

with $2\delta(1-\delta)^{-1}<\varepsilon/4M$. By Lemma 12, for such t, we have

$$|P_{r_n e^{it}}(\theta) - P_{a_k}(\theta)| \leqq \frac{\varepsilon}{4M} P_{r_n e^{it}}(\theta).$$

It follows that

$$\begin{aligned}
&\left|f(r_n e^{i\theta}) - (2n)^{-1} \sum_{k=1}^{2n} f(e^{i\psi_k}) P_{a_k}(\theta)\right| \\
&\leqq \left|\sum_{k=1}^{2n} \frac{1}{2\pi} \int_{(k-1)\pi/n}^{k\pi/n} (f(e^{it}) - f(e^{i\psi_k}))\, P_{r_n e^{i\theta}}(t)\, dt\right| \\
&\quad + \left|\sum_{k=1}^{2n} \frac{1}{2\pi} \int_{(k-1)\pi/n}^{k\pi/n} f(e^{i\psi_k})(P_{r_n e^{i\theta}}(t) - P_{a_k}(\theta)\, dt\right| \\
&< \varepsilon/4 + M \sum_{k=1}^{2n} \frac{1}{2\pi} \int_{(k-1)\pi/n}^{k\pi/n} |P_{r_n e^{it}}(\theta) - P_{a_k}(\theta)|\, dt \\
&< \varepsilon/2.
\end{aligned}$$

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