THE EFFECT OF A MAGNETIC FIELD ON STELLAR PULSATIONS AS A SINGULAR PERTURBATION PROBLEM

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ABSTRACT

The perturbation problem that describes the effect of a weak magnetic field on stellar adiabatic oscillation is considered. This perturbation problem is singular when the magnetic field does not vanish at the stellar surface, and a regular perturbation scheme fails where the magnetic pressure is comparable to the thermodynamic pressure. The application of the Method of Matched Asymptotic Expansion is used to obtain expressions for the eigenfunctions and the eigenfrequencies.

1. INTRODUCTION

The discovery of magnetic fields in a few pulsating stars calls for a theoretical discussion of pulsation in the presence of a magnetic field. As an initial step toward understanding the role of magnetic fields in pulsating stars, we present a first order perturbation approach to the effect of a weak, large scale magnetic field on the linear normal oscillation modes of a spherical non-magnetic star. The objective is to obtain expressions for both the eigenfrequencies and the eigenfunctions.

Magnetic fields with reasonably large scales are considered, so their normal natural decay times are short compared to stellar lifetimes. The magnetic field is assumed to be weak in the global sense that $M/|W| \ll 1$ with M being the magnetic and W the gravitational potential energy. In Section 2 it is recalled that the perturbation problem is singular for a magnetic field that is not identical zero on the stellar surface. In Section 3 the systems of differential equations are derived that govern the oscillations in the outer region where the regular perturbation scheme is valid. In Section 4 the concepts of inner region and inner variables are discussed, and an application to the "pseudo-radial"oscillation modes of Ferraro's model (Ferraro, 1954) is made. The effect of the magnetic field on the eigenfrequencies and eigenfunctions is discussed.

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2. SINGULAR CHARACTER OF THE PERTURBATION PROBLEM

The linear and adiabatic normal oscillation modes of a gaseous star with a magnetic field are the solutions of the eigenvalue problem

$$\sigma^2\vec{\xi} = L \cdot \vec{\xi} \quad \text{in } V, \qquad B \cdot \vec{\xi} = 0 \quad \text{on } S, \qquad (1\ ab)$$

where V is the non-spherical volume of the magnetic equilibrium configuration that is bounded by the surface S. $\vec{\xi}$ is the Lagrangian displacement, σ is the oscillation frequency, and the operators L and B are defined in Goossens (1972) and Goossens et al. (1976).

Denote the perturbation expansion parameter as $h \sim M/|W|$, and compare (1) with the corresponding eigenvalue problem of a spherical non-magnetic star. The latter is defined by

$$\sigma_0^2\vec{\xi}_0 = L_0 \cdot \vec{\xi}_0 \text{ in } V_0, \qquad B_0 \cdot \vec{\xi}_0 = 0 \quad \text{on } S_0, \qquad (2\ ab)$$

where the subscript "0" indicates a quantity of the spherical non-magnetic case. As shown by Goossens et al. (1976), the perturbation problem is singular when the magnetic field is non-zero on S. A regular perturbation scheme fails in a region where the magnetic pressure $H^2/8\pi \sim O(p)$ or larger, with p being the thermodynamic pressure. The singular perturbation problem is treated by the Method of Matched Asymptotic Expansions (M. M. A. E.). In this method a straightforward perturbation expansion, called outer expansion, is introduced in terms of the original variables, called outer variables. The outer expansion is valid in the outer region and fails in the inner region. The difficulty of the non-uniformity of the outer expansion in the inner region is overcome by the introduction of a new expansion, called inner expansion, in terms of new inner variables, which are of order unity in the inner region. The two expansions are matched according to the Asymptotic Matching Principle of Van Dyke (Van Dyke, 1964), and a composite expansion valid in the entire domain is constructed.

Of course, to perform a perturbation analysis, (1) has to be defined in V_0 and on S_0 (see Goossens, 1972; Goossens et al., 1976). In what follows all quantities and equations are defined in V_0.

3. PERTURBATION EQUATIONS FOR THE OUTER REGION

Consider now axisymmetric oscillations. Unlike linear adiabatic oscillations of a sphere, a normal mode is now not specified by an individual spherical harmonic. The displacement field is given by

$$\xi_r = \sum_{k=0}^{\infty} a_k(r)P_k(\mu), \quad \xi_\theta = -\sum_{k=1}^{\infty} b_k(r)P_k^1(\mu), \quad \xi_\varphi = \sum_{k=1}^{\infty} t_k(r)P_k^1(\mu). \quad (3)$$

$P_k(\mu)$ is the Legendre polynomial of degree k with $\mu = \cos\theta$; $a_k(r)$ $b_k(r)$, and $t_k(r)$ are functions of r; the functions a_k, b_k define the spheroidal part; and the functions t_k define the toroidal part of the

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displacement field. Use (3) to expand ρ', p', and ϕ' as

$$f'(r,\theta) = \sum_k f'_k(r)\, P_k(\mu) \tag{4}$$

where the functions $f'_k(r)$ consist of a zero order and a first order part

$$f'_k(r) = f'_{k,0}(r) + hf'_{k,1}(r) \tag{5}$$

simply because the equilibrium quantities consist of a zero-order and a first-order part.

As far as the displacement fields are concerned, $f'_{k,0}$ only involves the spheroidal displacement field associated with $P_k(\mu)$, but $f'_{k,1}$ involves spheroidal and toroidal displacement fields associated with various $P_\lambda(\mu)$ depending on the angular structure of the magnetic field. For example in the case of a purely poloidal dipole-type field, $f'_{k,1}$ involves spheroidal displacement fields associated with the Legendre polynomial of order $k-2$, k, $k+2$, and can be represented as

$$f'_{k,1} = f'_{k,1,-2} + f'_{k,1,0} + f'_{k,1,2} \tag{6}$$

where $f'_{k,1,-2}$ denotes the contribution of (a_{k-2}, b_{k-2}) to $f'_{k,1}$ etc.

So far the radial functions $a_k(r)$, $b_k(r)$, and $t_k(r)$ have not yet been specified. We now concentrate on the effect of the magnetic field on a particular spheroidal normal mode of the sphere associated with $P_\ell(\mu)$, and we expand the functions $a_k(r)$, $b_k(r)$, $t_k(r)$ in the outer region to the first order in h as

$$\begin{cases} a_\ell = a_{\ell,0} + ha_{\ell,1}\ , \\ b_\ell = b_{\ell,0} + hb_{\ell,1}\ , \end{cases} \quad \begin{cases} a_k = ha_{k,1}\ , \\ b_k = hb_{k,1}\ , \end{cases} \quad \text{for } k \neq \ell,\ t_k = ht_{k,1} \text{ for all } k. \tag{7}$$

$a_{\ell,0}(r)$ and $b_{\ell,0}(r)$ define the displacement field of the spheroidal normal oscillation mode of the sphere associated with $P_\ell(\mu)$.

Use expansion (7) to further evaluate the quantities f'_k and to obtain retaining terms up to the first order in h (for an axisymmetric purely poloidal dipole-type field)

$$f'_{\ell-2} = hf'_{\ell-2,0,1} + hf'_{\ell-2,1,2}\ ;\quad f'_\ell = f'_{\ell,0,0} + hf'_{\ell,0,1} + hf'_{\ell,1,0}\ ;$$
$$f'_{\ell+2} = hf'_{\ell+2,0,1} + hf'_{\ell+2,1,-2};\quad f'_k = hf'_{k,0,1} \text{ for } k \neq \ell-2, \ell, \ell+2. \tag{8}$$

$f'_{\ell,0,0}$ and $f'_{k,0,1}$ denote the contributions of $(a_{\ell,0}, b_{\ell,0})$ to $f'_{\ell,0}$ and of $(a_{k,1}, b_{k,1})$ to $f'_{k,0}$. Hence $f'_{\ell,0,0}$ is a quantity of the zero-order oscillation problem, while $f'_{k,0,1}$ is a quantity of the first-order oscillation problem. $f'_{\ell-2,1,2}$ and $f'_{\ell+2,1,2}$ now denote the contributions of $(a_{\ell,0}, b_{\ell,0})$ to the quantities indicated by the same notation in Equation (6), as the contributions of $(a_{\ell,1}, b_{\ell,1})$ to the latter quantities are of the second order in h. $f_{\ell-2,1,2}$ and $f_{\ell+2,1,-2}$

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depend only on the equilibrium and the zero-order oscillation properties. They give rise to non-homogeneous terms in the differential equations that govern the oscillations in the outer region. Expand the frequency as

$$\sigma^2 = \sigma_0^2 + h\sigma_1^2 .$$

As in the sphere, introduce dimensionless variables defined as

$$\vec{W}_\ell^{(0)} = \begin{pmatrix} \frac{1}{R} x\, a_{\ell,0} \\ \frac{R}{GM} \frac{p'_{\ell,0,0}}{\rho_0} \\ \frac{R}{GM} \phi'_{\ell,0,0} \\ x \frac{dW^{(0)}_{\ell,3}}{dx} \end{pmatrix} ; \qquad \vec{W}_k^{(1)} = \begin{pmatrix} \frac{1}{R} x\, a_{k,0} \\ \frac{R}{GM} \frac{p'_{k,0,1}}{\rho_0} \\ \frac{R}{GM}(\phi'_{k,0,1} + \phi'_{k,1,0}) \\ x \frac{dW^{(1)}_{k,3}}{dx} \end{pmatrix} . \qquad (9)$$

An axisymmetric poloidal dipole-type field introduces additional spheroidal displacement fields associated with $\ell - 2$, ℓ, $\ell + 2$; all other oscillation quantities are identically zero.

The zero-order variables $\vec{W}_\ell^{(0)}$ satisfy a system of four linear first-order differential equations

$$\frac{d\vec{W}_\ell^{(0)}}{dx} = A_\ell(x, \omega_0^2)\, \vec{W}_\ell^{(0)} \qquad (10)$$

where $A_\ell(x, \omega_0^2)$ is the matrix of coefficients that involve ℓ and the square of the dimensionless frequency ω_0^2 and $x = r/R$. The first order variables $\vec{W}_k^{(1)}$ $(k = \ell - 2, \ell, \ell + 2)$ satisfy systems of four linear non-homogeneous differential equations

$$\frac{d\vec{W}_{\ell-2}^{(1)}}{dx} = A_{\ell-2}(x,\omega_0^2)\vec{W}_{\ell-2}^{(1)} + \vec{F}_{\ell-2}(x) , \quad \frac{d\vec{W}_{\ell}^{(1)}}{dx} = A_{\ell}(x,\omega_0^2)\vec{W}_{\ell}^{(1)} + \vec{F}_{\ell}(x,\omega_1^2) ,$$

$$\frac{d\vec{W}_{\ell+2}^{(1)}}{dx} = A_{\ell+2}(x,\omega_0^2)\vec{W}_{\ell+2}^{(1)} + \vec{F}_{\ell+2}(x) . \qquad (11\ abc)$$

The homogeneous parts of (11 a) and (11 c) are identical to the equations that govern the oscillations of the sphere associated with $P_{\ell-2}$ and $P_{\ell+2}$, but with the eigenvalue parameter replaced by an eigenvalue ω_0^2 for an oscillation associated with P_ℓ in the sphere. The homogeneous part of (11 b) is identical to (10). The non-homogeneous part of (11 b) involves the correction ω_1^2 on the square of the dimensionless eigenfrequency. Equations (11) are the perturbation equations that describe the effect of the magnetic field on the oscillations in the outer region.

4. PERTURBATION EQUATIONS FOR THE INNER REGION

The definition of the inner region, inner variables, and inner expansions depends on the equilibrium distribution of pressure. The equations that govern the oscillations in the inner region can only be derived when the distribution of pressure is known. As an initial attempt to understand the mathematical problem, we consider "pseudo-radial" modes of Ferraro's model, which is a spheroid of constant density with an axisymmetric purely poloidal dipole-type magnetic field (Ferraro, 1954). The adjective "pseudo-radial" is used to indicate that the original radial oscillations of the sphere now also have a spheroidal displacement field associated with $P_2(\mu)$.

The distribution of pressure in Ferraro's model satisfies $p \sim 1 - x^2$, so that the inner region is defined by $1 - x = O(h)$ and has a radial thickness $\delta = O(h)$. The small perturbation parameter $h = e^2$ is choosen. To illustrate the application of the M.M.A.E., we list the partial results on the first overtone radial mode. The outer solutions are

$$w_{0,j}^{(0)} = \text{polynomials in } x^2 \text{ with terms up to } x^2 \,,$$

$$w_{0,j}^{(1)} = A_j w_{0,j}^{(0)} \, \ln(1 - x^2) + \sum_{\lambda=0}^{3} A_{2\lambda,j} \, x^{2\lambda} \,,$$

$$w_{2,j}^{(1)} = \text{polynomials in } x^2 \text{ with terms up to } x^6 \,, \qquad (12\text{ abc})$$

where A_j are known constants, and $A_{2\lambda,j}$ are constant that can be expressed linearly in the three as yet undetermined constants ω_1^2, $A_{2,1}$, and $A_{0,3}$.

The functions $w_{0,j}^{(1)}$ contain a term $\ln(1 - x^2)$ and are singular at $x = 1$. This clearly illustrates that the regular perturbation expansion breaks down in the inner region.

The inner variables are obtained by stretching the original variables by appropriate functions of h such that they are O(1) in the inner region. The inner independent variable η is defined as

$$\eta = (1 - x^2)/h \,. \qquad (13)$$

We do not stretch the dependent variables since they are already O(1) in the inner region. They are now denoted by $\vec{\Omega}$ with zero-order part $\vec{\Omega}^{(0)}$ and a first-order part $\vec{\Omega}^{(1)}$. Rewrite the equations in inner variables to obtain equations relative to the inner region. The solutions can be obtained in closed analytical form and read after application of the boundary condition on the conservation of momentum

$$\Omega_{0,j}^{(0)} = \text{constant}, \qquad (14)$$

$$\Omega_{0,j}^{(1)} = \text{analytical expressions in } \eta \text{ with two undetermined constants}. \qquad (15)$$

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The five remaining constants are determined by matching the 2-term outer expansion and the 2-term inner expansion by application of the continuity condition to the gravitational potential. The matching requires terms in h $\ln$ h.

We now have an outer solution valid in the outer region and an inner solution valid in the inner region. The method of additive composition is used to obtain the composite solution that is valid over the entire domain. As an example, the composite solution for $w_{0,1}$ reads as

$$(w_{0,1})^c_{2,2} = w^{(0)}_{0,1} + h \left\{ - \frac{14}{225\ \Gamma_1} (1 - x^2)(1 + \frac{7}{2} x^2) \ln(1 - x^2) \right.$$
$$+ \text{polynomials in } x^2 \text{ with terms up to } x^6$$
$$\left. + \frac{14}{225\ \Gamma_1} \ln[225\ \Gamma_1 (1 - x^2) + 4\ h] \right\}. \qquad (16)$$

The composite solution is regular over the total domain, also in x = 1. In x = 1, however, the composite solution contains a term in h $\ln$ h that is never included in a regular perturbation expansion but that, for small values of h, is more important than h. The eigenfrequency is

$$\omega^2 \doteq \frac{38}{3} [1 + (\frac{4549}{36575} - \frac{86}{1463} \Gamma_1)h] \ . \qquad (17)$$

Equation (17) shows that the eigenvalue of the radial first overtone increases with h. The magnetic field also has a distinct effect on the displacement field that is no longer purely spherically symmetric. Take, as an example, the amplitude of $\xi_r(r, \theta)$ for $e^2 = 0.05$. For $\theta < 62^\circ.9$ this amplitude is enhanced in the layers $0 \leqslant x \leqslant 0.88$ and reduced for $x \geqslant 0.88$, while for $\theta > 62^\circ.9$, the reverse effect occurs now with a critical value x = 0.82.

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DISCUSSION

J. COX: Are these last results for the constant density model?

GOOSSENS: Yes, only for the constant density model. It has a homogeneous constant density. The effect of the magnetic field will be more drastic for models with a density gradient. The inner region will be larger.

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