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EIGENVALUES OF SOME NON-LOCAL BOUNDARY-VALUE PROBLEMS

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Abstract Working on a suitable cone of continuous functions, we give new results for integral equations of the form $\lambda u(t) = \int_G k(t,s)f(s,u(s))\,\mathrm{d}s := Tu(t)$, where G is a compact set in $\mathbb{R}^n$ and k is a possibly discontinuous function that is allowed to change sign. We apply our results to prove existence of eigenvalues of some non-local boundary-value problems.

Keywords: fixed-point index; cone; non-zero solution

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Secondary 34B18; 47H10; 47H30

1. Introduction

In this paper we study the existence of eigenvalues of a Hammerstein integral equation of the form

$$\lambda u(t) = \int_G k(t,s)f(s,u(s))\,\mathrm{d}s := Tu(t), \tag{1.1}$$

where G is a compact set in $\mathbb{R}^n$ with $\mathrm{meas}(G) > 0$, k and f are allowed to be discontinuous and k may change sign. This type of problem, under a variety of conditions on the function f and with a positive kernel k, has been studied by several authors (see, for example, [**9**, **10**, **12**] and, with $\lambda = 1$, [**1**, **2**, **8**, **11**, **13**]). All papers cited above focus on particular problems where *positive* solutions of the integral equation exist. Here we deal with more general equations where positive solutions might not exist; nevertheless we are able to achieve existence of *non-trivial* solutions.

The tool we use is a well-known result for compact maps in order to establish existence of eigenvalues, working on the cone

$$K = \{u \in C(G) : \min\{u(t) : t \in G_0\} \geqslant c\|u\|\},$$

where G_0 is a closed subset of G. This type of cone was introduced by Infante and Webb in [**5**] and is a larger cone than the one used by Lan [**8**, **9**].

Our results apply to second-order differential equations of the form

$$\lambda u''(t) + f(t, u(t)) = 0 \quad (0 < t < 1), \tag{1.2}$$

subject to suitable boundary conditions (BCs).

Under a variety of BCs, Lan and Webb [**10**] and, more recently, Ma [**12**] studied eigenfunctions of the special case $f(t, u(t)) = g(t)h(u)$. Lan and Webb [**10**] proved that for every λ lying in a certain interval there exists a positive solution of equation (1.2). Our result is of a different kind. Under weaker assumptions we show that there exists a positive (or negative) λ such that equation (1.2) has a non-zero solution. This improves the results of Lan [**9**], who proved existence of positive solutions under separated BCs, but allowing positive kernels only.

In this paper we concentrate on the following non-local boundary-value problems (BVPs):

$$u'(0) = 0, \quad \alpha u'(\eta) = u(1), \quad 0 < \eta < 1, \tag{1.3 a}$$

$$u(0) = 0, \quad \alpha u'(\eta) = u(1), \quad 0 < \eta < 1, \tag{1.3 b}$$

$$u'(0) = 0, \quad \alpha u(\eta) = u(1), \quad 0 < \eta < 1, \tag{1.3 c}$$

$$u(0) = 0, \quad \alpha u(\eta) = u(1), \quad 0 < \eta < 1. \tag{1.3 d}$$

The boundary condition (1.3 b), under stronger assumptions on the growth of f and with $\lambda = 1$, has been studied by Infante and Webb in [**6**]. They proved existence of at least one and existence of multiple non-trivial solutions. Existence theorems under the two conditions (1.3 c) and (1.3 d) have been widely studied by Gupta and co-workers (see, for example, [**3**, **4**] and the references cited therein). Webb [**13**] studied the existence of positive solutions of (1.3 c), (1.3 d), improving a result of Ma [**11**], who dealt with the sublinear and superlinear case only for (1.3 d). Infante and Webb [**5**], with a technique similar to that of [**6**], also studied the conditions (1.3 c), (1.3 d), showing existence of multiple non-trivial solutions.

The conditions we impose on f are quite weak, a non-negative function that satisfies Carathéodory conditions and has suitable growth properties. In particular, we do not exclude the possibility that $f(t, 0) = 0$ for all t, so that 0 is a solution of equation (1.2) with the given BCs. Here we prove existence of non-trivial solutions that are positive (or negative) on an interval.

The results obtained are new and the BC (1.3 a) is studied for the first time.

2. Existence of eigenvalues of Hammerstein integral equations

We begin by giving some results for the following Hammerstein integral equation:

$$\lambda u(t) = \int_G k(t, s) f(s, u(s))\, \mathrm{d}s := Tu(t), \tag{2.1}$$

where G is a compact set in $\mathbb{R}^n$ of positive measure. We make the following assumptions on f and the kernel k for some $r > 0$; we assume throughout that the following conditions hold.

(C_1) $f : G \times [-r, r] \to [0, \infty)$ satisfies Carathéodory conditions on $G \times [-r, r]$ and there exists a measurable function $g_r : G \to [0, \infty)$ such that

$$f(t, u) \leqslant g_r(t) \quad \text{for almost every } t \in G \text{ and all } u \in [-r, r].$$

(C_2) $k : G \times G \to \mathbb{R}$ is measurable, and for every $\tau \in G$ we have

$$\lim_{t \to \tau} \int_G |k(t, s) - k(\tau, s)| g_r(s) \, \mathrm{d}s = 0.$$

(C_3) There exist a closed subset $G_0 \subset G$ with $\operatorname{meas}(G_0) > 0$, a measurable function $\Phi : G \to [0, \infty)$ and a constant $c \in (0, 1]$ such that

$$\begin{aligned} |k(t, s)| &\leqslant \Phi(s) \quad \text{for } t \in G \text{ and almost every } s \in G, \\ c\Phi(s) &\leqslant k(t, s) \quad \text{for } t \in G_0 \text{ and almost every } s \in G. \end{aligned}$$

(C_4) There is $M_r < \infty$ such that $\int_G \Phi(s) g_r(s) \, \mathrm{d}s \leqslant M_r$.

Assumptions (C_1)–(C_4) were also made in [**6**]. Hypothesis (C_3) means finding upper bounds for $|k|$ on G and lower bounds of the same form for k on G_0. In applications we have some freedom of choice in determining G_0 but we are constrained by needing $k(t, s)$ to be positive for almost every $t \in G_0$ and $s \in G$.

These hypotheses will allow us to work in the cone

$$K = \{u \in C(G) : \min\{u(t) : t \in G_0\} \geqslant c\|u\|\}.$$

This is a larger cone than used by Lan [**8**, **9**]. Note that functions in K are positive on the subset G_0 but may change sign on G.

Notation. We let $K_r = \{u \in K : \|u\| < r\}$, $\bar{K}_r = \{u \in K : \|u\| \leqslant r\}$ and $\partial K_r = \{u \in K : \|u\| = r\}$.

We need the following theorem.

Theorem 2.1 (see [6]). *Assume that* (C_1)–(C_4) *hold for some* $r > 0$. *Then* T *maps* $\bar{K}_r$ *into* K *and is compact.*

We use the following well-known result (see, for example, Lemma 1.1 in Chapter 5 of [**7**]).

Lemma 2.2. *Let* $T : \bar{K}_r \to K$ *be compact and suppose that*

$$\inf_{x \in \partial K_r} \|Tx\| > 0.$$

Then there exist $\lambda_0 > 0$ *and* $x_0 \in \partial K_r$ *such that* $\lambda_0 x_0 = Tx_0$.

The following theorem generalizes Lan's results, allowing operators with negative kernels.

Theorem 2.3. *Assume that there exists $\rho \in (0, r]$ such that*

(i) *there exists a measurable function $m_\rho : G_0 \to \mathbb{R}_+$ such that*

$$f(s, u) \geqslant m_\rho(s) \quad \text{for all } \ u \in [c\rho, \rho] \text{ and almost all } s \in G_0;$$

and

(ii) $\tau := \sup_{t \in G_0} \int_{G_0} k(t, s) m_\rho(s) \, \mathrm{d}s > 0.$

Then there exist λ_0 and $u_0 \in \partial K_\rho$ such that $\lambda_0 u_0 = T u_0$.

Proof. Since T satisfies the hypotheses of Theorem 2.1, $T : \bar{K}_r \to K$ is compact. Let $u \in \partial K_\rho$, then we have, for every $s \in G_0$, $c\rho \leqslant u(s) \leqslant \rho$. For $t \in G_0$ we have $k(t, s) \geqslant 0$ and

$$|Tu(t)| \geqslant \int_{G_0} k(t, s) f(s, u(s)) \, \mathrm{d}s \geqslant \int_{G_0} k(t, s) m_\rho(s) \, \mathrm{d}s.$$

Thus

$$\|Tu\| = \sup_{t \in G} |Tu(t)| \geqslant \sup_{t \in G_0} |Tu(t)| \geqslant \tau \quad \text{and} \quad \inf_{u \in \partial K_\rho} \|Tu\| > 0.$$

By Lemma 2.2 we obtain the existence of an eigenvalue $\lambda_0 > 0$. □

Remark 2.4. In [**9**], due to the positive nature of the kernel, Lan is able to take a larger τ, namely $\tau = \sup_{t \in G} \int_{G_0} k(t, s) m_\rho(s) \, \mathrm{d}s > 0$.

Remark 2.5. We shall see below that, for the integral equations corresponding to the BCs (1.3 c), (1.3 d) and certain values of the parameter α, the kernel $k(t, s)$ is negative for t in some interval G_0, for all s. In this case, assuming f is positive, we can show that a negative eigenvalue exists by studying the operator $-T$. Indeed, λ is an eigenvalue for

$$\lambda u(t) = \int_G k(t, s) f(s, u(s)) \, \mathrm{d}s$$

if and only if $\tilde{\lambda}$ is an eigenvalue of

$$\tilde{\lambda} u(t) = \int_G \tilde{k}(t, s) f(s, u(s)) \, \mathrm{d}s \equiv \tilde{T} u(t),$$

where $\tilde{k} = -k$ and $\tilde{\lambda} = -\lambda$. Hence we can obtain a result, the same as the one above, for the existence of negative eigenvalues. We do not state the obvious theorem thus obtained.

3. Eigenvalues of Problem (1.3 a)

As an application of the theory, we investigate in this section the existence of eigenvalues of equations of the form

$$\lambda u''(t) + f(t, u(t)) = 0 \quad \text{a.e. on } [0, 1], \tag{3.1}$$

with boundary conditions

$$u'(0) = 0, \quad \alpha u'(\eta) = u(1), \quad 0 < \eta < 1. \tag{3.2}$$

By an eigenvalue of this problem we mean an eigenvalue of the related Hammerstein integral equation

$$\lambda u(t) = \int_G k(t,s) f(s, u(s))\, \mathrm{d}s. \tag{3.3}$$

The solution of $u'' + y = 0$ with these BCs is

$$u(t) = \int_0^1 (1-s) y(s)\, \mathrm{d}s - \alpha \int_0^\eta y(s)\, \mathrm{d}s - \int_0^t (t-s) y(s)\, \mathrm{d}s,$$

with Green's function

$$k(t,s) = (1-s) + \begin{cases} -\alpha, & s \leqslant \eta, \\ 0, & s > \eta, \end{cases} - \begin{cases} t-s, & s \leqslant t, \\ 0, & s > t. \end{cases}$$

Note that, for $\alpha \neq 0$, the kernel is discontinuous on the line $s = \eta$. We shall study separately the cases $\alpha < 0$ and $\alpha > 1$. The case $\alpha = 0$ is included in the results of Lan [**9**], who studied separated BCs.

3.1. The case $\alpha < 0$

To simplify the calculations we write $-\beta$ in place of α, so that $\beta > 0$.

We have to exhibit $\Phi(s)$, a subinterval $[a,b] \subset [0,1]$ and a constant $c < 1$ such that

$$\begin{aligned} |k(t,s)| &\leqslant \Phi(s) \quad \text{for every } t, s \in [0,1], \\ k(t,s) &\geqslant c\Phi(s) \quad \text{for every } s \in [0,1],\ t \in [a,b]. \end{aligned}$$

We show that for these BCs we can take

$$\Phi(s) = (1-s)\left(1 + \frac{\beta}{1-\eta}\right).$$

Upper bounds.

Indeed,

$$k(t,s) \leqslant (1-s)\left(1 + \frac{\beta}{1-\eta}\right),$$

since

$$\frac{1-s}{1-\eta} \geqslant 1 \quad \text{for } s \leqslant \eta.$$

Lower bounds.

We show that we may take an arbitrary $[a,b] \subset [0,1)$.

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Case 1 ($s \leqslant \eta$). If $s > t$, then

$$k(t,s) = (1-s) + \beta \geqslant (1-s) = \left(1\bigg/\left(1+\frac{\beta}{1-\eta}\right)\right)\Phi(s).$$

If $s \leqslant t$, then

$$k(t,s) = (1-s) + \beta - (t-s),$$

which is a function decreasing in t and therefore the minimum is achieved when $t=1$. So

$$k(t,s) \geqslant \beta \geqslant \left(\beta\bigg/\left(1+\frac{\beta}{1-\eta}\right)\right)\Phi(s).$$

Case 2 ($s > \eta$). If $s > t$, then

$$k(t,s) = (1-s) = \left(1\bigg/\left(1+\frac{\beta}{1-\eta}\right)\right)\Phi(s).$$

If $s \leqslant t$, then

$$k(t,s) = (1-s) - (t-s) = 1 - t \geqslant \left(\left(\frac{1-b}{1-\eta}\right)\bigg/\left(1+\frac{\beta}{1-\eta}\right)\right)\Phi(s).$$

Thus we can take

$$c = \left(\min\left\{1, \beta, \frac{1-b}{1-\eta}\right\}\bigg/\left(1+\frac{\beta}{1-\eta}\right)\right). \tag{3.4}$$

We can now state the following result on the existence of eigenvalues of Equation (3.1) with the BC (3.2).

Theorem 3.1. *Let $\alpha < 0$, $[a,b] \subset [0,1)$, c be as in (3.4) and assume that there exists $\rho \in (0,r]$ such that*

(i) *there exists a measurable function $m_\rho : [a,b] \to \mathbb{R}_+$ such that*

$$f(s,u) \geqslant m_\rho(s) \quad \text{for all } u \in [c\rho, \rho] \text{ and almost all } s \in [a,b];$$

and

(ii) $\displaystyle\sup_{t\in[a,b]} \int_a^b k(t,s) m_\rho(s)\,\mathrm{d}s > 0.$

Then the BVP (3.1), (3.2) has a positive eigenvalue and a corresponding eigenfunction that is positive on $[a,b]$. Hence there exists an eigenfunction positive on $(0,1)$ if (i) and (ii) are satisfied for an arbitrary $[a,b] \subset [0,1)$.

3.2. The case $0 < \alpha < 1 - \eta$

When $\alpha > 0$ note that $k(1,s) = -\alpha < 0$ for every $s \in [0,\eta]$, and therefore the solution cannot be positive in all of $[0,1]$. We have to find Φ such that $|k(t,s)| \leqslant \Phi(s)$ for every $t,s \in [0,1]$ and show that there exist $[a,b] \subset [0,1]$ and a constant c such that $k(t,s) \geqslant c\Phi(s)$ for every $s \in [0,1]$ and $t \in [a,b]$. In fact we show that we can take

$$\Phi(s) = (1-s).$$

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Upper bounds.

Clearly, $k(t,s) \leqslant (1-s)$ in all cases. $k(t,s)$ is negative when $s \leqslant \eta$ and $t \geqslant s$ and $1-t-\alpha < 0$. In this case we then have

$$-k(t,s) = -1 + t + \alpha \leqslant \alpha < 1 - \eta \leqslant (1-s)$$

and we are done.

Lower bounds.

We will show that we may take $[a,b] \subset [0,\eta]$.

Case 1 ($s \leqslant \eta$). If $s > t$, then

$$k(t,s) = 1 - s - \alpha \geqslant (1-\eta-\alpha)(1-s).$$

If $s \leqslant t$, since we chose $\alpha < 1-\eta$, we obtain

$$k(t,s) = 1 - t - \alpha \geqslant 1 - \eta - \alpha \geqslant (1-\eta-\alpha)(1-s).$$

Case 2 ($s > \eta$). If $s > t$, then

$$k(t,s) = (1-s),$$

and we are done.

Since we take $b \leqslant \eta$, the case $s \leqslant t$ does not occur.

Therefore, we may set $c = (1-\eta-\alpha)$.

Theorem 3.2. *Let $0 < \alpha < 1-\eta$, $[a,b] \subset [0,\eta]$, $c = (1-\eta-\alpha)$ and assume that there exists $\rho \in (0,r]$ such that*

(i) *there exists a measurable function $m_\rho : [a,b] \to \mathbb{R}_+$ such that*

$$f(s,u) \geqslant m_\rho(s) \quad \text{for all } u \in [c\rho,\rho] \text{ and almost all } s \in [a,b];$$

and

(ii) $\displaystyle\sup_{t\in[a,b]} \int_a^b k(t,s) m_\rho(s)\,\mathrm{d}s > 0.$

Then the BVP (3.1), (3.2) has a positive eigenvalue and a corresponding eigenfunction that is positive on $[a,b]$.

We illustrate the theorem with two simple examples.

Example 3.3. Let $[a,b] = [0,\eta]$ and $f(s,u(s))$ be defined as

$$f(s,u) = \begin{cases} |u(s)|(\eta - s), & 0 \leqslant s \leqslant \eta, \\ 0, & \eta < s \leqslant 1. \end{cases}$$

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Take $0 < \rho \leqslant r < +\infty$ and $g_r = r\eta$. In this case we have $f(s,u) \leqslant g_r$ for every $u \in [-\rho, \rho]$ and $f(s,u) \geqslant c\rho(\eta - s)$ for $u \in [c\rho, \rho]$ and $s \in [0, \eta]$. Also

$$\int_0^\eta k(t,s)c\rho(\eta - s)\,\mathrm{d}s \geqslant c^2\rho \int_0^\eta (1-s)(\eta - s)\,\mathrm{d}s > 0.$$

By Theorem 3.2 we obtain the existence of a positive eigenvalue for the BVP (3.1), (3.2).

Example 3.4. Let $f(s,u) \equiv 2$. For every fixed $\rho > 0$, $\lambda = (1 - 2\alpha\eta)/\rho$ is a positive eigenvalue of the BVP (3.1), (3.2) with corresponding eigenfunction

$$u(t) = \frac{(1 - 2\alpha\eta) - t^2}{\lambda}.$$

$u(t)$ is positive on $[0, \eta]$ since $\alpha < 1 - \eta$ and u changes sign ($u(1) < 0$).

4. Eigenvalues of Problem (1.3 *b*)

We now investigate the second BVP,

$$\lambda u''(t) + f(t, u(t)) = 0 \quad (0 < t < 1), \tag{4.1}$$

with boundary conditions

$$u(0) = 0, \quad \alpha u'(\eta) = u(1), \quad 0 < \eta < 1, \quad \alpha < 1 - \eta. \tag{4.2}$$

The kernel in this case is

$$k(t,s) = \frac{t}{1-\alpha}(1-s) - \begin{cases} \dfrac{\alpha t}{1-\alpha}, & s \leqslant \eta, \\ 0, & s > \eta, \end{cases} - \begin{cases} t - s, & s \leqslant t, \\ 0, & s > t. \end{cases}$$

We study the cases when $\alpha < 0$ and $\alpha \leqslant 1 - \eta$ separately. The existence of positive eigenvalues when $\alpha = 0$ is covered by the results of Lan [**9**].

The case $\alpha < 0$

In [**6**] it has been shown that we can take

$$\Phi(s) = \max\left\{\frac{(1-\eta-\alpha)}{1-\eta}, -\frac{\alpha}{\eta}\right\}\frac{s(1-s)}{1-\alpha},$$

$[a,b] \subset (0,\eta]$ and $c = \min\{a, -\alpha\}/\max\{(1-\eta-\alpha), -\alpha/\eta\}$. Now it is clear that a theorem the same as Theorem 3.1 holds; we leave the statement to the reader.

The case $0 < \alpha < 1 - \eta$

In [**6**] it has been shown that we may take

$$\Phi(s) = \max\left\{1, \frac{\alpha}{\eta}\right\}\frac{s(1-s)}{1-\alpha}, c = \frac{\min\{a(1-\eta-\alpha), (1-b-\alpha)\}}{\max\{1, (\alpha/\eta)\}}$$

and $[a,b] \subset (0, 1-\alpha)$. A result similar to Theorem 3.2 holds. We omit the obvious statement.

5. Eigenvalues of Problem (1.3 c)

We now investigate the BVP

$$\lambda u''(t) + f(t, u(t)) = 0 \quad (0 < t < 1), \tag{5.1}$$

with boundary conditions

$$u'(0) = 0, \quad \alpha u(\eta) = u(1), \quad 0 < \eta < 1, \tag{5.2}$$

The kernel in this case is

$$k(t,s) = \frac{1}{1-\alpha}(1-s) - \begin{cases} \dfrac{\alpha}{1-\alpha}(\eta - s), & s \leqslant \eta, \\ 0, & s > \eta, \end{cases} - \begin{cases} t-s, & s \leqslant t, \\ 0, & s > t. \end{cases}$$

We shall study the cases $\alpha < 0$, $0 < \alpha < 1$ and $\alpha > 1$ separately. The case $\alpha = 0$ has been given by Lan in [**9**].

The case $\alpha < 0$

In [**5**] it has been shown that the kernel satisfies $|k(t,s)| \leqslant (1-s)$ for every $s, t \in [0,1]$ and $k(t,s) > c(1-s)$ for every $t \in [a,b]$ and $s \in [0,1]$, where $[a,b] \subset [0,\eta]$ and $c = (1-\eta)/(1-\alpha)$. Therefore, we can state the following theorem.

Theorem 5.1. *Let $\alpha < 0$, $[a,b] \subset [0,\eta]$, $c = (1-\eta)/(1-\alpha)$ and assume that there exists $\rho \in (0,r]$ such that*

(i) *there exists a measurable function $m_\rho : [a,b] \to \mathbb{R}_+$ such that*

$$f(s,u) \geqslant m_\rho(s) \quad \text{for all } u \in [c\rho, \rho] \text{ and almost all } s \in [a,b];$$

and

(ii) $\displaystyle\sup_{t\in[a,b]} \int_a^b k(t,s) m_\rho(s)\,\mathrm{d}s > 0.$

Then the BVP (5.1), (5.2) has a positive eigenvalue and a corresponding eigenfunction that is positive on $[a,b]$.

The case $0 < \alpha < 1$

In [**13**] Webb proved that we can take

$$\Phi(s) = \frac{1-s}{1-\alpha},$$

$[a,b] \subset [0,1]$ and $c = \alpha(1-\eta)$. Thus we can state a result similar to Theorem 5.1. We omit the obvious statement.

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The case $\alpha > 1$

For these BCs the kernel k is negative on an interval, so we apply Remark 2.5 and consider $-k$ in place of k. In [**5**] it has been shown that we may take

$$\Phi(s) = \frac{\alpha}{\alpha - 1}(1 - s)$$

and then $-k(t,s) > c\Phi(s)$ for $t \in [a,b]$ and $s \in [0,1]$, where $a = \eta$, $b \in (\eta,1]$ and $c = (1-\eta)/\alpha$. Therefore, we have the following result related to the existence of negative eigenvalues.

Theorem 5.2. *Let $\alpha > 1$, $[a,b]$ and c be as above and assume that there exists $\rho \in (0,r]$ such that*

(i) *there exists a measurable function $m_\rho : [a,b] \to \mathbb{R}_+$ such that*

$$f(s,u) \geqslant m_\rho(s) \quad \text{for all } u \in [c\rho,\rho] \text{ and almost all } s \in [a,b];$$

and

(ii) $\displaystyle \sup_{t\in[a,b]} \int_a^b -k(t,s) m_\rho(s)\,\mathrm{d}s > 0.$

Then the BVP (5.1), (5.2) has a negative eigenvalue and a corresponding eigenfunction that is negative on $[a,b]$.

We illustrate the theorem with the following example.

Example 5.3. Take $[a,b] = [\eta,1]$, $c = (1-\eta)/\alpha$ and let $f(s,u(s))$ be defined as

$$f(s,u) = \begin{cases} |u(s)|(s-\eta), & \eta \leqslant s \leqslant 1, \\ 1, & 0 \leqslant s < \eta. \end{cases}$$

The function f is positive and discontinuous, but satisfies Carathéodory conditions, and for $u \in [-r,r]$, $f(s,u)$ satisfies the condition (C_1) with $g_r = \max\{1,r\}$. Also $f(s,u) \geqslant c\rho(s-\eta)$ for $u \in [c\rho,\rho]$ and $s \in [\eta,1]$. Clearly, $\int_\eta^1 -k(t,s)(s-\eta)\,\mathrm{d}s > 0$. By Theorem 5.2 the BVP (5.1), (5.2) has a negative eigenvalue.

6. Eigenvalues of Problem (1.3 *d*)

We now investigate the BVP

$$\lambda u''(t) + f(t,u(t)) = 0 \quad (0 < t < 1), \tag{6.1}$$

with BCs

$$u(0) = 0, \quad \alpha u(\eta) = u(1), \quad 0 < \eta < 1. \tag{6.2}$$

The kernel in this case is

$$k(t,s) = \frac{1}{1-\alpha\eta} t(1-s) - \begin{cases} \dfrac{\alpha t}{1-\alpha\eta}(\eta - s), & s \leqslant \eta, \\ 0, & s > \eta, \end{cases} - \begin{cases} t-s, & s \leqslant t, \\ 0, & s > t. \end{cases}$$

We shall study the cases $\alpha\eta < 0$, $0 < \alpha\eta < 1$ and $\alpha\eta > 1$ separately. The case $\alpha = 0$ is covered by results of Lan [**9**].

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The case $\alpha\eta < 0$

In [**5**] it has been shown that we can take

$$\Phi(s) = (1-\alpha)\frac{s(1-s)}{1-\alpha\eta},$$

$[a,b] \subset (0,\eta]$ and $c = \min\{a, 1-\eta\}/(1-\alpha)$. Now it is clear that a theorem the same as Theorem 5.1 holds; we leave the statement to the reader.

The case $0 < \alpha\eta < 1$

In [**13**] Webb proved that we can take

$$\Phi(s) = \max\{1,\alpha\}\frac{1-s}{1-\alpha\eta},$$

$[a,b] \subset (0,1]$, and that for $\alpha < 1$ we may take $c = \min\{a, \alpha\eta, 4a(1-\eta), \alpha(1-\eta)\}$ and for $\alpha \geqslant 1$ we may take $c = \min\{a\eta, 4a(1-\alpha\eta)\eta, \eta(1-\alpha\eta)\}$. A result similar to Theorem 5.1 holds. We omit the obvious statement.

The case $\alpha\eta > 1$

For these BCs the kernel k is negative on an interval so we apply Remark 2.5 and consider $-k$ in place of k. In [**5**] it has been shown that we may take

$$\Phi(s) = \alpha\frac{s(1-s)}{\alpha\eta - 1}.$$

Indeed, $-k(t,s) > c\Phi(s)$ for $t \in [a,b]$ and $s \in [0,1]$, where $[a,b] \subset [\eta,1]$ and $c = \min\{a, 1-\eta\}/\alpha$. A theorem the same as Theorem 5.2 holds; we leave the statement to the reader.

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