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Meromorphic Functions Sharing the Same Zeros and Poles

Dedicated to Henri Cartan on his 100th birthday.

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Abstract. In this paper, Hinkkanen's problem (1984) is completely solved, *i.e.*, it is shown that any meromorphic function f is determined by its zeros and poles and the zeros of $f^{(j)}$ for $j = 1, 2, 3, 4$.

1 Introduction and Main Results

The uniqueness of meromorphic functions is an important research area. A natural problem is whether a meromorphic function $f(z)$ is determined by the zeros and poles of f and the zeros of its first few derivatives. For convenience, we say that two nonconstant meromorphic functions $f(z)$ and $g(z)$ share the value a CM when $f(z) - a$ and $g(z) - a$ have the same zeros with the same multiplicities.

For entire functions f and g with finite order, C. C. Yang [14] and G. G. Gundersen [7] studied the case where $f^{(j)}$ and $g^{(j)}$ share 0 CM for $j = 0, 1$.

For meromorphic functions f and g, we know that $f^{(j)}$ and $g^{(j)}$ share 0 and ∞ CM for each non-negative integer j whenever f and g satisfy one of the following four conditions:

(i) $f = cg$, $c \in \mathbb{C} - \{0\}$;
(ii) $f(z) = e^{az+b}$, $g(z) = e^{cz+d}$, $a, c \in \mathbb{C} - \{0\}$, $b, d \in \mathbb{C}$;
(iii) $f(z) = a(1 - be^{cz})$, $g(z) = d(e^{-cz} - b)$, $a, b, c, d \in \mathbb{C} - \{0\}$;
(iv) $f(z) = a/(1 - be^{\beta})$, $g(z) = a/(e^{-\beta} - b)$, $a, b \in \mathbb{C} - \{0\}$, β a non-constant entire function.

A. Hinkkanen [1, p. 492] proposed the following problem:

Question 1 **(Hinkkanen's Problem)** Does there exist an integer $n \geq 2$ such that f and g satisfy one of the conditions (i)–(iv) when $f^{(j)}$ and $g^{(j)}$ share the values 0 and ∞ CM for $j = 0, 1, \ldots, n$?

In 1989, L. Köhler [10] proved that $n = 6$ solves the problem. K. Tohge [13] in 1990 considered the case $n = 2, 3$ under restrictions on the growth of f and g.

In this paper, we shall provide a sharp answer to Hinkkanen's Problem by proving the following result see also [5]).

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Theorem 1 *The sharp answer to Hinkkanen's problem is $n = 4$.*

The following example shows that our theorem is best.

Example 1 Let $f = \exp(e^z)$ and $g = \exp(e^{-z})$. Then f and g do not satisfy (i)–(iv). It is easy to check that $f^{(j)}$ and $g^{(j)}$ share 0 and ∞ CM for $j = 0, 1, 2, 3$. However, from

$$f^{(4)} = (1 + 7e^z + 6e^{2z} + e^{3z}) \exp(z + e^z)$$

and

$$g^{(4)} = (1 + 6e^z + 7e^{2z} + e^{3z}) \exp(-4z + e^{-z})$$

we see that $f^{(4)}$ and $g^{(4)}$ have no common zeros. On the other hand, it is obvious that $f^{(4)}$ and $g^{(4)}$ have infinitely many zeros. Thus $f^{(4)}$ and $g^{(4)}$ do not share zeros.

To prove our result, the following strategy is used:

(1) Classify zeros of the functions f and g and their derivatives according to their multiplicities.
(2) Establish relations between the characteristic functions of f'/f and either the simple zeros of f and the zeros of f'' with multiplicities less than 109 or the simple zeros of f' and the zeros of f''' with multiplicities less than 109. The same is done for g'/g. The number 109 here can be replaced by any bigger number.
(3) Then restrict attention only to the kind of zeros listed in 2. This is done by considering several cases.

2 Nevanlinna's Theory

As a quantitative generalization of Picard's theorem, the theory of the distribution of values of meromorphic functions, developed by R. Nevanlinna and his student, L. Ahlfors, was one of the most outstanding achievements of mathematics in the 20th century (see [8, 11, 12]). The most important function in Nevanlinna's theory is Nevanlinna's characteristic function, which we now introduce.

Let $f(z)$ be meromorphic in $|z| \le R < \infty$. For $0 < r \le R$, we denote by $n(r, f)$ the number of poles of $f(z)$ in $|z| < r$, counted according to multiplicities. Setting $\log^+ x = \max(\log x, 0)$, we define

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t}\, dt + n(0, f) \log r,$$

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta,$$

$$T(r, f) = m(r, f) + N(r, f),$$

where $N(r, f)$, $m(r, f)$ and $T(r, f)$ are called counting function, proximity function and Nevanlinna characteristic function, respectively. One basic property is that $T(r, f)$ is a continuous and increasing convex function of $\log r$.

The order $\lambda(f)$ and the lower order $\rho(f)$ of f are defined, respectively, as follows:

$$\lambda(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}, \quad \rho(f) = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

Furthermore, the hyper-order of f is defined to be

$$\lambda_h(f) = \limsup_{r\to\infty} \frac{\log\log T(r, f)}{\log r}.$$

For example, e^z has order 1 and hyper-order 0.

Nevanlinna's First Fundamental Theorem: Let $f(z)$ be meromorphic in $|z| < R \leq \infty$. Then for any $a \in \mathbb{C}$ and $0 < r < R$,

$$T\Big(r, \frac{1}{f-a}\Big) = T(r, f) + O(1).$$

It is the following result that plays a key role in the Nevanlinna theory and its applications.

Nevanlinna's Second Fundamental Theorem: Suppose that f is a nonconstant meromorphic function in $|z| < R$. Let $a_1, \dots, a_q$ $(q \geq 3)$ be distinct values in $\overline{C}$. Then

$$(q-2)T(r, f) \leq \sum_{j=1}^{q} \overline{N}\Big(r, \frac{1}{f-a_j}\Big) + S(r, f),$$

where $S(r, f) = o(T(r, f))$ possibly outside a set r with finite linear measure if the order of f is infinite, and $\overline{N}$ is the counting function of the distinct roots of $f(z) = a_j$.

Nevanlinna's Small Function Theorem: Suppose that f is a nonconstant meromorphic function in $|z| < R$. Let $a_1(z)$, $a_2(z)$ and $a_3(z)$ be three distinct functions such that

$$T(r, a_i) = S(r, f), \quad i = 1, 2, 3.$$

Then

$$T(r, f) \leq \sum_{i=1}^{3} \overline{N}\Big(r, \frac{1}{f-a_j}\Big) + S(r, f).$$

This implies that if f is transcendental, and $f - a_1$ and $f - a_2$ have only finitely many zeros, then $f - a_3$ has infinitely many zeros.

Lemma on the Logarithmic Derivative: Suppose that f is a nonconstant meromorphic function in $|z| < R$. Then for any positive integer k and $0 < r < R$, we have

$$m\Big(r, \frac{f^{(k)}}{f}\Big) = S(r, f).$$

3 Notations

Let R be a relation, and let $N_{Rk}(r, f)$ "count" only those poles in $N(r, f)$ that have multiplicity p satisfying pRk. The symbol $\overline{N}_{Rk}$ means ignoring multiplicities in $N_{Rk}(r, f)$.

Set

$$(1)\qquad F_j := \frac{f^{(j+1)}}{f^{(j)}},\ G_j := \frac{g^{(j+1)}}{g^{(j)}},\ H_j := H_j(f,g) := F_j - G_j,\ (j = 0, 1, 2, \dots).$$

Obviously, for any $0 \le i \le j$,

$$(2)\qquad H_i(f^{(j-i)}, g^{(j-i)}) = H_j(f,g).$$

4 Fifteen Lemmas

The first lemma is a revised version of Clunie [3], (see, *e.g.*, Hua [9, Lemma 1]).

Lemma 1 *Let u be a meromorphic function and $Q[u]$ and $Q_0[u]$ be differential polynomials in u with coefficients a_i satisfying $m(r, a_i) = S(r, f)$. If the degree of $Q[u]$ is less than or equal to n and $u^n Q_0[u] = Q[u]$, then*

$$m(r, Q_0[u]) = S(r, u) + S(r, f).$$

Lemma 2 *For any positive integers n and q, if $f^{(n)} \not\equiv 0$, then*

$$(3)\qquad m\Big(r, \frac{f^{(n+1)}}{f^{(n)}}\Big) = m\Big(r, \frac{f'}{f}\Big) + S\Big(r, \frac{f'}{f}\Big),$$

$$(4)\qquad T\Big(r, \frac{f^{(n+1)}}{f^{(n)}}\Big) \le 2^n T\Big(r, \frac{f'}{f}\Big) + S\Big(r, \frac{f'}{f}\Big),$$

$$(5)\qquad \phi = S\Big(r, \frac{f^{(n)}}{f^{(m)}}\Big) \quad\Rightarrow\quad \phi = S\Big(r, \frac{f'}{f}\Big) \quad (0 \le m < n),$$

$$(6)\qquad \overline{N}_{=q}\Big(r, \frac{f^{(n-1)}}{f^{(n)}}\Big) = \overline{N}_{=q}\Big(r, \frac{1}{f^{(n)}}\Big) - \overline{N}_{=q+1}\Big(r, \frac{1}{f^{(n-1)}}\Big).$$

Proof (3) comes from [4, Lemma 2(v)]. (5) follows from (4). (6) can be easily checked. Now we prove (4). Since

$$\frac{f^{(n+1)}}{f^{(n)}} = h + \frac{h'}{h}, \quad h = \frac{f^{(n)}}{f^{(n-1)}},$$

we deduce from the First Fundamental Theorem and (3) that

$$\begin{aligned} N\Big(r, \frac{f^{(n+1)}}{f^{(n)}}\Big) &= N\Big(r, h + \frac{h'}{h}\Big) \\ &\le N(r,h) + \overline{N}(r, \frac{1}{h}) \le N(r,h) + T(r,h) + O(1) \\ &\le 2N(r,h) + m(r,h) + O(1) \\ &\le 2N\Big(r, \frac{f^{(n)}}{f^{(n-1)}}\Big) + m\Big(r, \frac{f'}{f}\Big) + S\Big(r, \frac{f'}{f}\Big). \end{aligned}$$

By induction, we get (4). ∎

Lemma 3 *If f is meromorphic and $f'' \not\equiv 0$, then*

$$m\Big(r, \frac{f'}{f}\Big) \le 2\Big[\overline{N}_{=1}\Big(r, \frac{1}{f}\Big) + \overline{N}\Big(r, \frac{1}{f''}\Big) - \overline{N}_{>2}\Big(r, \frac{1}{f}\Big)\Big] + S\Big(r, \frac{f'}{f}\Big).$$

Proof This lemma is essentially due to Frank and Hennekemper[4]. We present a simple proof here. Let

$$B_0 = \frac{(f''/f)'}{f''/f} \cdot \frac{(f'/f)'}{f'/f} - \frac{(f'/f)''}{f'/f} \tag{7}$$

and

$$B_1 = -\frac{(f''/f)'}{f''/f}. \tag{8}$$

Then

$$m(r, B_0) = S\Big(r, \frac{f'}{f}\Big), \quad m(r, B_1) = S\Big(r, \frac{f'}{f}\Big) \tag{9}$$

and

$$(B_1^2 + 2B_1' - 4B_0)\frac{f'}{f} = B_0B_1 + 2B_0'.$$

If $B_1^2 + 2B_1' - 4B_0 \not\equiv 0$, then by (9) and the above equation,

$$m\Big(r, \frac{f'}{f}\Big) \le N(r, B_1^2 + 2B_1' - 4B_0) + S\Big(r, \frac{f'}{f}\Big). \tag{10}$$

Now by (7) and (8),

$$B_1^2 + 2B_1' - 4B_0 = 2\,\frac{f'''}{f''}\,\frac{f'}{f} - 6\,\frac{f''}{f} + 3\left(\frac{f'}{f}\right)^2 - 2\,\frac{f^{(4)}}{f''} + 3\left(\frac{f'''}{f''}\right)^2.$$

It is easy to verify that any pole of f is not a pole of $B_1^2 + 2B_1' - 4B_0$. Thus poles of $B_1^2 + 2B_1' - 4B_0$ only occur at the zeros of f and f''. If z_0 is a zero of f of order $m \ge 2$, then, near $z = z_0$, $f(z)$ can be written in the form

$$f(z) = a_1(z - z_0)^m + a_2(z - z_0)^{m+1} + \cdots, \quad a_1 \ne 0.$$

This implies that, near $z = z_0$,

$$\frac{f^{(j+1)}}{f^{(j)}} = \frac{m - j}{z - z_0} + \frac{(m + 1)a_2}{(m + 1 - j)a_1} + O(z - z_0), \quad j = 0, 1, \ldots,$$

which yields

$$B_1^2 + 2B_1' - 4B_0|_{z_0} = O(1).$$

Thus z_0 is not a pole of $B_1^2 + 2B_1' - 4B_0$. The conclusion follows from (10). If $B_1^2 + 2B_1' - 4B_0 \equiv 0$, then $m\big(r, \frac{f'}{f}\big) = S\big(r, \frac{f'}{f}\big)$ by [4, pp. 52–53]. ∎

Lemma 4 *If f is meromorphic and $f''' \not\equiv 0$, then we have*

$$(11)\quad \overline{N}\Big(r,\frac{1}{f'''}\Big)-\overline{N}_{>2}\Big(r,\frac{1}{f'}\Big)\le\frac{6}{109}T\Big(r,\frac{f'}{f}\Big)+\sum_{i=1}^{108}\overline{N}_{=i}\Big(r,\frac{1}{f'''}\Big)+S\Big(r,\frac{f'}{f}\Big)$$

and

$$(12)\quad \overline{N}\Big(r,\frac{1}{f''}\Big)-\overline{N}_{>2}\Big(r,\frac{1}{f}\Big)\le\frac{3}{109}T\Big(r,\frac{f'}{f}\Big)+\sum_{i=1}^{108}\overline{N}_{=i}\Big(r,\frac{1}{f''}\Big)+S\Big(r,\frac{f'}{f}\Big).$$

Remark. The number 108 here can be replaced by any bigger number, but cannot be a smaller number. The aim is to get inequalities (56) and (57) below.

Proof The proof is similar to the one in [10, Lemma 9]. Note that, for any function $h \not\equiv 0$,

$$\overline{N}_{\ge q}(r,h)\le\frac{1}{q}T(r,h).$$

Then, by Lemma 2, (4), (6) and the First Fundamental Theorem, we have

$$\begin{aligned}
\overline{N}\Big(r,\frac{1}{f'''}\Big)-\overline{N}\Big(r,\frac{1}{f'}\Big)&=\Big[\overline{N}\Big(r,\frac{1}{f'''}\Big)-\overline{N}\Big(r,\frac{1}{f''}\Big)\Big]\\
&\quad+\Big[\overline{N}\Big(r,\frac{1}{f''}\Big)-\overline{N}\Big(r,\frac{1}{f'}\Big)\Big]\\
&=\overline{N}\Big(r,\frac{f''}{f'''}\Big)-\overline{N}_{=1}\Big(r,\frac{1}{f''}\Big)\\
&\quad+\overline{N}\Big(r,\frac{f'}{f''}\Big)-\overline{N}_{=1}\Big(r,\frac{1}{f'}\Big)\\
&\le\overline{N}_{\le 108}\Big(r,\frac{f''}{f'''}\Big)+\frac{1}{109}T\Big(r,\frac{f''}{f'''}\Big)-\overline{N}_{=1}\Big(r,\frac{1}{f''}\Big)\\
&\quad+\overline{N}_{\le 108}\Big(r,\frac{f'}{f''}\Big)+\frac{1}{109}T\Big(r,\frac{f'}{f''}\Big)-\overline{N}_{=1}\Big(r,\frac{1}{f'}\Big)\\
&\le\sum_{i=1}^{108}\Big[\overline{N}_{=i}\Big(r,\frac{1}{f'''}\Big)-\overline{N}_{=i+1}\Big(r,\frac{1}{f''}\Big)\Big]-\overline{N}_{=1}\Big(r,\frac{1}{f''}\Big)\\
&\quad+\sum_{i=1}^{108}\Big[\overline{N}_{=i}\Big(r,\frac{1}{f''}\Big)-\overline{N}_{=i+1}\Big(r,\frac{1}{f'}\Big)\Big]\\
&\quad-\overline{N}_{=1}\Big(r,\frac{1}{f'}\Big)+\frac{6}{109}T\Big(r,\frac{f'}{f}\Big)+S\Big(r,\frac{f'}{f}\Big)
\end{aligned}$$

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$$\leq \sum_{i=1}^{108} \overline{N}_{=i}\Big(r, \frac{1}{f'''}\Big) + \frac{6}{109} T\Big(r, \frac{f'}{f}\Big)$$
$$- \sum_{i=1}^{108} \overline{N}_{=i}(r, \frac{1}{f'}) + S\Big(r, \frac{f'}{f}\Big)$$

and (11) follows. The proof of (12) is similar and we omit it here. ■

Lemma 5 ([10, Lemma 7]) *If f and g share 0 and ∞ and if f'' and g'' share 0, then*

$$S\Big(r, \frac{g'}{g}\Big) = S\Big(r, \frac{f'}{f}\Big).$$

Lemma 6 *Suppose that f and g are not polynomials of degree less than 5. Then we have the following two conclusions.*

(A) *For any common zero z_0 of f and g with multiplicity m, if, near $z = z_0$,*

$$f(z) = a_1(z - z_0)^m + a_2(z - z_0)^{m+1} + a_3(z - z_0)^{m+2} + \cdots \tag{13}$$

and

$$g(z) = b_1(z - z_0)^m + b_2(z - z_0)^{m+1} + b_3(z - z_0)^{m+2} + \cdots, \tag{14}$$

then, near $z = z_0$,

$$H_0 = A(z_0) + [2B(z_0) - C(z_0)](z - z_0) + O((z - z_0)^2),$$
$$H_1 = \frac{m+1}{m}A(z_0) + \left[2\frac{m+2}{m}B(z_0) - \left(\frac{m+1}{m}\right)^2 C(z_0)\right](z - z_0) + O((z - z_0)^2),$$
$$H_j = \frac{m+1}{m+1-j}A(z_0)$$
$$+ \left[2\frac{(m+1)(m+2)}{(m+1-j)(m+2-j)}B(z_0) - \left(\frac{m+1}{m+1-j}\right)^2 C(z_0)\right](z - z_0)$$
$$+ O((z - z_0)^2), \quad (j < m+1),$$

where

$$A(z_0) = \frac{a_2}{a_1} - \frac{b_2}{b_1}, \quad B(z_0) = \frac{a_3}{a_1} - \frac{b_3}{b_1}, \quad C(z_0) = \left(\frac{a_2}{a_1}\right)^2 - \left(\frac{b_2}{b_1}\right)^2.$$

(B) *For any common pole p_0 of f and g with multiplicity m, if, near $z = p_0$,*

$$f(z) = \frac{c_1}{(z - p_0)^m} + \frac{c_2}{(z - p_0)^{m-1}} + \frac{c_3}{(z - p_0)^{m-2}} + \cdots$$

and

$$g(z) = \frac{d_1}{(z-p_0)^m} + \frac{d_2}{(z-p_0)^{m-1}} + \frac{d_3}{(z-p_0)^{m-2}} + \cdots, \tag{15}$$

then,near $z = p_0$,

$$H_0 = A(p_0) + [2B(p_0) - C(p_0)](z-p_0) + O((z-p_0)^2),$$

$$H_1 = \frac{m-1}{m}A(p_0) + \left[2\frac{m-2}{m}B(p_0) - \left(\frac{m-1}{m}\right)^2 C(p_0)\right](z-p_0)$$
$$+ O((z-p_0)^2),$$

$$H_j = \frac{m-1}{m-1+j}A(p_0)$$
$$+ \left[2\frac{(m-2)(m-1)}{(m-2+j)(m-1+j)}B(p_0) - \left(\frac{m-1}{m-1+j}\right)^2 C(p_0)\right](z-p_0)$$
$$+ O((z-p_0)^2), \quad j = 0, 1, 2, \ldots,$$

where

$$A(p_0) = \frac{c_2}{c_1} - \frac{d_2}{d_1}, \quad B(p_0) = \frac{c_3}{c_1} - \frac{d_3}{d_1}, \quad C(p_0) = \left(\frac{c_2}{c_1}\right)^2 - \left(\frac{d_2}{d_1}\right)^2.$$

Proof We need only prove (A). The proof of (B) is similar. For the zero, z_0, of f and g, from (13) and (14) we can easily deduce that, near $z = z_0$,

$$F_j = \frac{m-j}{z-z_0} + \frac{(m+1)a_2}{(m+1-j)a_1}$$
$$+ \left[2\frac{(m+1)(m+2)a_3}{(m+1-j)(m+2-j)a_1} - \left(\frac{(m+1)a_2}{(m+1-j)a_1}\right)^2\right](z-z_0)$$
$$+ O((z-z_0)^2),$$

$$G_j = \frac{m-j}{z-z_0} + \frac{(m+1)b_2}{(m+1-j)b_1}$$
$$+ \left[2\frac{(m+1)(m+2)b_3}{(m+1-j)(m+2-j)b_1} - \left(\frac{(m+1)b_2}{(m+1-j)b_1}\right)^2\right](z-z_0)$$
$$+ O((z-z_0)^2),$$

where $j = 0, 1, \ldots, m$. These two representations yield (A). ■

Lemma 7 *Suppose that f and g are meromorphic functions. Let*

$$f_j = \frac{1}{F_j}, \quad g_j = \frac{1}{G_j}, \quad (j \geq 0),$$

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where F_j and G_j are the same as in (1). *If*

$$F_j = e^u G_j, \quad F_{j+1} = e^v G_{j+1}, \quad F_{j+2} = e^w G_{j+2} \tag{16}$$

for three entire functions u and v and w, then

$$x_1 g_j + y_1 g_{j+1} + z_1 g_j g_{j+1} = r_1, \tag{17}$$

$$x_2 g_j + y_2 g_{j+1} + z_2 g_j g_{j+1} = r_2, \tag{18}$$

$$x_3 g_j + y_3 g_{j+1} + z_3 g_j g_{j+1} = r_3, \tag{19}$$

where

$$\begin{aligned}
x_1 &= (e^v - 1), \quad y_1 = -(e^u - 1), \quad z_1 = -u', \quad r_1 = 0,\\
x_2 &= (e^v - 1)\left[u'(e^w - 1) - v'e^v(e^w - 1) - u'(e^v - e^w)\right],\\
y_2 &= u'e^u(e^v - 1)(e^w - 1) - v'(e^u - 1)(e^v - 1) - u'(e^w - 1)(e^u - e^v),\\
z_2 &= -u'v'(e^v - 1) - u'^2(e^w - 1) + u''(e^v - 1)(e^w - 1),\\
r_2 &= (e^v - 1)\left[(e^u - 1)(e^v - e^w) - (e^w - 1)(e^u - e^v)\right],\\
x_3 &= x_2'(e^v - 1)(e^w - 1) - u'x_2(e^w - 1) - z_2(e^v - 1)(e^v - e^w),\\
y_3 &= y_2'(e^v - 1)(e^w - 1) - v'y_2(e^v - 1) - z_2(e^w - 1)(e^u - e^v),\\
z_3 &= -u'z_2(e^w - 1) - v'z_2(e^v - 1) + z_2'(e^v - 1)(e^w - 1),\\
r_3 &= x_2(e^w - 1)(e^u - e^v) + y_2(e^v - 1)(e^v - e^w) + r_2'(e^v - 1)(e^w - 1).
\end{aligned}$$

Proof By (1) it is easy to see that

$$F_{i+1} = F_i + \frac{F_i'}{F_i}, \quad G_{i+1} = G_i + \frac{G_i'}{G_i}, \tag{20}$$

for any non-negative integer i. By this and (16) we have

$$e^v G_{j+1} = F_{j+1} = F_j + \frac{F_j'}{F_j} = e^u G_j + u' + G_{j+1} - G_j,$$

i.e.,

$$(e^v - 1)G_{j+1} - (e^u - 1)G_j - u' = 0.$$

We thus obtain (17). To prove (18), we substitute (20) with $i = j$ into the above equation and obtain

$$(e^v - 1)g_j' = -u'g_j - (e^u - e^v). \tag{21}$$

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Similarly, by (16) and (20), we deduce that

$$e^w\Big(G_{j+1}+\frac{G'_{j+1}}{G_{j+1}}\Big)=e^wG_{j+2}=F_{j+2}$$
$$=F_{j+1}+\frac{F'_{j+1}}{F_{j+1}}=e^vG_{j+1}+v'+\frac{G'_{j+1}}{G_{j+1}}.$$

Thus

$$(e^w-1)g'_{j+1}=-v'g_{j+1}-(e^v-e^w). \tag{22}$$

Now differentiating (17) and substituting (21) and (22) into it, we get (18). By differentiating (18) and using (21) and (22) again, we obtain (19). ■

The following lemma is a revised version of the so-called Borel Unit Theorem, which can be found in Gross [6, Theorem 3.12].

Lemma 8 *Let $h_0,\dots,h_n$ be meromorphic functions and let $g_1,\dots,g_n$ be entire functions such that*

$$\sum_{j=1}^{k}h_j(z)e^{g_j(z)}=h_0(z).$$

Suppose that there exists a set I with infinite measure such that, for $r\in I$,

$$T(r,h_j)=o\{T(r,e^{g_k-g_i})\},\quad j=0,1,\dots,n;\ k,i=1,\dots,n;\ i\neq k.$$

Then $h_0=h_1=\cdots=h_n=0$.

Lemma 9 ([10, Lemma 8]) *Let f and g share 0 and ∞ CM, let f' and g' share 0 CM and let $f^{(n)}$ and $g^{(n)}$ share 0 CM for one $n>1$. If f'/f is rational, then either* (i) *or* (ii) *holds.*

The following lemma is a corollary in Tohge [12, p. 103].

Lemma 10 *Let f and g be meromorphic functions of hyper-order less than 2. If $f^{(j)}$ and $g^{(j)}$ share 0 and ∞ CM for $j=0,1,2,3$, then the possibilities for f and g are those of* (i)–(iv) *and* (v) *$f(z)=Ae^{\exp(az+b)}$, $g(z)=Be^{\exp(-az-b)}$, where A,B,a,b are constants and $ABa\neq0$.*

Lemma 11 *Let $\hat f$ and $\hat g$ be non-polynomial meromorphic functions. Suppose that $\hat f$ and $\hat g$ share 0 CM and that $\hat f'$ and $\hat g'$ share 0 CM. Then either*

$$\overline{N}_{=m}\Big(r,\frac{1}{\hat f'}\Big)\le\overline{N}_{=m+1}\Big(r,\frac{1}{\hat f}\Big)+T\big(r,H_1(\hat f,\hat g)-H_0(\hat f,\hat g)\big)$$
$$+T\big(r,H_2(\hat f,\hat g)-H_1(\hat f,\hat g)\big)+O(1)$$

or

$$H_1(\hat{f},\hat{g}) - H_0(\hat{f},\hat{g}) = m\big(H_2(\hat{f},\hat{g}) - H_1(\hat{f},\hat{g})\big)\,,$$

where m is a positive integer.

Proof Assume that

$$H_1(\hat{f},\hat{g}) - H_0(\hat{f},\hat{g}) \not\equiv m\big(H_2(\hat{f},\hat{g}) - H_1(\hat{f},\hat{g})\big)\,.$$

We shall deduce the desired inequality. Let z_0 be a zero of $\hat{f}'$ (and $\hat{g}'$) of order m. If $\hat{f}(z_0) = 0$, then the order is $m+1$. If $\hat{f}(z_0) \neq 0$, then $\hat{g}(z_0) \neq 0$. From (1) we can easily see that $H_0\big(\hat{f}(z_0), \hat{g}(z_0)\big) = 0$. Applying Lemma 6 to $f = \hat{f}'$ and $g = \hat{g}'$ and noting (2), we deduce that z_0 is a zero of $H_1(\hat{f},\hat{g}) - m(H_2(\hat{f},\hat{g}) - H_1(\hat{f},\hat{g})\big)$. Thus z_0 is a zero of $H_1(\hat{f},\hat{g}) - H_0(\hat{f},\hat{g}) - m(H_2(\hat{f},\hat{g}) - H_1(\hat{f},\hat{g})\big)$. Note that if z_0 is a zero of $\hat{f}'$ of order m and a zero of $\hat{f}$, then z_0 is a zero of $\hat{f}$ of order $m+1$. Therefore,

$$\overline{N}_{=m}\Big(r, \frac{1}{\hat{f}'}\Big) \le \overline{N}_{=m+1}\Big(r, \frac{1}{\hat{f}}\Big) + N\Big(r, \frac{1}{H_1(\hat{f},\hat{g}) - H_0(\hat{f},\hat{g}) - m\big(H_2(\hat{f},\hat{g}) - H_1(\hat{f},\hat{g})\big)}\Big)\,.$$

The conclusion follows from this and the First Fundamental Theorem. ■

Lemma 12 *Let $\hat{f}$ and $\hat{g}$ be non-polynomial meromorphic functions. Suppose that $\hat{f}$ and $\hat{g}$ share 0 CM. Then either*

$$\overline{N}_{=m}\Big(r, \frac{1}{\hat{f}}\Big) \le T\big(r, H_1(\hat{f},\hat{g}) - H_0(\hat{f},\hat{g})\big) + T\big(r, H_2(\hat{f},\hat{g}) - H_1(\hat{f},\hat{g})\big) + O(1)$$

or

$$H_1(\hat{f},\hat{g}) - H_0(\hat{f},\hat{g}) = \frac{m-1}{m+1}\big(H_2(\hat{f},\hat{g}) - H_1(\hat{f},\hat{g})\big)\,,$$

where $m \ge 2$ is a positive integer.

Proof We suppose that

$$H_1(\hat{f},\hat{g}) - H_0(\hat{f},\hat{g}) \not\equiv \frac{m-1}{m+1}\big(H_2(\hat{f},\hat{g}) - H_1(\hat{f},\hat{g})\big)\,.$$

Let z_0 be a zero of $\hat{f}$ (and $\hat{g}$) of order m. Applying Lemma 6 to $f = \hat{f}$ and $g = \hat{g}$, we obtain, near $z = z_0$,

$$H_1(\hat{f},\hat{g}) - H_0(\hat{f},\hat{g}) = \frac{1}{m} A(z_0) + O(z - z_0)$$

and

$$H_2(\hat{f},\hat{g}) - H_1(\hat{f},\hat{g}) = \frac{m+1}{(m-1)m} A(z_0) + O(z - z_0).$$

Thus z_0 is a zero of $H_1(\hat{f},\hat{g}) - H_0(\hat{f},\hat{g}) - \frac{m-1}{m+1}\big(H_2(\hat{f},\hat{g}) - H_1(\hat{f},\hat{g})\big)$. This implies that

$$\overline{N}_{=m}\Big(r,\frac{1}{\hat{f}}\Big) \leq N\Big(r,\frac{1}{H_1(\hat{f},\hat{g}) - H_0(\hat{f},\hat{g}) - \frac{m-1}{m+1}\big(H_2(\hat{f},\hat{g}) - H_1(\hat{f},\hat{g})\big)}\Big).$$

The conclusion follows from this and the First Fundamental Theorem. ■

Lemma 13 *Let $\hat{f}$ and $\hat{g}$ be non-polynomial meromorphic functions. Suppose that there exists an integer q such that*

$$H_1(\hat{f},\hat{g}) - H_0(\hat{f},\hat{g}) = q\big(H_2(\hat{f},\hat{g}) - H_1(\hat{f},\hat{g})\big) \not\equiv 0. \tag{23}$$

Then

$$\overline{N}_{\geq 2}\Big(r,\frac{1}{\hat{f}}\Big) \leq N\Big(r,\frac{1}{H_1(\hat{f},\hat{g}) - H_0(\hat{f},\hat{g})}\Big)$$

and

$$\overline{N}\Big(r,\frac{1}{\hat{f}'}\Big) \leq \overline{N}_{=q}\Big(r,\frac{1}{\hat{f}'}\Big) + 2N\Big(r,\frac{1}{H_1(\hat{f},\hat{g}) - H_0(\hat{f},\hat{g})}\Big).$$

Proof Let z_0 be a zero of $\hat{f}$ of multiplicity $m \geq 2$. Then from Lemma 6 we deduce that, near $z = z_0$,

$$H_1(\hat{f},\hat{g}) - H_0(\hat{f},\hat{g}) = \frac{1}{m}A(z_0) + O(z - z_0)$$

and

$$H_2(\hat{f},\hat{g}) - H_1(\hat{f},\hat{g}) = \frac{m+1}{(m-1)m}A(z_0) + O(z - z_0).$$

It follows from these two expressions and (23) that

$$\Big(1 - \frac{2}{m+1} - q\Big)A(z_0) = 0.$$

Since $m \geq 2$ and m and q are integers, $\frac{2}{m+1}$ is not an integer, and we deduce that $1 - \frac{2}{m+1} - q \neq 0$. Thus $A(z_0) = 0$ and z_0 is a zero of $H_1(\hat{f},\hat{g}) - H_0(\hat{f},\hat{g})$. This gives the first inequality.

Now, for any zero z_0 of $\hat{f}'$ of multiplicity $m \neq q$, if $\hat{f}(z_0) = 0$, then z_0 is a zero of $\hat{f}$ of order $m+1$. If $\hat{f}(z_0) \neq 0$, then z_0 is a zero of $H_0(\hat{f},\hat{g})$. Applying Lemma 6 to $f = \hat{f}$ and $g = \hat{g}$, we see that z_0 is a zero of $H_2(\hat{f},\hat{g}) - \frac{m+1}{m}H_1(\hat{f},\hat{g})$. It follows from this, (23) and $m \neq q$ that z_0 is a zero of $H_1(\hat{f},\hat{g})$. Thus

$$\overline{N}\Big(r,\frac{1}{\hat{f}'}\Big) - \overline{N}_{=q}\Big(r,\frac{1}{\hat{f}'}\Big) \leq \overline{N}_{\geq 2}\Big(r,\frac{1}{\hat{f}}\Big) + N\Big(r,\frac{1}{H_1(\hat{f},\hat{g}) - H_0(\hat{f},\hat{g})}\Big).$$

The second inequality follows from this and the first inequality. ■

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Lemma 14 *Let f and g be non-polynomial meromorphic functions. Suppose that there exists an integer q such that*

$$H_3 - H_2 = q(H_4 - H_3) \not\equiv 0. \tag{24}$$

Then

$$\overline{N}_{\geq 2}(r, f) \leq N\Big(r, \frac{1}{H_3 - H_2}\Big).$$

Proof Let p_0 be a pole of f with multiplicity $m \geq 2$. Then from Lemma 6 we deduce that, near $z = p_0$,

$$H_3 - H_2 = -\frac{m-1}{(m+1)(m+2)} A(z_0) + O(z - p_0)$$

and

$$H_4 - H_3 = -\frac{m-1}{(m+2)(m+3)} A(z_0) + O(z - p_0).$$

It follows from (24) that

$$\Big(1 + \frac{2}{m+1} - q\Big) A(p_0) = 0.$$

Since $m \geq 2$ and m and q are integers, $\frac{2}{m+1}$ is not an integer, and we deduce that $1 + \frac{2}{m+1} - q \neq 0$. Thus $A(p_0) = 0$, and so, p_0 is a zero of $H_3 - H_2$. ■

Lemma 15 *Let $\hat{f}$ and $\hat{g}$ be non-polynomial meromorphic functions such that $\hat{f}$ and $\hat{g}$ share 0 and ∞ CM. Assume that*

$$\frac{\hat{f}'}{\hat{f}} = c\,\frac{e^{\nu} - 1}{L}, \quad \frac{\hat{g}'}{\hat{g}} = \frac{e^{-\nu}(e^{\nu} - 1)}{L}, \tag{25}$$

where

$$L(z) = (1 - c)z + d$$

for two constants c and d and $\nu(z)$ is an entire function. Then $\hat{f}$ and $\hat{g}$ have no poles and they have at most one zero.

Proof Note that any zeros and poles of $\hat{f}$ will be poles of $\hat{f}'/\hat{f}$. If $c = 1$, then $L(z)$ is a constant and we see from (25) that $\hat{f}$ and $\hat{g}$ have no poles and no zeros. Next we suppose that

$$c \neq 1. \tag{26}$$

Then $L(z)$ has one zero. Thus, by (25), $\hat{f}$ and $\hat{g}$ have either one pole or one zero.

If $\hat{f}$ (and $\hat{g}$) has a pole p_0, then p_0 is the zero of L. Let

$$\hat{f} = \frac{1}{L} e^{\alpha}, \quad \hat{g} = \frac{1}{L} e^{\beta}, \tag{27}$$

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where α and β are entire functions. Note that $L'(z) = 1 - c$. We deduce from this, (1) and (25) that

$$\hat{f}' = c\,\frac{e^{\nu} - 1}{L^2}\,e^{\alpha}, \quad \hat{g}' = \frac{1 - e^{-\nu}}{L^2}\,e^{\beta}. \tag{28}$$

On the other hand, differentiating (27) gives

$$\hat{f}' = \frac{\alpha' L - L'}{L^2}\,e^{\alpha}, \quad \hat{g}' = \frac{\beta' L - L'}{L^2}\,e^{\beta}.$$

Combining these with (28) we obtain

$$\alpha' = \frac{ce^{\nu} - 2c + 1}{L}, \quad \beta' = \frac{2 - c - e^{-\nu}}{L}.$$

Since α' and β' are entire functions, then

$$c\,e^{\nu(p_0)} - 2c + 1 = 0, \quad 2 - c - e^{-\nu(p_0)} = 0.$$

This implies that $c = 1$, which contradicts (26). ∎

5 Proof of Theorem 1

By Example 1 in Section 1, we need only prove that $n = 4$ solves the problem. Obviously, we can suppose that f and g are not polynomials, otherwise, conclusion (i) holds since f and g have the same zeros and poles.

Let F_j, G_j and H_j be as in (1). By Lemma 6, all H_j ($j \le 4$) are entire functions. It follows from Lemma 5, (20) and the lemma on logarithmic derivatives that

$$\begin{aligned} T(r, H_{i+1} - H_i) &= m(r, H_{i+1} - H_i) \\ &\le m(r, F_{i+1} - F_i) + m(r, G_{i+1} - G_i) + O(1) \\ &= m\Big(r, \frac{F_i'}{F_i}\Big) + m\Big(r, \frac{G_i'}{G_i}\Big) + O(1) = S\Big(r, \frac{f'}{f}\Big) \end{aligned}$$

for $i = 0, 1, 2, 3$. Thus we have

$$T(r, H_j - H_i) = S\Big(r, \frac{f'}{f}\Big), \quad (0 \le i < j \le 4). \tag{29}$$

Next we distinguish four cases.

Case 1 $H_i \equiv 0$ for some i with $1 \le i \le 4$. We consider only $i = 4$, since the cases where $i < 4$ are easier and similar. From $H_4 \equiv 0$ and (2) we deduce that there exist constants c ($\neq 0$), c_1, c_2, c_3 and c_4 such that

$$f = cg + P \tag{30}$$

and

$$f' = cg' + P', \tag{31}$$

where $P(z) = c_1z^3 + c_2z^2 + c_3z + c_4$. Since f and g share 0 and ∞ CM, there exists an entire function $\beta(z)$ such that

$$g = e^{\beta} f. \tag{32}$$

If β is identically constant, then we obtain (i). If $\beta \not\equiv$ const., we deduce from (32) and (30) that

$$f(z) = \frac{P(z)}{1 - c\exp\bigl(\beta(z)\bigr)}, \quad g(z) = \frac{P(z)}{\exp\bigl(-\beta(z)\bigr) - c}. \tag{33}$$

If $P'(\beta'P - P') \not\equiv 0$, then $P' \not\equiv 0$ and $\beta'P - P' \not\equiv 0$. Differentiating the first equation in (33) we get

$$f'(z) = \frac{c(\beta'P - P')e^{\beta} + P'}{(1 - ce^{\beta})^2}.$$

Note that e^{β} has no zeros and poles. It thus follows from Nevanlinna's Small Function Theorem that f' has infinitely many zeros. Since f' and g' share 0 CM, it follows from (31) that P' has infinitely many zeros, and so $P' \equiv 0$, which is a contradiction. Thus $P'(\beta'P - P') \equiv 0$, which yields either $P' \equiv 0$ or $\beta'P - P' \equiv 0$. If $\beta'P - P' \equiv 0$, then $\beta' = P'/P$. Since β is entire, then $P(z)$ has to be a constant. If $P' \equiv 0$, then $P(z)$ is also a constant. We thus obtain (iv) from (33).

Case 2 $H_j - H_{j-1} \equiv 0$ for some j with $1 \leq j \leq 3$. By integration, it follows from (1) and (2) that there exists a non-zero constant c such that $f^{(j)}/g^{(j)} = cf^{(j-1)}/g^{(j-1)}$, *i.e.*,

$$F_{j-1} = cG_{j-1}.$$

Keeping in mind that $f^{(j-1)}$ and $g^{(j-1)}$ share 0 and ∞ CM, if $f^{(j-1)}$ and $g^{(j-1)}$ have either common zeros or common poles, by local expansions we deduce from the above equation that $c = 1$. Thus $H_{j-1} = F_{j-1} - G_{j-1} = 0$ and this reduces to Case 1. Now suppose that $f^{(j-1)} \neq 0, \infty$ and $g^{(j-1)} \neq 0, \infty$. Then, by the above equation, there exists an entire function $\alpha(z)$ such that

$$g^{(j-1)} = e^{\alpha}, \quad f^{(j-1)} = c_1\, e^{c\alpha}.$$

Differentiating these equations twice we get

$$\frac{f^{(j+1)}}{g^{(j+1)}} = c_1 c \frac{\Psi - c}{\Psi - 1}\, e^{(c-1)\alpha},$$

where $\Psi(z) = (1/\alpha')'$. Since $f^{(j+1)}$ and $g^{(j+1)}$ share 0 CM, Ψ does not assume c and 1. In addition, by definition, Ψ has no simple poles. It follows from Nevanlinna's Second Fundamental Theorem that

$$T(r, \Psi) \leq \overline{N}_{\geq 2}(r, \Psi) + S(r, \Psi) \leq \frac{1}{2} T(r, \Psi) + S(r, \Psi).$$

Thus Ψ is a constant, and so $\frac{1}{\alpha'}$ is linear. But α is entire, and so α' is a constant, and α is a linear polynomial. Hence

$$g^{(j-1)}(z) = e^{az+b}, \quad f^{(j-1)}(z) = e^{caz+d},$$

where a, b, c, d, are constants, $a \neq 0$ and $c \neq 1$. Thus f and g have hyper-order 0, so case (v) of Lemma 10 does not occur, and the conclusion follows from Lemma 10.

Case 3 $H_1 - 2H_0 \equiv 0$. From (1) we deduce that $f'/g' = c(f/g)^2$, *i.e.*, $f'/f^2 = cg'/g^2$. Integration yields that $1/f = c/g + c_1$ for some constant c_1. If $c_1 = 0$, then we obtain (i). If $c_1 \neq 0$, then f and g are entire and $f(c + c_1 g) = g$. Since f and g share 0 CM, there exists a non-constant entire function $\alpha(z)$ such that $c + c_1 g = e^{\alpha(z)}$. This implies that $f(z) = \frac{1}{c_1} - \frac{c}{c_1} e^{-\alpha(z)}$ and $g(z) = \frac{1}{c_1} e^{\alpha(z)} - \frac{c}{c_1}$. Thus

$$\frac{f''(z)}{g''(z)} = c\frac{\Psi(z) + 1}{\Psi(z) - 1} e^{-2\alpha(z)},$$

where $\Psi(z) = (1/\alpha')'$. Since f'' and g'' share 0 CM, Ψ does not assume 1 or -1. In addition, by definition, Ψ cannot have simple poles. It follows from Nevanlinna's Second Fundamental Theorem that

$$T(r, \Psi) \leq \overline{N}_{\geq 2}(r, \Psi) + S(r, \Psi) \leq \frac{1}{2} T(r, \Psi) + S(r, \Psi).$$

Thus Ψ is a constant. Since α is entire, α' is constant and so α is linear. This gives (iii).

Case 4 None of the above three cases holds, *i.e.*,

$$H_i \not\equiv 0 \ (i = 0, \dots, 4), \quad H_j - H_{j-1} \not\equiv 0 \ (j = 1, \dots, 3), \quad H_1 - 2H_0 \not\equiv 0. \tag{34}$$

If z_0 is a simple pole of f and g, then, by Lemma 6, $H_2(z_0) = H_1(z_0) = 0$. Thus by (29),

$$N_{=1}(r, f) \leq N\Big(r, \frac{1}{H_2 - H_1}\Big) = S\Big(r, \frac{f'}{f}\Big). \tag{35}$$

If z_0 is a simple zero of f and g, then by Lemma 6, $H_1(z_0) - 2H_0(z_0) = 0$. By (29),

$$\begin{aligned} N_{=1}\Big(r, \frac{1}{f}\Big) &\leq N\Big(r, \frac{1}{H_1 - 2H_0}\Big) \leq T(r, H_1 - 2H_0) + O(1) \\ &= m(r, H_0) + S\Big(r, \frac{f'}{f}\Big) \leq m\Big(r, \frac{f'}{f}\Big) + m(r, \frac{g'}{g}) + S\Big(r, \frac{f'}{f}\Big). \end{aligned} \tag{36}$$

Next, we deal with multiple zeros and poles of f or g. We shall prove that

$$\overline{N}_{\geq 2}(r, f) + \overline{N}_{\geq 2}\Big(r, \frac{1}{f}\Big) \leq 3m\Big(r, \frac{f'}{f}\Big) + 3m\Big(r, \frac{g'}{g}\Big) + S\Big(r, \frac{f'}{f}\Big). \tag{37}$$

To this end, we set

$$Q(z) = \frac{1}{H_0} - \frac{2}{H_1} + \frac{1}{H_2} \tag{38}$$

and consider two situations.

At first, we suppose that $Q(z) \not\equiv 0$. Let z_0 be a zero of f with multiplicity $m \geq 2$. Then, by assumption, z_0 is also a zero of g with multiplicity m. Suppose that, near $z = z_0$, f and g have expansions as in Lemma 6. If $A(z_0) = \frac{a_2}{a_1} - \frac{b_2}{b_1} = 0$, then $H_2(z_0) - H_1(z_0) = 0$. If $A(z_0) \neq 0$, then by Lemma 6, near $z = z_0$,

$$\frac{1}{H_0} = \frac{1}{A(z_0)} - \frac{1}{A(z_0)^2}\left[2B(z_0) - C(z_0)\right](z - z_0) + \cdots, \tag{39}$$

$$\frac{1}{H_1} = \frac{m}{m+1}\frac{1}{A(z_0)} - \frac{1}{A(z_0)^2}\left[2\frac{m(m+2)}{(m+1)^2}B(z_0) - C(z_0)\right](z - z_0) + \cdots, \tag{40}$$

$$\frac{1}{H_2} = \frac{m-1}{m+1}\frac{1}{A(z_0)} - \frac{1}{A(z_0)^2}\left[2\frac{(m-1)(m+2)}{m(m+1)}B(z_0) - C(z_0)\right](z - z_0) + \cdots. \tag{41}$$

It follows from (38)–(40) and the above equality that, near $z = z_0$,

$$Q(z) = \frac{4B(z_0)}{m(m+1)^2 A(z_0)^2}(z - z_0) + O\big((z - z_0)^2\big). \tag{42}$$

Thus z_0 is either a zero of $H_2 - H_1$ or a zero of $Q(z)$.

Similarly, for any pole p_0 of f and g with multiplicity $m \geq 2$, let f and g have the same expansions, near $z = p_0$, as in Lemma 6. If $A(p_0) = \frac{c_2}{c_1} - \frac{d_2}{d_1} = 0$, then $H_2(p_0) - H_1(p_0) = 0$. If $A(p_0) \neq 0$, then, near $z = p_0$,

$$\frac{1}{H_0} = \frac{1}{A(p_0)} - \frac{1}{A(p_0)^2}\left[2B(p_0) - C(p_0)\right](z - p_0) + \cdots, \tag{43}$$

$$\frac{1}{H_1} = \frac{m}{m-1}\frac{1}{A(p_0)} - \frac{1}{A(p_0)^2}\left[2\frac{m(m-2)}{(m-1)^2}B(p_0) - C(p_0)\right](z - p_0) + \cdots, \tag{44}$$

$$\frac{1}{H_2} = \frac{m+1}{m-1}\frac{1}{A(p_0)} - \frac{1}{A(p_0)^2}\left[2\frac{(m+1)(m-2)}{m(m-1)}B(p_0) - C(p_0)\right](z - p_0) + \cdots. \tag{45}$$

It follows from (37), (43)–(45) that, near $z = p_0$,

$$Q(z) = \frac{4B(p_0)}{m(m-1)^2 A(p_0)^2}(z - p_0) + O\big((z - p_0)^2\big). \tag{46}$$

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Thus, p_0 is either a zero of $H_2 - H_1$ or a zero of $Q(z)$.

Combining (29) and the above discussion about the zeros and poles of f, we obtain

$$\overline{N}_{\geq 2}(r, f) + \overline{N}_{\geq 2}\Big(r, \frac{1}{f}\Big) \leq N\Big(r, \frac{1}{Q}\Big) + N\Big(r, \frac{1}{H_2 - H_1}\Big) \leq N\Big(r, \frac{1}{Q}\Big) + S\Big(r, \frac{f'}{f}\Big).$$

Since all H_i ($i \leq 4$) are entire, then (37) follows from (3), (1), (2), (38) and the First Fundamental Theorem.

Now we consider the case $Q(z) \equiv 0$. Thus, (38) gives

$$H_1H_2 - 2H_0H_2 + H_0H_1 \equiv 0. \tag{47}$$

We write this in the form

$$H_2 = HH_0 \tag{48}$$

for

$$H = \frac{H_2 - H_1}{H_1 - H_0}. \tag{49}$$

From (29) we see that

$$T(r, H) \leq T(r, H_2 - H_1) + T(r, H_1 - H_0) + O(1) = S\Big(r, \frac{f'}{f}\Big).$$

Now by (1) and (20),

$$\frac{f'''}{f''} = P\Big(\frac{f'}{f}\Big) + \frac{f'}{f}, \quad \frac{g'''}{g''} = P\Big(\frac{g'}{g}\Big) + \frac{g'}{g},$$

where

$$P\Big(\frac{f'}{f}\Big) = \frac{(f''/f')'}{f''/f'} + \frac{(f'/f)'}{f'/f}, \quad P\Big(\frac{g'}{g}\Big) = \frac{(g''/g')'}{g''/g'} + \frac{(g'/g)'}{g'/g}.$$

Thus by (1) and (2), $H_2 = H_0 + P(\frac{f'}{f}) - P(\frac{g'}{g})$. Substituting this into (48), we obtain

$$(H-1)H_0 = P\Big(\frac{f'}{f}\Big) - P\Big(\frac{g'}{g}\Big). \tag{50}$$

If $H \equiv 1$, then $H_2 = H_0$ by (48), and so, $H = -1$ by (49), which is a contradiction. Thus, $H \not\equiv 1$. Now by (3), Lemma 5 and the expressions of $P(\frac{f'}{f})$ and $P(\frac{g'}{g})$, we have

$$m\Big(r, P\Big(\frac{f'}{f}\Big)\Big) = S\Big(r, \frac{f'}{f}\Big), \quad m\Big(r, P\Big(\frac{g'}{g}\Big)\Big) = S\Big(r, \frac{f'}{f}\Big).$$

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It follows from (50) that

$$T(r,H_0)=m(r,H_0)\le T(r,H)+m\Big(r,P\Big(\frac{f'}{f}\Big)\Big)+m\Big(r,P\Big(\frac{g'}{g}\Big)\Big)+O(1)$$
$$=S\Big(r,\frac{f'}{f}\Big),$$

which, with (29), implies

$$T(r,H_i)=S\Big(r,\frac{f'}{f}\Big),\quad (i=0,1,2). \tag{51}$$

Let

$$\Phi=\frac{1}{H_0}-\frac{1}{H_1}. \tag{52}$$

We claim that $\Phi'\not\equiv 0$. Otherwise, there exists a constant c such that

$$\frac{1}{H_0}-\frac{1}{H_1}=c. \tag{53}$$

By (38) and the assumption that $Q\equiv 0$, we have

$$\frac{1}{H_1}-\frac{1}{H_2}=c. \tag{54}$$

If $c=0$, then $H_2-H_1=0$, which contradicts (34). If $c\neq 0$, then from (53) and (54) and the fact that all H_j $(j\le 4)$ are entire, it follows that $H_1\neq -\frac{1}{c},\frac{1}{c},\infty$. Thus H_1 is a non-zero constant, so are H_0 and H_2 by (53) and (54). Hence, there exist three non-zero constants c_i $(i=0,1,2)$ such that $H_i=c_i$. By integration, there exist constants d_i $(i=0,1,2)$ such that

$$f=e^{c_0z+d_0}g,\quad f'=e^{c_1z+d_1}g',\quad f''=e^{c_2z+d_2}g''.$$

By differentiating the first equation twice and the second equation once and using all six equations we get

$$c_0^2e^{(2c_1-c_0-c_2)z+2d_1-d_0-d_2}+c_0(c_1-c_0)e^{(c_1-c_0)z+d_1-d_0}-c_0(c_1-c_0)e^{(c_1-c_2)z+d_1-d_2}=c_0^2.$$

It follows from Lemma 8 that $c_0=c_1=c_2$, *i.e.*, $H_0=H_1=H_2$, which contradicts (34). Thus $\Phi'\not\equiv 0$.

Let z_0 be a zero of f and g with multiplicity $m\ge 2$ such that

$$H_2(z_0)-H_1(z_0)\neq 0.$$

Then $A(z_0)\neq 0$ by Lemma 6. From (42) and $Q\equiv 0$ we deduce that $B(z_0)=0$. Combining this with (39), (40) and (52), we obtain, near $z=z_0$,

$$\Phi'(z)=\frac{1}{(m+1)A(z_0)}+O\big((z-z_0)^2\big),$$

and so, $\Phi'(z_0) = 0$. Similarly, if p_0 is a multiple pole of f and g such that $H_2(p_0) - H_1(p_0) \neq 0$, then we deduce from Lemma 6, (43), (44), (46) and (52) that $\Phi'(p_0) = 0$. It follows from (3), Lemma 5, (31), (51), (52) and the above discussion that

$$\overline{N}_{\geq 2}(r, f) + \overline{N}_{\geq 2}\Big(r, \frac{1}{f}\Big) \leq N\Big(r, \frac{1}{H_2 - H_1}\Big) + N\Big(r, \frac{1}{\Phi'}\Big) = S\Big(r, \frac{f'}{f}\Big).$$

This also proves (37).

Now, by (3) and (35)–(37) we have

$$T\Big(r, \frac{f'}{f}\Big) \leq 5m\Big(r, \frac{f'}{f}\Big) + 4m\Big(r, \frac{g'}{g}\Big) + S\Big(r, \frac{f'}{f}\Big). \tag{55}$$

From (3) Lemma 3 and Lemma 5, we have

$$\begin{aligned} m\Big(r, \frac{f'}{f}\Big) &\leq m\Big(r, \frac{f''}{f'}\Big) + S\Big(r, \frac{f'}{f}\Big) \\ &\leq 2\Big[N_{=1}\Big(r, \frac{1}{f'}\Big) + \overline{N}\Big(r, \frac{1}{f'''}\Big) - \overline{N}_{>2}\Big(r, \frac{1}{f'}\Big)\Big] + S\Big(r, \frac{f'}{f}\Big) \end{aligned}$$

and

$$\begin{aligned} m\Big(r, \frac{g'}{g}\Big) &\leq m\Big(r, \frac{g''}{g'}\Big) + S\Big(r, \frac{g'}{g}\Big) \\ &\leq \Big[N_{=1}\Big(r, \frac{1}{g'}\Big) + \overline{N}\Big(r, \frac{1}{g'''}\Big) - \overline{N}_{>2}\Big(r, \frac{1}{g'}\Big)\Big] + S\Big(r, \frac{f'}{f}\Big) \\ &= 2\Big[N_{=1}\Big(r, \frac{1}{f'}\Big) + \overline{N}\Big(r, \frac{1}{f'''}\Big) - \overline{N}_{>2}\Big(r, \frac{1}{f'}\Big)\Big] + S\Big(r, \frac{f'}{f}\Big). \end{aligned}$$

Substituting these two inequalities into (55) and using Lemma 4, we get

$$T\Big(r, \frac{f'}{f}\Big) \leq 1962\Big[N_{=1}\Big(r, \frac{1}{f'}\Big) + \sum_{i=1}^{108} \overline{N}_{=i}\Big(r, \frac{1}{f'''}\Big)\Big] + S\Big(r, \frac{f'}{f}\Big). \tag{56}$$

In addition, by (55), Lemmas 3 and 4,

$$T\Big(r, \frac{f'}{f}\Big) \leq 36\Big[N_{=1}(r, \frac{1}{f}) + \sum_{i=1}^{108} \overline{N}_{=i}\Big(r, \frac{1}{f''}\Big)\Big] + S\Big(r, \frac{f'}{f}\Big). \tag{57}$$

Similarly, by $N(r, g'/g) = \overline{N}(r, g) + \overline{N}(r, 1/g) = N(r, f'/f)$ we have

$$T\Big(r, \frac{g'}{g}\Big) \leq 1962\Big[N_{=1}\Big(r, \frac{1}{f'}\Big) + \sum_{i=1}^{108} \overline{N}_{=i}\Big(r, \frac{1}{f'''}\Big)\Big] + S\Big(r, \frac{g'}{g}\Big)$$

and

$$T\Big(r, \frac{g'}{g}\Big) \leq 36\Big[N_{=1}(r, \frac{1}{f}) + \sum_{i=1}^{108} \overline{N}_{=i}\Big(r, \frac{1}{f''}\Big)\Big] + S\Big(r, \frac{g'}{g}\Big).$$

Next, we consider two subcases.

Case 4.1

$$N_{=1}\Big(r,\frac{1}{f'}\Big) = S\Big(r,\frac{f'}{f}\Big). \tag{58}$$

If $\sum_{i=1}^{108}\overline{N}_{=i}\big(r,\frac{1}{f'''}\big) = S\big(r,\frac{f'}{f}\big)$, then $\frac{f'}{f}$ is constant by (56), so is $\frac{g'}{g}$. This is (ii). If $\sum_{i=1}^{108}\overline{N}_{=i}\big(r,\frac{1}{f'''}\big) \neq S\big(r,\frac{f'}{f}\big)$, then there exists a positive integer q $(1 \le q \le 108)$ such that

$$\overline{N}_{=q}\Big(r,\frac{1}{f'''}\Big) \neq S\Big(r,\frac{f'}{f}\Big). \tag{59}$$

Now, we discuss three situations:

Case 4.1.1

$$\overline{N}_{=1}\Big(r,\frac{1}{f''}\Big) + \overline{N}_{=q+1}\Big(r,\frac{1}{f''}\Big) = S\Big(r,\frac{f'}{f}\Big). \tag{60}$$

Applying Lemma 11 with $m = q$, and $\hat{f} = f''$ and $\hat{g} = g''$, we have from (59), (60), (29) and (2) that

$$H_3 - H_2 = q(H_4 - H_3). \tag{61}$$

By applying Lemma 13 to $\hat{f} = f''$ and $\hat{g} = g''$ and noting (2), we deduce from (29), (35), (60), (61) and Lemmas 13 and 14 that

$$\overline{N}(r,f) + \overline{N}\Big(r,\frac{1}{f''}\Big) = S\Big(r,\frac{f'}{f}\Big) \tag{62}$$

and

$$\overline{N}\Big(r,\frac{1}{f'''}\Big) \le \overline{N}_{=q}\Big(r,\frac{1}{f'''}\Big) + S\Big(r,\frac{f'}{f}\Big). \tag{63}$$

Now by (3), (1), Lemma 3 and the First Fundamental Theorem,

$$\begin{aligned}
N_{=q}\Big(r,\frac{1}{f'''}\Big) &= N_{=q}\Big(r,\frac{1}{F_2}\Big) + N_{=q+1}\Big(r,\frac{1}{f''}\Big) - \overline{N}_{q+1}\Big(r,\frac{1}{f''}\Big)\\
&\le m(r,F_2) + \overline{N}(r,f) + (q+1)\overline{N}\Big(r,\frac{1}{f''}\Big) - m\Big(r,\frac{1}{F_2}\Big) + S\Big(r,\frac{f'}{f}\Big)\\
&= m(r,F_1) - m\Big(r,\frac{1}{F_2}\Big) + \overline{N}(r,f) + (q+1)\overline{N}\Big(r,\frac{1}{f''}\Big) + S\Big(r,\frac{f'}{f}\Big)\\
&\le 2N_{=1}\Big(r,\frac{1}{f'}\Big) + 2\overline{N}\Big(r,\frac{1}{f'''}\Big) + \overline{N}(r,f) + (q+1)\overline{N}\Big(r,\frac{1}{f''}\Big)\\
&\quad - m\Big(r,\frac{1}{F_2}\Big) + S\Big(r,\frac{f'}{f}\Big).
\end{aligned} \tag{64}$$

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It follows from (58), (60), (62) and (63) that

$$(65)\qquad (q-2)\overline{N}_{=q}\Big(r,\frac{1}{f'''}\Big)+m\left(r,\frac{1}{F_2}\right)=S\Big(r,\frac{f'}{f}\Big).$$

By the same reasoning we obtain

$$(66)\qquad (q-2)\overline{N}_{=q}\Big(r,\frac{1}{g'''}\Big)+m\Big(r,\frac{1}{G_2}\Big)=S\Big(r,\frac{f'}{f}\Big).$$

From (59) and (65) we see that $q \le 2$. Next, we discuss $q = 2$ and $q = 1$, respectively.

If $q = 2$, then (66) and (65) give

$$(67)\qquad m\Big(r,\frac{1}{F_2}\Big)=S\Big(r,\frac{f'}{f}\Big),\quad m\Big(r,\frac{1}{G_2}\Big)=S\Big(r,\frac{f'}{f}\Big).$$

From (61), (2) and (20) we obtain

$$\frac{F_2'}{F_2}-\frac{G_2'}{G_2}=q\left(\frac{F_3'}{F_3}-\frac{G_3'}{G_3}\right)=2\left(\frac{F_3'}{F_3}-\frac{G_3'}{G_3}\right).$$

An integration gives

$$\frac{F_2}{G_2}=c\Big(\frac{F_3}{G_3}\Big)^2,$$

where c is a non-zero constant. It is easy to verify that

$$F_3=F_2+\frac{F_2'}{F_2},\quad G_3=G_2+\frac{G_2'}{G_2}.$$

Thus we have

$$(68)\qquad cF_2-G_2=2\frac{G_2'}{G_2}+\frac{1}{G_2}\Big(\frac{G_2'}{G_2}\Big)^2-2c\frac{F_2'}{F_2}+\frac{c}{F_2}\Big(\frac{F_2'}{F_2}\Big)^2.$$

Note that $m(r,F_2'/F_2)=S(r,F_2)=S(r,f'/f)$ by (1) and (5). Thus the same conclusion also holds for G_2. It follows from this, Lemma 5, (68) and (67) that

$$(69)\qquad m(r,cF_2-G_2)=S\Big(r,\frac{f'}{f}\Big).$$

Set $U = cF_2 - G_2$. Then $m(r,U)=S\Big(r,\frac{f'}{f}\Big)$ by (69). We rewrite (68) in the form

$$U=2\frac{cF_2'-U'}{cF_2-U}+\frac{1}{cF_2-U}\Big(\frac{cF_2'-U'}{cF_2-U}\Big)^2-2c\frac{F_2'}{F_2}+\frac{c}{F_2}\Big(\frac{F_2'}{F_2}\Big)^2.$$

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Multiplying both sides by $F_2(cF_2 - U)^3$ gives

$$\begin{aligned} UF_2(cF_2-U)^3 &= (cF_2' - U')F_2(cF_2-U)^2 + (cF_2'-U')^2F_2 \\ &\quad - 2cF_2'(cF_2-U)^3 + c\Big(\frac{F_2'}{F_2}\Big)^2(cF_2-U)^3 \\ &= 2\Big(c\frac{F_2'}{F_2}F_2 - U'\Big)F_2(cF_2-U)^2 + \Big(c\frac{F_2'}{F_2}F_2 - U'\Big)^2F_2 \\ &\quad - 2c\frac{F_2'}{F_2}F_2(cF_2-U)^3 + c\Big(\frac{F_2'}{F_2}\Big)^2(cF_2-U)^3. \end{aligned}$$

Now by expanding this equality and putting all the terms with F_2^4 to the left-hand side we get

$$c^3\Big[U - 2(1-c)\frac{F_2'}{F_2}\Big]F_2^4 = P(F_2),$$

where $P(F_2)$ is a differential polynomial of F_2 with coefficients in U, U', c and F_2'/F_2, and the degree of $P(F_2)$ with respect to F_2 is less than or equal to 3. Lemma 1 implies

$$m\Big(r, \Big[U - 2(1-c)\frac{F_2'}{F_2}\Big]F_2\Big) = S\Big(r, \frac{f'}{f}\Big), \quad m\Big(r, U - 2(1-c)\frac{F_2'}{F_2}\Big) = S\Big(r, \frac{f'}{f}\Big).$$

If $U - 2(1-c)\frac{F_2'}{F_2} \not\equiv 0$, then

$$\begin{aligned} m(r, F_2) &\le m\Big(r, \Big[U - 2(1-c)\frac{F_2'}{F_2}\Big]F_2\Big) + m\left(r, \frac{1}{U - 2(1-c)\frac{F_2'}{F_2}}\right) + O(1) \\ &\le N\Big(r, U - 2(1-c)\frac{F_2'}{F_2}\Big) + S\Big(r, \frac{f'}{f}\Big) \\ &\le \overline{N}(r, f) + \overline{N}\Big(r, \frac{1}{f''}\Big) + \overline{N}\Big(r, \frac{1}{f'''}\Big) + S\Big(r, \frac{f'}{f}\Big). \end{aligned}$$

It follows from (64) with $q = 2$, (62) and (63) that

$$N_{=2}\Big(r, \frac{1}{f'''}\Big) \le \overline{N}_{=2}\Big(r, \frac{1}{f'''}\Big) + S\Big(r, \frac{f'}{f}\Big).$$

Since $N_{=2}\big(r, \frac{1}{f'''}\big) = 2\overline{N}_{=2}\big(r, \frac{1}{f'''}\big)$, we obtain $\overline{N}_{=2}\big(r, \frac{1}{f'''}\big) = S\big(r, \frac{f'}{f}\big)$, which contradicts (59). Thus $U - 2(1-c)\frac{F_2'}{F_2} \equiv 0$. If $c \ne 1$, then we have

$$\begin{aligned} \overline{N}\Big(r, \frac{1}{f'''}\Big) &\le N\Big(r, \frac{F_2'}{F_2}\Big) = N(r, U) = N(r, cF_2 - G_2) \\ &= \overline{N}(r, f) + \overline{N}\Big(r, \frac{1}{f''}\Big), \end{aligned}$$

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and so, by (62), $\overline{N}\big(r, \frac{1}{f'''}\big) = S\big(r, \frac{f'}{f}\big)$, which contradicts (59). If $c = 1$, then $U = 0$ and $F_2 = G_2$, *i.e.*, $H_2 \equiv 0$, which contradicts (34).

If $q = 1$, then (59) and (61) are

$$\overline{N}_{=1}\Big(r, \frac{1}{f'''}\Big) \neq S\Big(r, \frac{f'}{f}\Big) \tag{70}$$

and

$$H_3 - H_2 = H_4 - H_3,$$

respectively. Integrating the above equation we get

$$\frac{F_2}{G_2} = c\frac{F_2 + F_2'/F_2}{G_2 + G_2'/G_2}, \quad c \in \mathbb{C} - \{0\}. \tag{71}$$

We rewrite this in the form

$$c + c\frac{F_2'}{F_2^2} = 1 + \frac{G_2'}{G_2^2}.$$

By integration we have

$$L(z) = \frac{1}{G_2} - \frac{c}{F_2}, \quad L(z) = (1-c)z + d, \tag{72}$$

for some constant d. If $L(z) \equiv 0$, then $c = 1$ and $d = 0$. By (72), $F_2 = G_2$, *i.e.*, $H_2 = 0$, which contradicts (34). Next, we consider the case

$$L(z) \not\equiv 0. \tag{73}$$

By our assumptions, there exist entire functions $u(z)$ and $v(z)$ such that

$$F_1 = e^u G_1, \quad F_2 = c\, G_2\, e^v. \tag{74}$$

We deduce from (72) that

$$F_2 = c\frac{e^v - 1}{L}, \quad G_2 = \frac{e^{-v}(e^v - 1)}{L}. \tag{75}$$

Applying Lemma 15 with $\hat{f} = f''$ and $\hat{g} = g''$, we imply that f'' and g'' have no poles and have at most one zero which is the zero of L. Hence we may suppose that

$$f'' = Le^{\alpha}, \quad g'' = Le^{\beta},$$

where α and β are entire. From this, (1) and (75) we deduce that

$$f''' = c\,(e^v - 1)\, e^{\alpha}, \quad g''' = \big(1 - e^{-v}\big)\, e^{\beta}. \tag{76}$$

Since

$$F_2 = F_1 + \frac{F_1'}{F_1}, \quad G_2 = G_1 + \frac{G_1'}{G_1},$$

by (74) we have

$$F_2 = e^u G_1 + u' + \frac{G_1'}{G_1} = e^u G_1 + u' + G_2 - G_1.$$

Substituting (75) into this equation we get

$$G_1(1 - e^u) = u' + \frac{1}{L}(1 - e^{-v})(1 - c\,e^v).$$

Differentiating this equation and using (75) and $G_2 = G_1 + \frac{G_1'}{G_1}$ we obtain

$$h_1\,e^v - h_2\,e^u + h_3\,e^{u-v} - e^{u-2v} + h_4\,e^{-v} - c^2\,e^{2v} + h_5\,e^{u+v} - h_6 = 0, \tag{77}$$

where

$$\begin{aligned} h_1 &= 2cLu' + cLv' + 3c^2, \\ h_2 &= 2Lu' + L^2u'^2 + cLu' - L^2u'' + 2c + c^2, \\ h_3 &= 2Lu' + Lv' + 3, \\ h_4 &= Lu' - Lv' - 1 + 2c, \\ h_5 &= cL(u - v)' + 2c - c^2, \\ h_6 &= L^2u'^2 + (1 + 2c)Lu' + 2c + 2c^2 - 1 + L^2u''. \end{aligned} \tag{78}$$

Next, we consider three subcases.

Case 4.1.1.1 There exists a set I with infinite measure such that

$$T(r, e^u) = o\big(T(r, e^v)\big) \quad r \in I.$$

We rewrite (77) in the form

$$-c^2\,e^{2v} + (h_1 + h_5\,e^u)\,e^v + (h_3\,e^u + h_4)\,e^{-v} - e^u\,e^{-2v} - (h_2\,e^u + h_6) = 0.$$

From Lemma 8 we deduce that $c = 0$ which contradicts (71).

Case 4.1.1.2 There exists a set I with infinite measure such that

$$T(r, e^v) = o\big(T(r, e^u)\big) \quad r \in I.$$

We rewrite (77) in the form

$$(-h_2 + h_3\,e^{-v} - e^{-2v} + h_5\,e^v)e^u + (-h_1\,e^v + h_4e^{-v} - c^2\,e^{2v} - h_6) = 0.$$

This and Lemma 8 imply

$$-h_2 + h_3\, e^{-v} - e^{-2v} + h_5\, e^{v} = 0, \quad -h_1\, e^{v} + h_4\, e^{-v} - c^2\, e^{2v} - h_6 = 0.$$

By (78), these two equations are equivalent to

$$\begin{aligned} -L^2 u'^2 + L^2 u'' + [-(2+c)L + 2Le^{-v} + cL\, e^{v}]u' \\ + [-2c - c^2 + (Lv' + 3)\, e^{-v} - e^{-2v} - (cLv' - 2c + c^2)\, e^{v}] = 0 \end{aligned}$$

and

$$\begin{aligned} -L^2 u'^2 - L^2 u'' + [-(1+2c)L + L\, e^{-v} + 2cL\, e^{v}]u' \\ + [2c + 2c^2 - 1 - (Lv' + 1 - 2c)\, e^{-v} - c^2\, e^{2v} + (cLv' + 3c^2)\, e^{v}] = 0, \end{aligned}$$

respectively. From these two equations we get

$$\begin{aligned} 2L^2 u'' + L(c - 1 + e^{-v} - c\, e^{v})u' + 1 - 4c - 3c^2 \\ + 2(Lv' + 2 - c)\, e^{-v} - e^{-2v} + c^2\, e^{2v} 2c(Lv' - 1 + 2c)\, e^{v} = 0 \end{aligned}$$

and

$$\begin{aligned} -2L^2 u'^2 + 3L(-1 - c + e^{-v} + c\, e^{v})u' + c^2 - 1 \\ + 2(c+1)\, e^{-v} - e^{-2v} - c^2\, e^{2v} + 2c(1+c)\, e^{v} = 0. \end{aligned}$$

Differentiating the last equation and eliminating u' and u'' in the three equations, we obtain

$$24L^8 e^{-6v} + \sum_{j=-5}^{j=5} P_j(L, v') e^{jv} - 24c^6 L^8 e^{6v} = 0,$$

where $P_j(L, v')$ are differential polynomials in v' and L. If v is a constant, then $N\big(r, \frac{1}{f'''}\big) = 0$ by (76), which contradicts (70). If v is non-constant, then from the above equation and Lemma 8 it follows that $L \equiv 0$, which contradicts (73).

Case 4.1.1.3 Neither Case 4.1.1.1 nor Case 4.1.1.2 holds. Then

$$T(r, e^u) = O\big(T(r, e^v)\big), \quad T(r, e^v) = O\big(T(r, e^u)\big)$$

for all $r > 0$ except possibly for a set of values of r with finite linear measure. From (78) we see that

$$T(r, h_j) = o\big(T(r, e^u), T(r, e^v)\big), \quad (r \notin E,\ j = 1, \dots, 6),$$

where E is a set of finite measure.

Case 4.1.1.3.1 There exists a set I with infinite measure such that

$$T(r, e^{u-v}) = o\big(T(r, e^{u})\big), \quad T(r, e^{u-v}) = o\big(T(r, e^{v})\big), \quad r \in I.$$

We rewrite (77) in the form

$$(h_5\, e^{u-v} - c^2)\, e^{2v} + (h_1 - h_2\, e^{u-v})\, e^{v} + (h_4 - e^{u-v})\, e^{-v} - h_6 = 0.$$

This and Lemma 8 assert that

$$h_6 = 0, \quad h_5\, e^{u-v} - c^2 = 0, \quad h_1 - h_2\, e^{u-v} = 0, \quad h_4 - e^{u-v} = 0. \tag{79}$$

Eliminating e^{u-v} we obtain

$$h_5 h_4 - c^2 = 0, \quad h_1 - h_2 h_4 = 0. \tag{80}$$

From (78), (79) and Lemma 1 we imply that $u' =$ const. and $L =$ const., *i.e.,* $c = 1$, $L = d$. It follows from this, (78) and (80) that v' is constant. Noting that $e^{u-v} = h_4 = d(u-v)' + 1$, we deduce that $u - v =$ const. Combining all these facts with (80) we get $u' = v' = 0$, *i.e.,* v is a constant. By this and (76), $f''' \neq 0$, which contradicts (70).

Case 4.1.1.3.2 There exists a set I with infinite measure such that

$$T(r, e^{u-2v}) = o\{T(r, e^{u}), T(r, e^{v})\}, \quad r \in I.$$

We rewrite (77) in the form

$$h_5\, e^{u-2v} e^{3v} - (h_2\, e^{u-2v} + c^2)\, e^{2v} + (h_1 + h_3\, e^{u-2v})\, e^{v} + h_4\, e^{-v} - (e^{u-2v} + h_6) = 0.$$

By Lemma 8,

$$h_5 = h_4 = 0, \quad h_1 + h_3 e^{u-2v} = 0,$$
$$h_2 e^{u-2v} + c^2 = 0, \quad e^{u-2v} + h_6 = 0.$$

From $h_5 = h_4 = 0$ and the expressions for h_4 and h_5 in (78) we deduce that $c = 1$ and so $L \equiv d$ by (72). Now eliminating e^{u-2v} in the above equations gives

$$h_2 h_6 = c^2 = 1.$$

Substituting the expressions for h_2 and h_6 in (78) into this, we obtain

$$d^4 (u')^4 + P_3(u') = 0,$$

where $P_3(u')$ is a differential polynomial of u' of degree ≤ 3. This and Lemma 1 imply that u' is a constant, and so v' is also a constant by $h_5 \equiv 0$ (in fact, we can further deduce that $du' = -2$, $dv' = -1$ and $e^{u-2v} = -1$). Thus v is linear. Therefore

f'' and g'' have hyper-order 1 by (76), and so, f and g also have hyper-order 1 by Chuang [2]. By Lemma 10, we need only consider case (v):

$$f(z) = Ae^{\exp(az+b)}, \quad g(z) = Be^{\exp(-az-b)}, \quad (ABa \neq 0).$$

Differentiating the equation for f twice we obtain

$$f''(z) = Aa^2(e^{az+b} + 1)e^{az+b+\exp(az+b)}.$$

Thus f'' has infinitely many zeros, which contradicts (76).

Case 4.1.1.3.3 There exists a set I with infinite measure such that

$$T(r, e^{u+v}) = o\{T(r, e^u), T(r, e^v)\}.$$

We rewrite (77) in the form

$$-c^2e^{2v} + h_1e^v + (h_5e^{u+v} - h_6) - (h_2\, e^{u+v})e^{-v} + (h_3\, e^{u+v})e^{-2v} - e^{u+v}e^{-3v} = 0.$$

By Lemma 8, $c = 0$, which is a contradiction to (71).

Case 4.1.1.3.4 None of the above three subcases hold. Then Lemma 8 and (77) give a contradiction.

Case 4.1.2

$$\overline{N}_{=q+1}\Big(r, \frac{1}{f''}\Big) \neq S\Big(r, \frac{f'}{f}\Big). \tag{81}$$

Applying Lemma 12 to $m = q + 1$, $\hat{f} = f''$ and $\hat{g} = g''$, it follows from (29), (81) and (2) that

$$H_3 - H_2 = \frac{q}{q+2}(H_4 - H_3). \tag{82}$$

By the assumptions of Theorem 1, there exist three entire functions $u(z)$, $v(z)$ and $w(z)$ such that

$$F_1(z) = e^{u(z)}G_1(z), \quad F_2(z) = e^{v(z)}G_2(z), \quad F_3(z) = e^{w(z)}G_3(z). \tag{83}$$

Integrating (82) gives

$$\frac{F_2}{G_2} = c_2\Big(\frac{F_3}{G_3}\Big)^{q/(q+2)},$$

where c_2 is a non-zero constant. This and (83) imply

$$v(z) = \frac{q}{q+2} w(z) + d_1, \tag{84}$$

where d_1 is a constant. Now we consider two subcases.

Case 4.1.2.1

$$\overline{N}_{q+2}\Big(r, \frac{1}{f'}\Big) \neq S\Big(r, \frac{f'}{f}\Big).$$

Applying Lemma 12 with $m = q + 2$, $\hat{f} = f'$ and $\hat{g} = g'$, we have from (29) and (2) that

$$H_2 - H_1 = \frac{q+1}{q+3}(H_3 - H_2),$$

which, upon an integration, becomes

$$\frac{F_1}{G_1} = c_1 \Big(\frac{F_2}{G_2}\Big)^{(q+1)/(q+3)}, \tag{85}$$

where c_1 is a non-zero constant. It follows from (83)–(85) that

$$u(z) = \frac{q(q+1)}{(q+2)(q+3)}\, w(z) + d_2, \tag{86}$$

where d_2 is a constant. Applying Lemma 7 with $j = 1$ we obtain

$$x_1 g_1 + y_1 g_2 + z_1 g_1 g_2 = r_1, \tag{87}$$

$$x_2 g_1 + y_2 g_2 + z_2 g_1 g_2 = r_2, \tag{88}$$

$$x_3 g_1 + y_3 g_2 + z_3 g_1 g_2 = r_3, \tag{89}$$

where x_i, y_i, z_i, r_i $(i = 1, 2, 3)$ are as in Lemma 7, and by (84) and (86), each member in x_i, y_i, z_i, r_i $(i = 1, 2, 3)$ has a representation of the form

$$\sum_{i=1}^{K} P_i[w']\, e^{\alpha_i w}, \quad K \in \mathbb{N},$$

where $\alpha_1 > \alpha_2 > \cdots \geq 0$, $P_i[w']$ are differential polynomials in w' and

$$T(r, P[w']) = S\Big(r, \frac{f'}{f}\Big).$$

By simple calculations we see that

$$
\begin{aligned}
x_1 &= \exp\Big\{\frac{q}{q+2}w+d_1\Big\}-1,\\
x_2 &= -v'\exp\Big\{\Big(1+2\frac{q}{q+2}\Big)w+2d_1\Big\}+\cdots,\\
x_3 &= -[v'(2v'+w')+v'']\exp\Big\{\Big(2+3\frac{q}{q+2}\Big)w+3d_1\Big\}+\cdots,\\
y_1 &= -\exp\Big\{\frac{q(q+1)}{(q+2)(q+3)}w+d_2\Big\}+1,\\
y_2 &= u'\exp\Big\{\Big(1+\frac{q}{q+2}+\frac{q(q+1)}{(q+2)(q+3)}\Big)w+d_1+d_2\Big\}+\cdots,\\
y_3 &= [u''+u'(u'+v'+w')]\exp\Big\{\Big(2+2\frac{q}{q+2}+\frac{q(q+1)}{(q+2)(q+3)}\Big)w+2d_1+d_2\Big\}\\
&\qquad +\cdots,\\
z_1 &= -u',\\
z_2 &= u''\exp\Big\{\Big(1+\frac{q}{q+2}\Big)w+d_1\Big\}+\cdots,\\
z_3 &= [(v'+w')u''+u''']\exp\Big\{2\Big(1+\frac{q}{q+2}\Big)w+2d_1\Big\}+\cdots,\\
r_1 &= 0,\\
r_2 &= \exp\Big\{\Big(1+2\frac{q}{q+2}\Big)w+2d_1\Big\}+\cdots,\\
r_3 &= (3v'+w')\exp\Big\{\Big(2+3\frac{q}{q+2}\Big)w+3d_1\Big\}+\cdots,
\end{aligned}
$$

where we only list the largest term with respect to α_i. Combining all these representations with (84) we deduce from (87)–(89) (where we look at (87)–(89) as a system of linear equations in the unknowns g_1, g_2 and g_1g_2) that the largest term with respect to α_i in the determinant of the coefficients of (87)–(89) is

$$L(D)=P_1\exp\Big\{\Big(3+4\frac{q}{q+2}+\frac{q(q+1)}{(q+2)(q+3)}\Big)w+4d_1+d_2\Big\},$$

where

$$P_1=-\frac{2}{q+1}\Big(u'u'''-u''^2-\Big(1+\frac{q+3}{q+1}\Big)u''u'^2+\frac{q+3}{q+1}u'^4\Big).$$

If the determinant of the coefficients of (87)–(89) is vanishing, then by Lemma 8, $P_1\equiv 0$. Thus,

$$uu'''-u''^2-\Big(1+\frac{q+3}{q+1}\Big)u''u'^2+\frac{q+3}{q+1}u'^4\equiv 0.$$

By Lemma 1, u' is constant. It follows from the above equation that $u' = 0$, and so $v' = w' = 0$ by (84) and (86). Thus u, v, w are constants. If $e^v \neq 1$, then from (15) with $j = 1$ and (83) we deduce that F_1 and G_1 are constants, thus $f'' \neq 0$, which contradicts (81). If $e^v = 1$, then $F_1 = G_1$ by (83), and so $H_1 \equiv 0$, which contradicts (34). Therefore the determinant of the coefficients of (87)–(89) is non-vanishing. By Cramer's rule we obtain

$$g_1 = \frac{\det(\vec{r}, \vec{y}, \vec{z})}{\det(\vec{x}, \vec{y}, \vec{z})}, \quad g_2 = \frac{\det(\vec{x}, \vec{r}, \vec{z})}{\det(\vec{x}, \vec{y}, \vec{z})}, \quad g_1 g_2 = \frac{\det(\vec{x}, \vec{y}, \vec{r})}{\det(\vec{x}, \vec{y}, \vec{z})}.$$

Thus,

$$\det(\vec{r}, \vec{y}, \vec{z}) \det(\vec{x}, \vec{r}, \vec{z}) = \det(\vec{x}, \vec{y}, \vec{z}) \det(\vec{x}, \vec{y}, \vec{r}). \tag{90}$$

By (84) and the representations of $x_1, \ldots, r_3$ above, the highest term with respect to α_i on the right-hand and the left-hand sides of (90) are

$$t(r) = P \exp\Big\{ \Big(6 + 9\frac{q}{q+2} + 2\frac{q(q+1)}{(q+2)(q+3)}\Big) w + 9d_1 + 2d_2\Big\}$$

and

$$t(l) = P_0 \exp\Big\{ \Big(6 + 9\frac{q}{q+2} + \frac{q(q+1)}{(q+2)(q+3)}\Big) w + 9d_1 + d_2\Big\},$$

respectively, where

$$P = -\Big(\frac{2}{q+1}\Big)^2 \Big[u'u''' - u''^2 - \Big(1 + \frac{q+3}{q+1}\Big) u''u'^2 + \frac{q+3}{q+1}u'^4\Big] \times \Big[u'' - \Big(1 + \frac{q+3}{q+1}\Big) u'^2\Big]$$

and

$$P_0 = \Big[u''' - \Big(2\frac{q+3}{q+1} + 1\Big) u'u'' + \Big(2\frac{q+3}{q+1} - 1\Big) u'^3\Big] \times \Big[u''' - \Big(2\frac{q+3}{q+1} + 1\Big) u'u'' + \Big(\frac{q+3}{q+1}\Big)^2 u'^3\Big].$$

It follows from these two equations, (90) and Lemma 8 that $P \equiv 0$, and so,

$$\Big[u'u''' - u''^2 - \Big(1 + \frac{q+3}{q+1}\Big) u''u'^2 + \frac{q+3}{q+1}u'^4\Big] \Big[u'' - \Big(1 + \frac{q+3}{q+1}\Big) u'^2\Big] \equiv 0.$$

This and Lemma 1 give $m(r, u') = S(r, u')$. Note that, by the above equality, u' has no poles. Thus $T(r, u') = S(r, u')$. This implies that u' is a constant. Using the above equality again, we imply that $u' \equiv 0$, and so $v' = w' = 0$ by (84) and (86). Thus, u, v, w, are constants. If $e^v \neq 1$, then from (15) with $j = 1$ and (83) we deduce that F_1 and G_1 are constants, thus $f'' \neq 0$, which contradicts (81). If $e^v = 1$, then $F_1 = G_1$ by (83), and so $H_1 \equiv 0$, which contradicts (34).

Case 4.1.2.2 Assume

$$\overline{N}_{=q+2}\Big(r, \frac{1}{f'}\Big) = S\Big(r, \frac{f'}{f}\Big).$$

Applying Lemma 11 to $\hat{f} = f$ and $\hat{g} = g$ with $m = q + 1$, we have from the above equality, (29), (81) and (2) that

$$H_2 - H_1 = (q + 1)(H_3 - H_2).$$

Integrating the last equation gives

$$\frac{F_1}{G_1} = c_1\Big(\frac{F_2}{G_2}\Big)^{(q+1)},$$

where c_1 is a non-zero constant. It follows from (83) and (84) that

$$u(z) = \frac{q(q+1)}{q+2}w(z) + d_2, \tag{91}$$

where d_2 is a constant. If $q = 1$, then (84) and (91) become

$$v(z) = \frac{1}{3}w(z) + d_1, \quad u(z) = \frac{2}{3}w(z) + d_2.$$

By the same reasoning as in Case 4.1.2.1 we arrive at the desired result, where we replace $\frac{q}{q+2}$ and $\frac{q(q+1)}{(q+2)(q+3)}$ in (84) and (86) by $\frac{1}{3}$ and $\frac{2}{3}$, respectively. If $q \geq 2$, then (84) and (91) give

$$w(z) = \frac{q+2}{q(q+1)}u(z) + e_1, \quad v(z) = \frac{1}{q+1}u(z) + e_2, \tag{92}$$

where e_1 and e_2 are constants. Similarly as in Case 4.1.2.1, we replace (84) and (86) by (92) and obtain the desired result.

Case 4.1.3 Assume

$$\overline{N}_{=q+1}\Big(r, \frac{1}{f''}\Big) = S\Big(r, \frac{f'}{f}\Big), \quad \overline{N}_{=1}\Big(r, \frac{1}{f''}\Big) \neq S\Big(r, \frac{f'}{f}\Big). \tag{93}$$

Applying Lemma 11 with $m = q$, $\hat{f} = f''$ and $\hat{g} = g''$, it follows from (29), (59) and (2) that

$$H_3 - H_2 = q(H_4 - H_3), \tag{94}$$

which, upon integration, becomes

$$\frac{F_2}{G_2} = c\Big(\frac{F_3}{G_3}\Big)^q, \tag{95}$$

where c is a constant. If $q = 1$, then

$$\frac{F_2}{G_2} = c\,\frac{F_3}{G_3} = c\Big(F_2 + \frac{F_2'}{F_2}\Big) \Big/ \Big(G_2 + \frac{G_2'}{G_2}\Big).$$

We rewrite this in the form

$$1 - c = c\,\frac{F_2'}{F_2^2} - \frac{G_2'}{G_2^2}.$$

By integration,

$$L(z) = -\frac{c}{F_2} + \frac{1}{G_2},$$

where $L(z) = (1-c)z + d$ and d is a constant. If $L(z) \equiv 0$, then $c = 1$ and $F_2 = G_2$, which contradicts (34). If $L(z) \not\equiv 0$, then by (83) we obtain

$$\frac{f'''}{f''} = F_2 = \frac{e^{v} - c}{L(z)}.$$

This implies that f'' has at most one zero. However, this contradicts (93) unless $\frac{f'}{f}$ is rational. By Lemma 9 we obtain the desired results. Next we let $q \geq 2$ and consider two subcases.

Case 4.1.3.1 Assume

$$\overline{N}_{=2}\Big(r, \frac{1}{f'}\Big) \neq S\Big(r, \frac{f'}{f}\Big). \tag{96}$$

Applying Lemma 12 to $m = 2$, $\hat{f} = f'$ and $\hat{g} = g'$, it follows from (29), (96) and (2) that

$$H_2 - H_1 = \frac{1}{3}(H_3 - H_2).$$

From this, (83) and (95) we deduce that

$$u(z) = \frac{1}{3}v(z) + d_1, \quad w(z) = \frac{1}{q}v(z) + d_2, \tag{97}$$

where d_1 and d_2 are constants. Similarly, as in Case 4.1.2.1, we can derive the desired results.

Case 4.1.3.2 Assume

$$\overline{N}_{=2}\Big(r, \frac{1}{f'}\Big) = S\Big(r, \frac{f'}{f}\Big),$$

Applying Lemma with to $m = 1$, $\hat{f} = f'$ and $\hat{g} = g'$, we deduce from (29) and (93) that

$$H_2 - H_1 = H_3 - H_2,$$

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which, upon integration, becomes

$$\frac{F_1}{G_1} = c\Big(\frac{F_2}{G_2}\Big) = c\Big(F_1 + \frac{F_1'}{F_1}\Big) \Big/ \Big(G_1 + \frac{G_1'}{G_1}\Big),$$

where c is a constant. We rewrite this in the form

$$1 - c = c\,\frac{F_1'}{F_1^2} - \frac{G_1'}{G_1^2}.$$

By integration,

$$L(z) = -\frac{c}{F_1} + \frac{1}{G_1},$$

where $L(z) = (1-c)z + d$, and d is a constant. If $L(z) \equiv 0$, then $c = 1$ and $H_1 = F_1 - G_1 = 0$, which contradicts (34). If $L(z) \not\equiv 0$, then by (83),

$$F_1 = \frac{e^u - c}{L}, \quad G_1 = \frac{1 - c\,e^{-u}}{L}. \tag{98}$$

It follows from (1) and Lemma 15 that f' and g' have at most one zero which is the zero of L. Suppose that

$$f' = L\,e^{\alpha}, \quad g' = L\,e^{\beta}.$$

From this and (98), we deduce that

$$\alpha' = \frac{1}{L}(e^u - 1), \quad \beta' = \frac{c}{L}(1 - e^{-u})$$

and

$$f'' = (e^u - c)\,e^{\alpha}, \quad g'' = (1 - ce^{-u})\,e^{\beta}.$$

Differentiating this gives

$$f''' = \frac{1}{L}[e^{2u} + (Lu' - 1 - c)e^u + c]\,e^{\alpha}, \quad g''' = \frac{c}{L}\,e^{-2u}[e^{2u} + (Lu' - 1 - c)\,e^u + c]\,e^{\beta}.$$

Differentiating the above equations, we get

$$\begin{aligned} f^{(4)} = \frac{e^{\alpha}}{L^2}\big[\,& e^{3u} + (3Lu' - u - 3 + c)e^{2u} \\ &+ \big(2 + L^2u'' + L^2u'^2 - (2+c)Lu' + (2-c)u\big)\,e^u - 2c + c^2\big], \end{aligned}$$

$$\begin{aligned} g^{(4)} = \frac{ce^{\beta-3u}}{L^2}\big[\,& (2c-1)e^{3u} \\ &+ \big((1+2c)Lu' - L^2u'^2 + L^2u'' + 1 - 2c - 2c^2\big)\,e^{2u} + 3c(c - Lu')\,e^u - c^2\big]. \end{aligned}$$

From this and the Second Fundamental Theorem we can easily verify that $f^{(4)}$ and $g^{(4)}$ do not share 0 CM, which is a contradiction.

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Case 4.2 Suppose

$$\overline{N}_{=1}\Big(r, \frac{1}{f'}\Big) \neq S\Big(r, \frac{f'}{f}\Big). \tag{99}$$

We consider two subcases.

Case 4.2.1

$$\overline{N}_{=2}\Big(r, \frac{1}{f}\Big) = S\Big(r, \frac{f'}{f}\Big).$$

Let z_0 be a simple zero of f' which is not a zero of f. Then by Lemma 6, z_0 is a zero of $H_1 - H_0 - (H_2 - H_1)$. It follows from (29), (99) and Lemma 11 that

$$H_1 - H_0 = H_2 - H_1.$$

Integrating this, we get

$$\frac{F_0}{G_0} = c\,\frac{F_1}{G_1} = c\frac{F_0 + F_0'/F_0}{G_0 + G_0'/G_0},$$

where c is a nonzero constant. We rewrite this in the form

$$c - 1 + c\,\frac{F_0'}{F_0^2} = \frac{G_0'}{G_0^2}.$$

By integration we obtain

$$L(z) = \frac{1}{G_0} - \frac{c}{F_0}, \tag{100}$$

where $L(z) = (1 - c)z + d$ for some constant d. If $L(z) \equiv 0$, then $c = 1$ and $F_0 = G_0$ by (100), which implies $H_0 = 0$, a contradiction to (34). If $L(z) \not\equiv 0$, then similarly as the Case 4.1.3.2, we can derive a contradiction.

Case 4.2.2

$$\overline{N}_{=2}\Big(r, \frac{1}{f}\Big) \neq S\Big(r, \frac{f'}{f}\Big). \tag{101}$$

Applying Lemma 12 with $m = 2$, $\hat{f} = f$ and $\hat{g} = g$, it follows from (2), (29) and (101) that

$$H_1 - H_0 = \frac{1}{3}\,(H_2 - H_1). \tag{102}$$

Integrating this gives

$$\Big(\frac{F_0}{G_0}\Big)^3 = c\Big(\frac{F_1}{G_1}\Big),$$

where c is a non-zero constant. Let z_0 be a zero of f of order 2. Then by the above equation, near $z = z_0$,

$$1 = c + O(z - z_0).$$

Thus $c = 1$ and

$$\left(\frac{F_0}{G_0}\right)^3 = \frac{F_1}{G_1}. \tag{103}$$

By our assumptions, there exists an entire function $u(z)$ such that

$$F_0 = e^u G_0. \tag{104}$$

Since

$$G_1 = G_0 + \frac{G_0'}{G_0} = e^{-u}F_0 - u' + \frac{F_0'}{F_0} = e^{-u}F_0 - u' + F_1 - F_0,$$

we deduce from the above three equations that

$$(e^{-u} - 1)F_0 + (1 - e^{-3u})F_1 - u' = 0. \tag{105}$$

If, near $z = z_0$, f has a simple zero z_0 and f and g have the expansions given by Lemma 6, then near $z = z_0$,

$$F_0 = \frac{1}{z - z_0} + \frac{a_2}{a_1} + O(z - z_0),$$

$$F_1 = \frac{2a_2}{a_1} + O(z - z_0).$$

From (105) we get

$$\frac{e^{-u(z_0)} - 1}{z - z_0} + \frac{a_2}{a_1}\left(e^{-u(z_0)} - 1\right) + 2\frac{a_2}{a_1}\left(1 - e^{-3u(z_0)}\right) - u'(z_0)\left(e^{-u(z_0)} + 1\right) + O(z - z_0) = 0.$$

Thus,

$$e^{-u(z_0)} - 1 = 0$$

and

$$\frac{a_2}{a_1}\left(e^{-u(z_0)} - 1\right) + 2\frac{a_2}{a_1}\left(1 - e^{-3u(z_0)}\right) - u'(z_0)\left(e^{-u(z_0)} + 1\right) = 0.$$

This implies that $u'(z_0) = 0$. On the other hand, by (104),

$$H_1 - H_0 = F_1 - G_1 - (F_0 - G_0) = F_1 - F_0 - (G_1 - G_0) = \frac{F_0'}{F_0} - \frac{G_0'}{G_0} = u'.$$

Thus $H_1(z_0) - H_0(z_0) = 0$. Therefore

$$N_{=1}\Big(r, \frac{1}{f}\Big) \le N\Big(r, \frac{1}{H_1 - H_0}\Big) = S\Big(r, \frac{f'}{f}\Big) \tag{106}$$

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by (31). If $\sum_{i=1}^{108} \overline{N}_{=i}\big(r, \frac{1}{f''}\big) = S\big(r, \frac{f'}{f}\big)$, then $\frac{f'}{f}$ is rational by (57) and (106), and the conclusion follows. If $\sum_{i=1}^{108} \overline{N}_{=i}\big(r, \frac{1}{f''}\big) \neq S\big(r, \frac{f'}{f}\big)$, then there exists q with $1 \leq q \leq 108$ such that

$$\overline{N}_{=q}\Big(r, \frac{1}{f''}\Big) \neq S\Big(r, \frac{f'}{f}\Big). \tag{107}$$

By the assumptions of Theorem 1, there exist three entire functions, $u(z)$, $v(z)$ and $w(z)$, such that

$$F_0(z) = e^{u(z)}G_0(z), \quad F_1(z) = e^{v(z)}G_1(z), \quad F_2(z) = e^{w(z)}G_2(z). \tag{108}$$

Substituting (108) into (103) we get

$$u(z) = \frac{1}{3}\, v(z) + d_1, \tag{109}$$

where d_1 is a constant. Next, we discuss two cases.

Case 4.2.2.1

$$\overline{N}_{=q+1}\Big(r, \frac{1}{f'}\Big) \neq S\Big(r, \frac{f'}{f}\Big). \tag{110}$$

Applying Lemma 12 with $m = q+1$, $\hat{f} = f'$ and $\hat{g} = g'$, we deduce from (110), (29) and (2) that

$$H_2 - H_1 = \frac{q}{q+2}(H_3 - H_2). \tag{111}$$

Integrating (111) and then substituting (108) into it we obtain

$$v(z) = \frac{q}{q+2} w(z) + d_2, \tag{112}$$

where d_2 is a constant. We apply Lemma 7 with $j = 0$ and use (109) and (112). By the same reasoning as in Case 4.1.2.1, we complete this case.

Case 4.2.2.2

$$\overline{N}_{=q+1}\Big(r, \frac{1}{f'}\Big) = S\Big(r, \frac{f'}{f}\Big).$$

Applying Lemma 11 with $m = q$, $\hat{f} = f'$ and $\hat{g} = g'$, it follows from the above equality, (107), (29) and (2) that

$$H_2 - H_1 = q(H_3 - H_2).$$

Integrating this equation and then substituting (108) into it we obtain

$$w(z) = \frac{1}{q}\, v(z) + d_2.$$

By the same reasoning as in Case 4.1.2.1 where (84) and (86) were replaced by (109) and the above equation, applying Lemma 7 with $j = 0$ we solve this case.

This completes the proof of the theorem. ■

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