Correspondence

DEAR EDITOR,

I was pleasantly surprised to stumble across yet another example of the same discovery made independently in different parts of the globe: The divisibility test for 19 given in the November 1998 issue of the *Mathematical Gazette* by Humphreys and Macharia was also offered, as part of a more general result, by a high school student in India, Apoorva Khare, in 'Divisibility Tests' by A. Khare, *Furman University Electronic Journal of Undergraduate Mathematics*, **Volume 3,** 1997, pp 1-5. That is not to take anything away from the H-M article, which I found explained the special case more usefully.

Yours sincerely,

DINO SURENDRAN

University of Zimbabwe, Harare, Zimbabwe

DEAR EDITOR,

In a recent note entitled 'The convergence of a Lucas series' [*Math. Gaz.* **83** (July 1999) pp. 273-274], T. Koshy undertook to prove that, for integral $k \geqslant 2$, the ratio $(2k - 1)/(k^2 - k - 1)$ is integral if, and only if, $k = 2$ or $k = 3$. Koshy's approach was unnecessarily involved. Here is a more direct proof of this result.

For $k \geqslant 2$, $(2k - 1)/(k^2 - k - 1)$ is positive. Furthermore, this ratio can only be integral if $(k^2 - k - 1) \leqslant (2k - 1)$, i.e. if $k(k - 3) \leqslant 0$. The only solutions for integral $k \geqslant 2$ are then obviously $k = 2, 3$.

Yours sincerely,

N. GAUTHIER

Department of Physics, Royal Military College of Canada, Kingston, Ontario, Canada

DEAR EDITOR,

With regard to the article [1] by K. Robin McLean, an interesting variation on 'Diffy' is to play 'Quiffy', in which one finds the larger quotient of each number with its successor in the cycle, ending with a cycle of ones e.g.

$$\begin{array}{cccc} 4 & 7 & 2 & 1 \\ 7/4 & 7/2 & 2 & 4 \\ 2 & 7/4 & 2 & 16/7 \\ 8/7 & 8/7 & 8/7 & 8/7 \\ 1 & 1 & 1 & 1 \end{array}$$

The reason that the process works is simple, since, if we take logarithms of each number, we are then playing 'Diffy' (unsigned differences) and reaching a cycle of zeros ($\ln 1 = 0$).

An interesting sidelight is a very simple proof that, if the cycle of four positive numbers is not equivalent to a purely increasing sequence, the process terminates after at most six steps. (Cycles are equivalent when cycled, reversed, multiplied by a constant, or raised to a power.)

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Proof

A little thought determines that there are only two distinct cycles to consider:

$$1 \quad b \quad c \quad a \qquad \text{and} \qquad 1 \quad c \quad a \quad b$$

where $1 \leqslant a \leqslant b \leqslant c$, all cycles being equivalent to one of them or to a strictly increasing sequence.

Lemma

Any sequence $\{1, x, 1, y\}$ terminates in at most four steps. We need only consider three cases: $x \leqslant y \leqslant 1$, $x \leqslant 1 \leqslant y$ and $1 \leqslant x \leqslant y$ (since all other cases can be obtained by cycling and reversal). For $1 \leqslant x \leqslant y$

$$\begin{array}{cccc} 1 & x & 1 & y \\ x & x & y & y \\ 1 & y/x & 1 & y/x \\ 1 & 1 & 1 & 1 \end{array}$$

with similar sequences in the other cases.

For the main result, we have

$$\text{Cycle 1:} \quad \begin{array}{cccc} 1 & b & c & a \\ b & c/b & c/a & a \\ [b^2/c] & b/a & [a^2/c] & b/a \end{array} \qquad \text{Cycle 2:} \quad \begin{array}{cccc} 1 & c & a & b \\ c & c/a & b/a & b \\ a & c/b & a & c/b \end{array}$$

where $[x/y] = \max(x/y, y/x)$. In each case the last line is equivalent to $\{1, x, 1, y\}$.

I believe 'Quiffy' has practical value for students learning to use a calculator as the result in practice always arrives in a small number of steps.

Reference

1. K. Robin McLean, Playing Diffy with real sequences, *Math. Gaz.* **83** (March 1999) pp. 58-68.

Yours sincerely,

BRIAN STOKES

54 Pine Avenue, Melville, Hamilton, New Zealand

P.S. Returning to the original 'Diffy', it is quite easy to obtain a sequence of four numbers which takes as many steps as you like to terminate. Just take a sequence from the so-called Tribonacci numbers, where

$$T_1 = T_2 = T_3 = 1, \; T_{n+3} = T_n + T_{n+1} + T_{n+2} \text{ for } n > 0,$$

giving:

$$1, 1, 1, 3, 5, 9, 17, 31, 57, 105, 193, 355, 653, \ldots .$$

The limit of T_n / T_{n+1} as $n \to \infty$ is h, where $h^3 + h^2 + h + 1 = 0$, the reciprocal of the number pointed out by McLean.

In fact, each set of three steps of a run of Tribonacci numbers brings it to double the run two terms earlier; for example,

57	105	193	355	
48	88	162	298	
40	74	136	250	
34	62	114	210	(= 2 × [17, 31, 57, 105])

– the sequence terminating after 16 steps.

DEAR EDITOR,

Mathematical Pie (No. 146, I think) gave a fascinating result which I had not previously come across: if squares are drawn on the sides of any quadrilateral, the joins of the centres of opposite squares are equal and perpendicular.

I can prove this by either of two techniques (complex numbers and vector products) by which half-sides of the quadrilateral may be turned through a right-angle. (Incidentally, these methods make it clear that the squares may be drawn either all outwardly or all inwardly.) But I feel that such a seemingly 'elementary' result should admit of an 'elementary' proof and such I have failed to find. Perhaps a *Gazette* reader can supply a proof that Euclid would have understood.

It may be of interest to note that a pretty result emerges by regarding a triangle as, in three different ways, a quadrilateral with a zero side. One of my grandchildren has pointed out to me that a line segment may also be regarded as a degenerate quadrilateral: the result still holds.

Yours sincerely,

ERIC BARTON

202 Old Bath Road, Cheltenham GL53 9EQ

DEAR EDITOR,

The article by M. N. Brearley et al, in the November 1998 *Gazette* pp. 389-404, reminds me of some work done at Cambridge in the 1930s, which should not perhaps go unrecorded. Professor Sir Charles Inglis, then Head of the Department of Engineering, had the idea of multiple-phase rowing in order to even out the acceleration of a racing shell. To avoid conflict of the blades, he developed the 'syncopated six', with three phases, and trained a crew to use it in a specially fitted boat.

A race was then arranged between an eight and his six. Though I was not present, I understand that the six was substantially faster and pulled away from the eight. I believe the Cambridge authorities then banned this form of rowing in races.

Sir Charles was still Head of Department when I was up as an undergraduate, and I heard the story from him myself. As far as I know, nothing was published about this work. It would be a pity for it to be forgotten.

Yours sincerely,

ROBERT MACMILLAN

1 The Empire, Grand Parade, Bath BA2 4DF