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Another proof of Jakobson's Theorem and related results

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Abstract. The author shows that any family C^2-close to $f_\alpha(x) = 1 - \alpha x^2$ $(2 - \varepsilon \le \alpha \le 2)$ satisfies Jakobson's theorem: For a positive measure set of α the transformation f_α has an absolutely continuous invariant measure. He also indicates some generalizations.

0. *Introduction*

In recent years there has been major interest in the following theorem of Jakobson:

THEOREM. *Let $f_r(x) = rx(1-x)$, $0 \le r \le 4$, be a one parameter family of mappings of the unit interval. There is a positive measure set of those r for which f_r has an absolutely continuous invariant measure* (*abbreviation*: *a.c.i.m.*).

The author of the current paper uses his earlier ideas from [**5**] to give another proof of this theorem. It seems to be less technical than other existing proofs (see [**2**], [**1**]) and therefore it yields some interesting generalizations (§ 6). In particular, any family C^2-close to the one above satisfies Jakobson's theorem. Also, the families that contain $f_4(x) = 4x(1-x)$ and do not satisfy Jakobson's theorem form a 'set of codimension ∞' in the set of C^2-families that contain f_4 (or any mapping C^2-close to f_4 with the property that f^2(critical point) = fixed point). In particular, any analytic family of this type satisfies Jakobson's theorem.

One reason to understand Jakobson's result is a possible generalization to higher dimensions. Similar phenomena seem to accompany every period-doubling bifurcation, when we pass the critical value of the parameter (and the 'chaos' is born!). So far M. Rees has found an analogue for rational mappings of the Riemann sphere [**4**]. (Probably our proof can be modified to work in that case also.)

Let us say a few things about our notation. The Lebesgue measure of a set A is denoted $|A|$. Variables as subscripts mean differentiation. Occasionally we use prime for the derivative over x or when the parameter is the only variable. Sometimes we do not say explicitly that an object depends on the parameter. We also call a set an interval, where obviously the set has two components. For technical reasons we work with the family $f(x, \alpha) = 1 - \alpha x^2$ and interval $[-1, 1]$.

The author would like to apologize if some ideas are being used without references. It is difficult, though, to write a proof disjoint with the existing work.

We would like to express our thanks to the referee, whose work allowed us to eliminate many mistakes and other deficiencies of the first version.

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1. *Sketch of the proof*

For $\alpha=2$ the point $f^2(0)=-1$ is a repelling fixed point.

Let us define $I_n=[-3^{-n},3^{-n}]$ for $n=0,1,\ldots$. Let us fix $\rho\in(0,1)$ very close to 1. Let $V_n=f(\rho^{-1}I_n)$. It is easy to show (see Appendix) that $f^{n+1}|\rho^{-1}I_n\backslash\rho I_{n+1}$ is expanding with constant $\Lambda_0=\frac{4}{3}\rho$. It is also easy to show that the set $V_{n+1}\cup f(V_{n+1})\cup\cdots\cup f^n(V_{n+1})\subset\{x:|f'(x)|\geq 3\}$. Let $W_n=f^n(V_{n+1})$. We will show that $|W_n|\leq\rho^{-1}(\frac{4}{9})^{n-1}$, so $|W_n|\to 0$, as $n\to\infty$.

Let us define $T:I\to I$ by

$$T(x)=f^n(x), \qquad \text{as } x\in M_n\overset{\text{def}}{=}I_{n-1}\backslash I_n \tag{1.1}$$

for $n=1,2,\ldots$. This transformation is piecewise expanding. From [**5**] we know that T has an a.c.i.m. ν_0 with bounded density. We can easily verify that the measure

$$\nu=\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}f_*^k(\nu_0|M_n) \tag{1.2}$$

is an a.c.i.m. for f. It is finite, since

$$\nu(I)=\sum_{n=1}^{\infty}n\nu_0(M_n)\leq\text{const.}\sum_{n=1}^{\infty}n|M_n|<\infty. \tag{1.3}$$

We would like to point out that this measure is well known and has density const.$(1-x^2)^{-1/2}$. The construction we have just presented is a starting point to our construction of a.c.i.m. for $\alpha\neq 2$. It also produces a.c.i.m. for families like $f(x)=1-2x^2+\xi x^2(1-x^2)$, where ξ is a small parameter. The author does not know any explicit formula for the density of the a.c.i.m. in this case.

Let us start with the observation that given arbitrarily large $N\in\mathbb{Z}^+$ there is $\alpha_N<2$ such that $f^{n+1}|\rho^{-1}I_n\backslash\rho I_{n+1}$ is an expanding for $n\leq N$ and $\alpha\in[\alpha_N,2]$. For $\alpha=2$ the set W_n contains -1 for every n. For $\alpha\neq 2$ the set W_n approaches the critical point. The extreme case is when for some n we have $f^n(0)=0$. In this situation 0 attracts a.e. orbit of f and no a.c.i.m. exists. Our hope is that by varying α we can push W_n away from 0. Actually, we will put W_n into a set $\rho^{-1}I_k\backslash\rho I_{k+1}$ for some $k<n$.

Let us fix N sufficiently large and let $\alpha_N<2$ be such that for all $\alpha\in[\alpha_N,2]$ and $n=0,1,\ldots,N$ we have $W_n\subset\{x:|f'(x)\geq 3\}$. Let $T_{(n)}=f^{n+1}|\rho^{-1}I_n\backslash\rho I_{n+1}$ for $n=0,1,\ldots,N-1$ and let $S_{(n)}=f^n|V_n$ for $n=1,2,\ldots,N$. We have

$$T_{(n)}=S_{(n)}\circ f \quad\text{and}\quad S_{(n+1)}=T_{(0)}\circ S_{(n)} \tag{1.4}$$

for $n=0,1,\ldots,N-1$.

We will try to extend the definition of expandings $T_{(n)}$ and $S_{(n)}$ on $n>N$ (at least for some parameters). We are going to use induction.

Suppose that I_m, $S_{(m)}$ have been defined for $m\leq n$ $(n\geq N)$ and let $T_{(m)}=S_{(m)}\circ f$ for $m<n$. Let us fix $\tau\in(\frac{1}{2},1)$ to the solution of the equation $4^\tau=3$ and let $a=\frac{1}{3}$.

We define

$$I_{n+1}=[-a|S'_{(n)}(1)|^{-\tau}, a|S'_{(n)}(1)|^{-\tau}], \tag{1.5}$$
$$V_{n+1}=f(\rho^{-1}I_{n+1}).$$

Now we can define $T_{(n)}=S_{(n)}\circ f|\rho^{-1}I_n\backslash\rho I_{n+1}$ and $W_n=S_{(n)}(V_{n+1})$.

Suppose that $W_n\subset\rho^{-1}I_k\backslash\rho I_{k+1}$ for some $k=k(n)<n$. In this case we set

$$S_{(n+1)}=T_{(k)}\circ S_{(n)}|V_{n+1} \qquad (k=k(n)). \tag{1.6}$$

As we have already mentioned, for some parameters there is no k with the property $W_n\subset\rho^{-1}I_k\backslash I_{k+1}$. We need to discard those parameters to proceed with the next step of our induction. Let us describe in detail how we do it.

Let $(s_m)_{m=0}^n$ be a sequence of integers such that $T_{(m)}=f^{s_m}$ (on $\rho^{-1}I_m\backslash\rho I_{m+1}$). We impose an extra condition in our construction.

We fix a sufficiently small $\beta>0$ and require

$$s_{k(m)}\le\max(\beta m, 1) \qquad (m=0, 1, \ldots, n). \tag{1.7}$$

Along with I_n and $S_{(n)}$ we construct families of intervals $\mathcal{A}_n$ in the space of parameters. This is the corresponding inductive definition:

(i) $\mathcal{A}_N$ consists of a single interval $[\alpha_N, 2]$.

(ii) If $J\in\mathcal{A}_n$ then for $\alpha\in J$ all n steps of our construction work and yield the same sequence $(s_m)_{m=0}^{n-1}$. Let us consider intervals

$$J_k=\{\alpha\in J: S_{(n)}(1,\alpha)\in I_k\backslash I_{k+1}\}. \tag{1.8}$$

The family $\mathcal{A}_{n+1}$ consists of all intervals J_k for all $J\in\mathcal{A}_n$ and k such that $s_k\le\beta n$.

We will show that if $\alpha\in J_k$ then $W_n\subset\rho^{-1}I_k\backslash\rho I_{k+1}$. This is possible because the length of $|W_n|$ decays much faster than the length of $I_{k(n)}$.

The set of parameters $J\backslash\bigcup J_k$ is in general nonempty, but we will see that there is $\lambda\in(0,1)$ such that

$$J\backslash\bigcup J_k\subset\{\alpha\in J: \operatorname{dist}(S_{(n)}(1,\alpha), 0)\le a\lambda^{\sqrt{n}}\}. \tag{1.9}$$

Let $A_n=\bigcup\mathcal{A}_n$ and let B_n be the union of the sets on the right-hand side of (1.9). The crucial part of the proof is to show that $|B_n|\le\text{const.}\,\lambda^{\sqrt{n}}|A_N|$. Once we have obtained this estimate, we can write (in view of $A_n\backslash A_{n+1}\subset B_n$):

$$|A_n|\ge\left|A_N\Big\backslash\bigcup_{m=N}^{n-1}B_m\right|\ge|A_N|-\sum_{m=N}^{n-1}|B_n|\ge|A_N|\left(1-\text{const.}\sum_{m=N}^{n-1}\lambda^{\sqrt{m}}\right). \tag{1.10}$$

Let $A_\infty=\bigcap_{n=N}^{\infty}A_n$. Letting $n\to\infty$ in (1.10) we get

$$|A_\infty|\ge|A_N|\left(1-\text{const.}\sum_{m=N}^{\infty}\lambda^{\sqrt{m}}\right). \tag{1.11}$$

By fixing N sufficiently large we can make the ratio $|A_\infty|/|A_N|$ arbitrarily close to 1, in particular $|A_\infty|>0$.

From our estimates it easily follows that if $\alpha\in A_\infty$ then f has an a.c.i.m. First we construct a piecewise expanding $T(x)=f^{s_n}(x)$, as $x\in I_n\backslash I_{n+1}$ $(n=0, 1, \ldots)$ which has an a.c.i.m. ν_0 with bounded density. Formulas analogous to (1.2) and (1.3) yield an a.c.i.m. ν for f.

2. *Certain consequences of the Chain Rule*

Let u and v be functions of x and α, where α is a parameter. For example, $w = v \circ u$ means that $w(x, \alpha) = v(u(x, \alpha), \alpha)$. We can easily verify

LEMMA 2.1. *The following formulas hold:*

(i) $w_x = (v_x \circ u)u_x$;

(ii) $\dfrac{w_\alpha}{w_x} = \left(\dfrac{v_\alpha}{v_x} \circ u\right)\dfrac{1}{u_x} + \dfrac{u_\alpha}{u_x}$;

(iii) $\dfrac{w_{xx}}{w_x^2} = \dfrac{v_{xx}}{v_x^2} \circ u + \left(\dfrac{1}{v_x} \circ u\right)\dfrac{u_{xx}}{u_x^2}$;

(iv) $\dfrac{w_{\alpha x}}{w_x^2} = \left(\dfrac{v_{\alpha x}}{v_x^2} \circ u\right)\dfrac{1}{u_x} + \left(\dfrac{v_{xx}}{v_x^2} \circ u\right)\dfrac{u_\alpha}{u_x} + \left(\dfrac{1}{v_x} \circ u\right)\dfrac{u_{\alpha x}}{u_x^2}$;

(v) $\dfrac{w_{\alpha\alpha}}{w_x^2} = \left(\dfrac{v_{\alpha\alpha}}{v_x^2} \circ u\right)\dfrac{1}{u_x^2} + 2\left(\dfrac{v_{\alpha x}}{v_x^2} \circ u\right)\dfrac{u_\alpha}{u_x}\dfrac{1}{u_x} + \left(\dfrac{v_{xx}}{v_x^2} \circ u\right)\left(\dfrac{u_\alpha}{u_x}\right)^2 + \left(\dfrac{1}{v_x} \circ u\right)\dfrac{u_{\alpha\alpha}}{u_\alpha^2}$.

Let us introduce the following notation:

$$\begin{aligned}
&\Delta_{xx}(u) = \frac{|u_{xx}|}{|u_x|^2}, \quad \Delta_{\alpha x}(u) = \frac{|u_{\alpha x}|}{|u_x|^2}, \quad \Delta_{\alpha\alpha}(u) = \frac{|u_{\alpha\alpha}|}{|u_x|^2},\\
&\Delta(u) = \max\,(\Delta_{xx}(u), \Delta_{\alpha x}(u), \Delta_{\alpha\alpha}(u)),\\
&\delta(u) = \frac{u_\alpha}{u_x},\\
&R_{xx}(u) = 1,\\
&R_{\alpha x}(u) = \left|\frac{1}{u_x}\right| + \left|\frac{u_\alpha}{u_x}\right|,\\
&R_{\alpha\alpha}(u) = \frac{1}{|u_x|^2} + 2\left|\frac{u_\alpha}{u_x}\right|\frac{1}{|u_x|} + \left|\frac{u_\alpha}{u_x}\right|^2,\\
&R(u) = \max\,(R_{xx}(u), R_{\alpha x}(u), R_{\alpha\alpha}(u)).
\end{aligned} \tag{2.1}$$

LEMMA 2.2. *Let $w = v \circ u$. Then*

$$\Delta(w) \le \left(\frac{1}{|v_x|} \circ u\right)\Delta(u) + R(u)(\Delta(v) \circ u). \tag{2.2}$$

Moreover, this formula holds with $\Delta(w)$, $\Delta(u)$ and $R(u)$ (but not $\Delta(v)$) subscripted with 'xx', 'αx' or 'αα'.

Proof. By inspection of formulas (iii)-(v) of Lemma 2.1. □

Remark 2.1. If $|u_x| \ge 1$ then $R(u) \le 4\max\,(1, |\delta(u)|^2)$.

3. *Basic concepts*

Let f be a C^2-function of x and α. As a function of x, f is a transformation $I \to I$, where $I = [-1, 1]$. We will deal with transformations $T = f^d : \mathscr{D}(T) \to \mathscr{R}(T)$, where $\mathscr{D}(T)$ and $\mathscr{R}(T)$ are subintervals of I. The integer $d \ge 1$ is called the *degree* of T and denoted by $\deg(T)$.

Let $\Lambda_0 > 1$. We will call T *Λ_0-expanding*, if $|T'| \geq \Lambda_0$.

Definition 3.1. A sequence of expanding maps $(T_i)_{i=1}^n$ is called *β-homogeneous* ($\beta \geq 0$) if $\deg(T_i) \leq \max(\beta i, 1)$ for $i = 1, 2, \ldots, n$.

This notion proves useful because of the following:

LEMMA 3.1. *Let $(T_i)_{i=1}^n$ be a β-homogeneous sequence of Λ_0-expandings and let $S_i = T_i \circ T_{i-1} \circ \cdots \circ T_1$ for $i = 1, 2, \ldots, n$. Suppose that $\beta \leq 1$. Then*

$$|T_i'| \circ S_{i-1} \leq R_0 |S_{i-1}'|^\varepsilon, \tag{3.1}$$

where R_0 is a constant such that $\sup |f'| \leq R_0$ and $\varepsilon = \beta \log R_0 / \log \Lambda_0$.

Proof. $|T_i'| \leq R_0^{\deg(T_i)} \leq R^{\max(\beta i, 1)} \leq \max(R_0, R_0^{\beta i})$. Also, $R_0 |S_{i-1}|^\varepsilon \geq R_0 (\Lambda_0^{i-1})^\varepsilon = R_0 R_0^{\beta(i-1)} \geq R_0^{\beta i}$. □

Remark 3.1. (3.1) holds with T_i replaced by any T verifying $\deg(T) \leq \max(\beta i, 1)$.

Definition 3.2. A sequence of functions $(\varphi_i)_{i=1}^n$ is called (C, q)-stable ($C \geq 0$, $q \in (0, 1)$) if

$$|\varphi_i - \varphi_j| \leq C q^{\min(i,j)} \tag{3.2}$$

for $i, j = 1, 2, \ldots, n$.

Definition 3.3. (i) We say that T has *rank* (μ, A) if

$$|\delta(T)| \leq A |T_x|^\mu. \tag{3.3}$$

(ii) We say that T has *type* (σ, B) if

$$\Delta(T) \leq B |T_x|^\sigma. \tag{3.4}$$

THEOREM 3.1. *Suppose $(T_i)_{i=1}^n$ is a β-homogeneous sequence of Λ_0-expandings and T_i has rank (μ, A) for $i = 1, 2, \ldots, n$. Let $S_i = T_i \circ \cdots \circ T_1$ for $i \leq n$ and let $S = S_n$, $S_0 = \mathrm{id}$. There are constants C_1 and $q \in (0, 1)$ independent of A, μ or n such that for sufficiently small β we have:*

(i) $\delta(S) \leq C_1$ (*equivalently, S has rank* $(0, C_1)$).

(ii) *The sequence of functions $(\delta(S_i))_{i=0}^n$ is (C_1, q)-stable.*

Proof. (ii) From Lemma 2.1 we derive by induction the following formula ($i < j$)

$$\delta(S_j) = \sum_{l=i+1}^{j} (\delta(T_l) \circ S_{l-1})(S_{l-1}')^{-1} + \delta(S_i). \tag{3.5}$$

This gives

$$|\delta(S_j) - \delta(S_i)| \leq \sum_{l=i+1}^{j} (|\delta(T_l)| \circ S_{l-1}) |S_{l-1}'|^{-1}. \tag{3.6}$$

For $l \leq [\beta^{-1}]$ we have $|\delta(T_l)| \leq R_1$ where R_1 depends on f only (note: $\deg(T_l) = 1$). For $l > [\beta^{-1}]$ we have

$$|\delta(T_l)| \circ S_{l-1} \leq A |T_l'|^\mu \circ S_{l-1} \leq A R_0^\mu |S_{l-1}'|^{\varepsilon\mu}$$

by Lemma 3.1. We always have $|S_{l-1}'| \geq \Lambda_0^{l-1}$. Therefore,

$$|\delta(S_j) - \delta(S_i)| \leq \sum_{1+i \leq l \leq [\beta^{-1}]} R_1 \Lambda_0^{-(l-1)} + \sum_{[\beta^{-1}] < l < j} R_0^\mu A \Lambda_0^{-(1-\varepsilon\mu)(l-1)}. \tag{3.7}$$

Suppose $\varepsilon\mu < \frac{1}{3}$ (i.e. $\beta < \log \Lambda_0 / 3\mu \log R_0$). Let $q = \Lambda_0^{-\frac{1}{3}}$ and let β be such that

$R_0^\mu A \leq R_1 q^{-[\beta^{-1}]}$. The right-hand side of (3.7) does not exceed

$$\sum_{l=i+1}^{j} R_1 q^{l-1} \leq R_1(1-q)^{-1} q^i$$

(the terms of the first sum are obviously $\leq R_1 q^{l-1}$ and the second sum $\leq R_1 q^{2(l-1)-[\beta^{-1}]} \leq R_1 q^{l-1}$, since $l-1 \geq [\beta^{-1}]$). Therefore, the Lemma holds with $C_1 = R_1(1-q)^{-1}$. We notice that (i) can be easily obtained from (ii), since $\delta(S_0)=0$. □

THEOREM 3.2. *Let ν be an arbitrary positive number. Suppose that the assumptions of Theorem 3.1 are satisfied and, in addition, T_i has type (σ, B) for $i=1,2,\ldots,n$. There is a constant C_2 independent of A, B, μ, ν, σ or n such that if β is sufficiently small then S has type (ν, C_2).*

Proof. Lemma 2.2, Remark 2.1 and Theorem 3.1(i) give

$$\Delta(S_n) \leq \Lambda_0^{-1}\Delta(S_{n-1}) + C_3(\Delta(T_n)\circ S_{n-1}), \tag{3.8}$$

where $C_3 = 4\max(1, C_1^2)$. Therefore, by induction we get

$$\Delta(S_n) \leq C_3 \sum_{i=1}^{n} \Lambda_0^{-(n-i)} \Delta(T_i)\circ S_{i-1} \leq C_3(1-\Lambda_0^{-1})^{-1} \max_{1\leq i\leq n} \Delta(T_i)\circ S_{i-1}. \tag{3.9}$$

We can assume that for $i \leq [\beta^{-1}]$ we have $\Delta(T_i) \leq R_2$ and for $i > [\beta^{-1}]$ we have $\Delta(T_i)\circ S_{i-1} \leq B|T_i'|^\sigma \circ S_{i-1} \leq R_0^\sigma B|S'|^{\varepsilon\sigma}$. Therefore,

$$\Delta(S) \leq \frac{C_3}{1-\Lambda_0^{-1}} \max(R_2, R_0^\sigma B|S'|^{\varepsilon\sigma}). \tag{3.10}$$

Suppose that $\varepsilon\sigma < \nu/2$ (i.e. $\beta < \nu \log \Lambda_0 / 2\sigma \log R_0$) and β is such that $R_2 \Lambda_0^{[\beta^{-1}]\nu/2} \geq R_0^\sigma B$. Then

$$R_0^\sigma B|S'|^{\varepsilon\sigma} \leq R_0^\sigma B|S'|^{\nu/2} \leq (R_0^\sigma B/|S'|^{\nu/2})|S'|^\nu.$$

We also have

$$R_0^\sigma B/|S'|^{\nu/2} \leq R_0^\sigma B \Lambda_0^{-n\nu/2} \leq R_0^\sigma B \Lambda_0^{-[\beta^{-1}]\nu/2} \leq R_2.$$

Therefore

$$\Delta(S) \leq \frac{C_3}{1-\Lambda_0^{-1}} R_2 |S'|^\nu$$

and we can set C_2 to $R_2 C_3/(1-\Lambda_0^{-1})$. □

Definition 3.4. Let $V \subset \mathscr{D}(T)$. The number $\sup_V |T'|/\inf_V |T'|$ is called the *distortion* of T on V. We omit V, if $V = I$.

The next lemma is well known in the theory of expandings.

LEMMA 3.2. *Let $S = T_n \circ \cdots \circ T_1$, where T_i is a Λ_0-expanding for $i=1,2,\ldots,n$ and $\deg(T_i)=1$. The distortion of S does not exceed $C_4 = \exp(2R_2(1-\Lambda_0^{-1})^{-1})$.*

Proof. In a similar way as in the proof of Theorem 3.2 we get $|S''/(S')^2| \leq C_5 = R_2(1-\Lambda_0^{-1})^{-1}$. Therefore, for $y, z \in \mathscr{D}(S)$ we have

$$\left|\ln\frac{S'(y)}{S'(z)}\right| \leq \int_{[y,z]} \left|\frac{S''}{S'}\right| \leq C_5 \int_{[y,z]} |S'| = C_5|S(y)-S(z)| \leq 2C_5. \tag{3.11}$$

This gives $S'(y)/S'(z) \leq \exp(2C_5) = C_4$ (note: $S'(y)$, $S'(z)$ have the same sign). □

Possessing type (σ, B) does not imply bounded distortion on the whole of I, though the distortion is bounded on sufficiently small intervals.

THEOREM 3.3. *Let S be an expanding type (ν, C_2). For every $\theta_0 > 1$ there is $\eta_0 > 0$ s.t. if for some interval $V \subset \mathscr{D}(S)$ and $y_0 \in V$ we have $|V| \leq \eta_0 |S'(y_0)|^{-(1+\nu)}$ then the distortion of S on V is bounded by θ_0.*

Proof. Let $y \in V$. We can assume that $S' \geq 0$. We have

$$\left|\frac{1}{S'(y)^{1+\nu}} - \frac{1}{S'(y_0)^{1+\nu}}\right| \leq (1+\nu)\int_{[y,y_0]} \left|\frac{S_{xx}}{S_x^{2+\nu}}\right|$$
$$\leq (1+\nu)C_2|V| \leq (1+\nu)\eta_0 C_2 |S'(y_0)|^{-(1+\nu)}. \tag{3.12}$$

Multiplying (3.13) by $|S'(y_0)|^{1+\nu}$ yields

$$\left|\left(\frac{S'(y_0)}{S'(y)}\right)^{1+\nu} - 1\right| \leq (1+\nu)\eta_0 C_2. \tag{3.13}$$

Now it's clear that the distortion is arbitrarily close to 1, if η_0 is sufficiently small. □

Now let us go back to the construction of § 1. We will apply our results to the β-homogeneous sequence $T_i = T_{(k(i))}$ $(i = 1, 2, \ldots, n)$ and $\alpha \in J \in \mathscr{A}_n$. Obviously, $S_i = T_i \circ \cdots \circ T_1 = S_{(i)}$. From now on we will often write S_n instead of $S_{(n)}$, which is consistent with the formula and simplifies our notation.

THEOREM 3.4. *Suppose that $S = S_n$ has rank $(0, C_1)$, type (ν, C_2) and distortion $\leq C_4$. Let $T = T_{(n)} = S \circ f | \rho^{-1} I_n \backslash \rho I_{n+1}$. There exist constants C_7, A, B, μ, σ independent of n such that T has rank (μ, A) and type (σ, B). Moreover, $|T'| \geq C_7 |S'(1)|^{1-\tau}$.*

Proof. Let us fix C_0 such that

$$\begin{gathered} C_0^{-1}|x| \leq f_x| \leq C_0|x|, \\ \max(R(f), \Delta(f)) \leq C_0|x|^{-2}, \\ |\delta(f)| \leq C_0. \end{gathered} \tag{3.14}$$

We have $|T'| = (|S'| \circ f)|f'| \geq C_4^{-1}|S'(1)|C_0^{-1}|x| \geq C_4^{-1}C_0^{-1}a|S'(1)|^{1-\tau}$ (see (1.5)). We set $C_7 = C_4^{-1}C_0^{-1}a$. We notice that

$$|S'(1)| \leq C_8 |T'|^{1/(1-\tau)}, \tag{3.15}$$

where $C_8 = C_7^{-1/(1-\tau)}$.

Lemma 2.2 gives

$$\begin{aligned} \Delta(T) &\leq (|S'|^{-1} \circ f)\Delta(f) + (\Delta(S) \circ f) \cdot R(f) \\ &\leq \max(\Delta(f), R(f)) \times (1 + \Delta(S) \circ f) \\ &\leq C_0|x|^{-2}(1 + C_2(C_4|S'(1)|)^\nu) \\ &\leq C_0(1 + C_2C_4^\nu)a^{-2}|S'(1)|^{2\tau+\nu} \\ &\leq C_0(1 + C_2C_4^\nu)a^{-2}C_8^{2\tau+\nu}|T'|^{(2\tau+\nu)/(1-\tau)}. \end{aligned}$$

This yields $\sigma = (2\tau + \nu)/(1-\tau)$ and $B = C_0(1 + C_2C_4^\nu)a^{-2}C_8^{2\tau+\nu}$. We have

$$\begin{aligned} |\delta(T)| &= |(\delta(S) \circ f)f_x^{-1} + \delta(f)| \\ &\leq C_1 \cdot C_0|x|^{-1} + C_0 \leq a^{-1}C_1C_0|S'(1)|^\tau + C_0 \\ &\leq (1 + a^{-1}C_1)C_0C_8^\tau|T'|^{\tau/(1-\tau)}. \end{aligned}$$

This yields $A = (1 + a^{-1}C_1)C_0C_8^\tau$, $\mu = \tau/(1-\tau)$. □

Remark 3.2. We can apply the method of the proof to $\Delta_{xx}(T)$, $\Delta_{x\alpha}(T)$, $\Delta_{\alpha\alpha}(T)$ separately. We obtain useful estimates

$$\begin{aligned}\Delta_{xx}(T) &\le C_9|S'(1)|^{\max(\nu,2\tau-1)}\\ \Delta_{\alpha x}(T) &\le C_9|S'(1)|^{\nu+\tau}\\ \Delta_{\alpha\alpha}(T) &\le C_9|S'(1)|^{\nu+2\tau}.\end{aligned} \tag{3.18}$$

From now on we assume $\nu<2\tau-1$, which reduces the first exponent to $2\tau-1$.

Remark 3.3. There exists $\theta_0>1$ such that for sufficiently large N, $\alpha\in[\alpha_N,2]$ and all $n\ge 1$ we have $|S'_n(1)|\ge\theta_0^n$ and $|T'_{(n)}|\ge\max(\theta_0, C_7\theta_0^{n(1-\tau)})$. In particular, $\inf|T'_{(n)}|$ grows exponentially.

Proof. Let us pick $\theta_0\in(1,\frac{4}{3}\rho)$ arbitrarily and let N be large enough, so that for all $\alpha\in A_N=[\alpha_N,2]$ we have $|T'_{(n)}|\ge\theta_0$ for all n satisfying $(*)$ $C_7\theta_0^{n(1-\tau)}<\theta_0$. The number of such n is bounded, so this is possible by the Appendix and C^1-continuity of our family (since $\alpha_N\to 2$, as $N\to\infty$).

We will show by induction that

(i) $|S'_{(n)}(1)|\ge\theta_0^n$,

(ii) $|T'_{(k)}|\ge\theta_0$ for all $k<n$.

This is obvious for $n=1$. Using the induction hypothesis and Theorem 3.4, we get $|T'_{(n)}|\ge C_7\theta_0^{n(1-\tau)}$, which is $\ge\theta_0$, if n does not satisfy $(*)$. If n does satisfy $(*)$ then (ii) is obvious for $k=n$. Now we apply the definition $S_{(n+1)}=T_{(k)}\circ S_{(n)}(k<n)$ and get $|S'_{n+1}(1)|\ge\theta_0^{n+1}$, which completes the proof. □

COROLLARY 3.1. *There exists constants C_1, C_2, C_4, A, B, Λ_0, ν, μ (all positive and $\Lambda_0>1$) such that for N sufficiently large and β sufficiently small and for every $n\ge1$ and $\alpha\in A_{\max(n,N)}$ the mapping $S_{(n)}$ has rank $(0, C_1)$, type (ν, C_2), distortion $\le C_4$ and is Λ_0^n-expanding, and $T_{(n)}$ has rank (μ, A), type (σ,β) and is Λ_0-expanding.*

Proof. Induction. Suppose that $T_{(k)}$ has rank (μ, A) and type (σ, B) for all $k<n$. Theorems 3.1, 3.2 (applied to the sequence $T_i=T_{(k(i))}$) and Remark 3.3 imply that $S_{(n)}$ has rank $(0, C_1)$ and type (ν, C_2). If $n\le N$ then Lemma 3.2 implies that the distortion of $S_{(n)}$ does not exceed C_4.

Let us notice that $\mathscr{D}(S_{(n)})=V_n$ and $|V_n|\le\frac{1}{2}C_0\rho^{-2}|I_n|^2\le\frac{1}{2}C_0\rho^{-2}(2a)^2|S'_{(n)}|^{-2\tau}$. This means that Theorem 3.3 applies to $S_{(n)}$ for $n\ge N$ with $\theta_0=C_4$, if N is large enough (note: $2\tau>1+\nu$).

Eventually from Theorem 3.4 we get that $T_{(n)}$ has rank (μ, A) and type (σ, B). This ends the proof. □

4. *Families with a prerepelling critical point*

Here we consider families a little more general than $f(x,\alpha)=1-\alpha x^2$.

Suppose that f has a critical point $c(\alpha)$ for α close to α_0 and suppose that for $\alpha=\alpha_0$ and some $m\in\mathbb{Z}^+$ the point $f^m(c(\alpha_0),\alpha_0)=x_0$ is a repelling point of period κ. We consider the following nondegeneracy condition (cf. [4]). Let $x(\alpha)=f^m(c(\alpha))$ be a differentiable function and let

$$\left.\frac{d}{d\alpha}\right|_{\alpha=\alpha_0}(f^\kappa(x(\alpha))-x(\alpha))\ne 0.$$

Let us define a sequence of functions

$$\chi_n(\alpha)=\left(\frac{d}{d\alpha}f^n(x(\alpha),\alpha)\right)\Big/(f^n)_x(x(\alpha),\alpha). \tag{4.2}$$

PROPOSITION 4.1. *The limit* $\chi=\lim_{n\to\infty}\chi_n(\alpha_0)$ *exists. It is* $\neq 0$ *iff* (4.1) *is satisfied.*

Proof. Let $T=f^\kappa$. Let $n=l\kappa+l_1$, where $0\le l_1\le\kappa-1$. We have by Lemma 2.1(ii):

$$\delta(f^n)=\sum_{i=1}^{l}(\delta(T)\circ T^{i-1})\frac{1}{(T^{i-1})'}+\frac{1}{(T^l)'}\left(\sum_{j=1}^{l_1}(\delta(f)\circ f^{j-1})\cdot\frac{1}{(f^{j-1})'}\right)\circ T^l. \tag{4.3}$$

Substituting (x_0,α_0) and letting $l\to\infty$ yields:

$$\lim_{n\to\infty}\delta(f^n)(x_0,\alpha_0)=\delta(T)(x_0,\alpha_0)(1-T'(x_0)^{-1})^{-1}=\bar{\chi}. \tag{4.4}$$

Condition (4.1) is equivalent to

$$(T_x(x_0,\alpha_0)-1)x'(\alpha_0)\neq -T_\alpha(x_0,\alpha_0) \tag{4.5}$$

or $x'(\alpha_0)\neq-\bar{\chi}$. On the other hand,

$$\begin{aligned}\frac{d}{d\alpha}\bigg|_{\alpha=\alpha_0}&f^n(x(\alpha),\alpha)/(f^n)_x(x_0,\alpha_0)\\&=x'(\alpha_0)+\delta(f^n)(x_0,\alpha_0)\to x'(\alpha_0)+\bar{\chi}=\chi,\qquad\text{as } n\to\infty.\end{aligned} \tag{4.6}$$

This completes the proof. □

THEOREM 4.1. *Let* $\chi\neq0$. *For every* $\eta>0$ *there is* $n_0\in\mathbb{Z}^+$ *and* $\delta>0$ *such that for every* $n\ge n_0$ *and* $\alpha\in(\alpha_0-\delta,\alpha_0+\delta)$: *if for some interval* $U\subset I$, $f(\cdot,\alpha)|U$ *is* Λ_0*-expanding and* $\{x(\alpha),f(x(\alpha),\alpha),\dots,f^n(x(\alpha),\alpha)\}\subset U$ *then*

$$|\chi_n(\alpha)/\chi-1|<\eta. \tag{4.7}$$

Proof. First, we choose n_0 large enough, so that

$$\begin{gathered}|\chi_{n_0}(\alpha_0)-\chi|<\tfrac14\eta|\chi|\\|\delta(f^{n_0})(x,\alpha)-\delta(f^n)(x,\alpha)|<C_1q^{n_0}<\tfrac14\eta|\chi|\end{gathered} \tag{4.8}$$

(we applied Theorem 3.1).

We pick δ such that if $|\alpha-\alpha_0|<\delta$ then

$$\begin{gathered}|\delta(f^{n_0})(\chi,\alpha)-\delta(f^{n_0})(x_0,\alpha_0)|<\tfrac14\eta|\chi|\\|x'(\alpha)-x'(\alpha_0)|<\tfrac14\eta|\chi|.\end{gathered} \tag{4.9}$$

We get Theorem 4.1 by the triangle inequality. □

Remark 4.1. We will not dwell on the general case, leaving the details to the reader. We notice that in the case of $f(\alpha,x)=1-\alpha x^2$ we have $\chi_n(\alpha)=-2\alpha\delta(S_{(n+1)})(1,\alpha)$. It is easy to see that condition (4.1) holds for this family.

COROLLARY 4.1. *Let* $D=|\chi/(2\alpha_0)|$. *For every* $\theta>1$ *we can choose sufficiently large* n_1 *such that for every* $n\ge n_1$ *and every* α, α_1, $\alpha_2\in A_n$

$$\begin{gathered}h\le|\delta(S_{(n)})(1,\alpha)|\le H.\\\theta^{-1}\le\frac{\delta(S_{(n)})(1,\alpha_1)}{\delta(S_{(n)})(1,\alpha_2)}\le\theta,\end{gathered} \tag{4.10}$$

where $h=\theta^{-1}D$, $H=\theta D$ (*see Remark* 4.1 *and Theorem* 3.1(ii)).

Let us finish this section with one more distortion estimate, this time over α.

LEMMA 4.1. *For every $\theta_1 > 1$ there is $\eta_1 > 0$ such that if $|\alpha_2 - \alpha_1| \le \eta_1 |S_n(1, \alpha_1)|^{-(1+\nu)}$, $\alpha_1, \alpha_2 \in A_n$, then*

$$\theta_1^{-1} \le \frac{(S_n)_\alpha(1, \alpha_2)}{(S_n)_\alpha(1, \alpha_1)} \le \theta_1.$$

Proof. We notice that

$$\left|\frac{(S_n)_{\alpha\alpha}}{(S_n)_\alpha^{2+\nu}}\right| = |\delta(S_n)|^{-(2+\nu)} \left|\frac{(S_n)_{\alpha\alpha}}{(S_n)_x^{2+\nu}}\right| \le h^{-(2+\nu)} C_2. \tag{4.11}$$

Integrating over α, we get

$$\begin{aligned}\left|\frac{1}{(S_n)_\alpha^{1+\nu}(\alpha_2)} - \frac{1}{(S_n)_\alpha^{1+\nu}(\alpha_1)}\right| &\le (1+\nu) h^{-(2+\nu)} C_2 |\alpha_2 - \alpha_1| \\ &\le (1+\nu) h^{-(2+\nu)} C_2 \eta_1 |S_n(1, \alpha_1)|^{-(1+\nu)}.\end{aligned}$$

Multiplying by $|(S_n)(1, \alpha_1)|^{1+\nu}$ leads to:

$$\left|\left(\frac{(S_n)_\alpha(1, \alpha_1)}{(S_n)_\alpha(1, \alpha_2)}\right)^{1+\nu} - 1\right| \le (1+\nu) h^{-(2+\nu)} C_2 \eta_1. \tag{4.12}$$

If η_1 is sufficiently small, (4.12) yields the lemma. □

COROLLARY 4.2. *Using the assumptions and notation of Lemma* 4.1 *we have*

$$\theta_1^{-1} \theta^{-1} \le \frac{(S_n)_x(1, \alpha_2)}{(S_n)_x(1, \alpha_1)} \le \theta_1 \theta. \tag{4.13}$$

Remark 4.2. The sets J_k have not more components than $I_k \backslash I_{k+1}$, since by simple differentiation we can see that the absolute value of the derivative of $S_{(n)}(1, \alpha)$ over α is > absolute value of the derivative of the ends of $I_k \backslash I_{k+1}$.

Proof. Indeed, we ask if

$$\begin{aligned} a\left|\frac{d}{d\alpha} |S_k'(1, \alpha)|^{-\tau}\right| &= a\tau |(S_k)_{\alpha x}(1, \alpha)| \cdot |S_k'(1, \alpha)|^{-(\tau+1)} \\ &< |(S_n)_\alpha(1, \alpha)|. \end{aligned} \tag{4.14}$$

This is equivalent to

$$a\Delta_{\alpha x}(S_k) \cdot \tau |S_k'|^{-(\tau-1)} < |\delta(S_n)| \, |S_n'| \tag{4.15}$$

and because of $\Delta_{\alpha x}(S_k) \le C_2 |S_k'|^\nu$ we need

$$a\tau C_2 |S_k'|^{\nu - \tau + 1} < |S_n'| \, |\delta(S_n)|. \tag{4.16}$$

We have $|S_k'| \le |S_n'|$. Also $|\delta(S_n)| \ge h > 0$ for large n. Therefore (4.16) holds for large n, since $\nu < \tau$ (note: $\nu < 2\tau - 1 < \tau$).

5. *The measure of A_∞*

We use the notation of § 1.

PROPOSITION 5.1. (i) *Suppose that $N \ge 2\beta^{-1}$. The numbers s_m, $m = N, N+1, \ldots, n$ satisfy the inequality*

$$s_m \le \frac{\beta}{2} m^2 + 1. \tag{5.1}$$

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(ii) *If* $\alpha \in J_k \in \mathcal{A}_{n+1}$ *then* $W_n \subset \rho^{-1} I_k \backslash \rho I_{k+1}$.
(iii) *There is* $\lambda \in (0, 1)$ *such that* (1.9) *is satisfied.*

Proof. (i) Induction. We notice that $s_N = N+1 \leq (\beta/2)N^2+1$, if $N \geq 2/\beta$. The induction step:

$$s_{n+1} = s_n + s_{k(n)} \leq \frac{\beta}{2} n^2 + 1 + \beta n \leq \frac{\beta}{2}(n+1)^2 + 1.$$

(ii) We notice that since $S_{(n)}(1) \in I_k \backslash I_{k+1}$:

$$\begin{aligned} &\operatorname{dist}(S_{(n)}(1), \rho I_{k+1}) \geq \tfrac{1}{2}(1-\rho)|I_{k+1}| \\ &\operatorname{dist}(S_{(n)}(1), \rho^{-1} I_k) \geq \tfrac{1}{2}(\rho^{-1}-1)|I_k|. \end{aligned} \tag{5.2}$$

It suffices to show that

$$\begin{aligned} 2|W_n| &\leq \min((1-\rho)|I_{k+1}|, (\rho^{-1}-1)|I_k|) \\ &= (1-\rho)|I_{k+1}|. \end{aligned} \tag{5.3}$$

Obviously, we have

$$\begin{aligned} |W_n| &\leq \sup |S'_{(n)}| \cdot |V_{n+1}| \leq C_4 |S'_{(n)}(1)| \cdot \tfrac{1}{2}\rho^{-2} C_0 |I_{n+1}|^2 \\ &= \tfrac{1}{2} C_0 \rho^{-2} C_4 a^{-2} |S'_{(n)}(1)|^{1-2\tau} \leq \tfrac{1}{2} C_0 \rho^{-2} C_4 a^{-2} \Lambda_0^{n(1-2\tau)}. \end{aligned} \tag{5.4}$$

On the other hand, $|I_{k+1}| = 2a|S'_{(k)}(1)|^{-\tau} \geq 2aR_0^{-\tau s_k}$, since $\deg(S_{(k)}) = s_k - 1 \leq s_k$. Because $s_k \leq \beta n$ for $n \geq N$, we get $R_0^{-\tau s_k} \geq R_0^{-\tau\beta n} = \Lambda_0^{-\varepsilon\tau n}$. Suppose that $\varepsilon\tau < 2\tau - 1$. For sufficiently large n we have

$$2a(1-\rho)\Lambda_0^{-\varepsilon n\tau} \geq C_0 \rho^{-2} C_4 a^{-2} \Lambda_0^{-n(2\tau-1)} \tag{5.5}$$

and (ii) holds if N is sufficiently large.

(iii) Suppose that $S_{(n)}(1, \alpha) \in I_k$ and $\deg(T_{(k)}) > \beta n$. Then

$$\operatorname{dist}(S_{(n)}(1, \alpha), 0) \leq \tfrac{1}{2}|I_k| = a|S'_{(k)}(1)|^{-\tau} \leq a\Lambda_0^{-k\tau}.$$

From (i) we now have that $\deg(T_{(k)}) = s_k \leq (\beta/2)k^2+1$. Hence, $\beta n < (\beta/2)k^2+1$ or $k > \sqrt{2n - 2/\beta} \geq \sqrt{n}$, since $n \geq N \geq 2/\beta$. This yields (iii) with $\lambda = \Lambda_0^{-\tau}$. □

Let us consider the transformation ψ_n defined on every $J \in \mathcal{A}_n$ by $\psi_n(\alpha) = S_n(1, \alpha)$. We assume $N \geq 2\beta^{-1}$.

THEOREM 5.1. *There is a constant* C_{10} *such that for every* $n \geq N$

$$\sum_{J \in \mathcal{A}_n} \sup_J \frac{1}{|\psi'_n|} \leq C_{10}|A_N|. \tag{5.6}$$

Proof. Let π be the partition of I into intervals of equal length d ($d^{-1} \in \mathbb{Z}$). Let $\mathcal{A}'_n$ be the family of intervals defined in a similar way to $\mathcal{A}_n$, except we replace the definition of J_k with

$$J_k(P) = \{\alpha \in J : S_n(1, \alpha) \in (I_k \backslash I_{k+1}) \cap P\}, \tag{5.7}$$

where $P \in \pi$. Since every $J \in \mathcal{A}_n$ is a union of elements of $\mathcal{A}'_n$, it is sufficient to prove (5.6) for $\mathcal{A}'_n$ instead of $\mathcal{A}_n$. We will be able to do it, if d is sufficiently small.

Let us introduce two sequences

$$\begin{aligned}\gamma_n &= \sum_{J\in\mathcal{A}'_n} \sup_J \frac{1}{|\psi'_n|}\\ \eta_n &= \sum_{J\in\mathcal{A}'_n} \sup_J \left|\frac{\psi''_n}{(\psi'_n)^2}\right| |J|.\end{aligned} \tag{5.8}$$

We are going to derive estimates of $(\gamma_{n+1}, \eta_{n+1})$ in terms of (γ_n, η_n).

We fix an arbitrary $\theta > 1$. First, by Corollary 4.1 we have for n large enough

$$\begin{aligned}\sup_{J_k(P)} \frac{1}{|\psi'_{n+1}|} &\le \frac{1}{h} \sup_{J_k(P)} \frac{1}{|S'_{n+1}(1)|}\\ &\le \frac{1}{h} \sup \frac{1}{|T'_{(k)}|} \sup_{J_k(P)} \frac{1}{|S'_n(1)|}\\ &\le \frac{1}{h} \sup \frac{1}{|T'_{(k)}|} H \sup_{J_k(P)} \frac{1}{|\psi'_n|}\\ &= \theta^2 \sup |T'_{(k)}|^{-1} \sup_{J_k(P)} |\psi'_n|^{-1}.\end{aligned} \tag{5.9}$$

The first supremum is over $\alpha \in J_k(P)$ and $x \in P \cap (I_k \backslash I_{k+1})$ (note: I_k depends on α). There is $k_1 \in \mathbb{Z}^+$ such that

$$\sum_{k\ge k_1} \sup |T'_{(k)}|^{-1} \le \Lambda_0^{-1} \tag{5.10}$$

(see Remark 3.3).

Let us choose d small enough, so that for every $P \in \pi$ one of the two alternatives holds: either $(1^0) P \subset I_{k_1}$ or $(2^0) P$ intersects not more than two of the sets $I_k \backslash I_{k+1}$.

We get:

$$\sum_k \sup_{J_k(P)} \frac{1}{|\psi'_{n+1}|} \le 2\theta^2 \Lambda_0^{-1} \sup_{J(P)} \frac{1}{|\psi'_n|}, \tag{5.11}$$

where $J(P) = J \cap \psi_n^{-1}(P)$.

We notice that $\psi_n(J(P)) \ne P$ for at most two $P \in \pi$, namely those for which $(*) P \cap \partial\psi_n(J) \ne \varnothing$. Summing up over these P, we get

$$\sum_{k,P}{}^{*} \sup_{J_k(P)} \frac{1}{|\psi'_{n+1}|} \le 4\theta^2 \Lambda_0^{-1} \sup_{\alpha\in J} \frac{1}{|\psi'_n|}. \tag{5.12}$$

For those P that $(**) \psi_n(J(P)) = P$ we have

$$\sup_{J(P)} \frac{1}{|\psi'_n|} \le \frac{1}{\psi_n(J(P))} \int_{\psi_n(J(P))} \frac{1}{|\psi'_n|} \circ \psi_n^{-1} + \sup_{\alpha\in J(P)} \left|\frac{\psi''_n}{(\psi'_n)^2}\right| |J(P)|, \tag{5.13}$$

according to the rule 'maximum $\le$ average + max. of derivative $\times$ length of the interval' applied to $(1/|\psi'_n|) \circ \psi_n^{-1}$ on $P = \psi_n(J(P))$; we notice that the average is just $d^{-1}|J(P)|$. Totalling over P satisfying $(**)$ we get

$$\sum_{P,k}{}^{**} \sup_{J_k(P)} \frac{1}{|\psi'_{n+1}|} \le 2\theta^2 \Lambda_0^{-1} \left(\frac{1}{d} |J| + \sup_J \left|\frac{\psi''_n}{(\psi'_n)^2}\right| \cdot |J| \right). \tag{5.14}$$

Inequalities (5.12) and (5.14) and summation over J yield

$$\gamma_{n+1} \le 4\theta^2 \Lambda_0^{-1} \gamma_n + 2\theta^2 \Lambda_0^{-1} \eta_n + \frac{2\theta^2 \Lambda_0^{-1}}{d} |A_N|. \tag{5.15}$$

Now we intend to estimate η_{n+1}. We have $\psi_{n+1}(\alpha)=T_{(k)}(\psi_n(\alpha),\alpha)$ and this yields

$$\begin{aligned}\psi''_{n+1}(\alpha)=(T_{(k)})_{\alpha\alpha}(\psi_n(\alpha),\alpha)\\&+2(T_{(k)})_{\alpha x}(\psi_n(\alpha),\alpha)\psi'_n(\alpha)\\&+(T_{(k)xx}(\psi_n(\alpha),\alpha)(\psi'_n(\alpha))^2\\&+(T_{(k)})_x(\psi_n(\alpha),\alpha)\psi''_n(\alpha).\end{aligned}\tag{5.16}$$

We also have $\psi'_n(\alpha)=(S_n)_\alpha(1,\alpha)$. Therefore

$$\begin{aligned}\frac{|\psi''_{n+1}|}{(\psi'_{n+1})^2}\le \delta(S_{n+1})^{-2}\Bigg[\Delta_{xx}(T_{(k)})\delta(S_n)^2\\&+\frac{2}{|S'_n|}\Delta_{\alpha x}(T_{(k)})|\delta(S_n)|+\frac{1}{(S'_n)^2}\Delta_{\alpha\alpha}(T_{(k)})\Bigg]\\&+\frac{1}{|(T_{(k)})_x|}\,\frac{\delta(S_n)^2}{\delta(S_{n+1})^2}\,\frac{|\psi''_n|}{(\psi'_n)^2},\end{aligned}\tag{5.17}$$

where we omitted the arguments for simplicity. This yields

$$\left|\frac{\psi''_{n+1}}{(\psi'_{n+1})^2}\right|\le \Phi_n(T_{(k)})+\Lambda_0^{-1}\theta^2\left|\frac{\psi''_n}{(\psi'_n)^2}\right|\tag{5.18}$$

where θ^2 comes from Corollary 4.1 and

$$\Phi_n(T_{(k)})=\delta(S_{n+1})^{-2}\left[\Delta_{xx}(T_{(k)})\delta(S_n)^2+\frac{2}{|S'_n|}\Delta_{\alpha x}(T_{(k)})\delta(S_n)+\frac{1}{(S'_n)^2}\Delta_{\alpha\alpha}(T_{(k)})\right].\tag{5.19}$$

Therefore, we can write

$$\eta_{n+1}=\sum_{J,k,P}\sup_{J_k(P)}\left|\frac{\psi''_{n+1}}{(\psi'_{n+1})^2}\right||J_k(P)|\le\sum_{J,k,P}\Phi_n(T_{(k)})|J_k(P)|+\Lambda_0^{-1}\theta^2\eta_n\tag{5.20}$$

Because of the obvious inequality

$$|J_k(P)|\le\sup_{J_k(P)}\frac{1}{|\psi'_n|}\cdot\left|P\cap\bigcup_{\alpha\in J}(I_k\backslash I_{k+1})\right|\tag{5.21}$$

we get

$$\eta_{n+1}\le r\cdot\gamma_n+\Lambda_0^{-1}\theta^2\eta_n\tag{5.22}$$

where

$$r=\sup_n\max_{P\in\pi}\sum_k\sup_{\alpha,x}\Phi_n(T_{(k)})\left|P\cap\bigcup_\alpha(I_k\backslash I_{k+1})\right|.\tag{5.23}$$

We shall soon see that r is finite and even arbitrarily small. Inequalities (5.15) and (5.22) imply the theorem, if the eigenvalues of the matrix

$$\mathbf{P}=\begin{bmatrix}4\theta^2\Lambda_0^{-1} & 2\theta^2\Lambda_0^{-1}\\ r & \theta^2\Lambda_0^{-1}\end{bmatrix}\tag{5.24}$$

have modulus <1. As a matter of fact, we can write (5.15) and (5.22) as a single matrix inequality

$$\boldsymbol{\xi}_{n+1}\le\mathbf{P}\boldsymbol{\xi}_n+\mathbf{c}\qquad(n\le N),\tag{5.25}$$

where $\boldsymbol{\xi}_n=(\gamma_n,\eta_n)$. Here $\mathbf{c}$ is a constant vector and $\|\mathbf{c}\|\le\text{const.}\,|A_N|$. Here and in what follows, const. means an unspecified constant independent of N. Also we can easily see that $\|\xi_N\| \le \text{const.}\,|A_N|$. In fact, it is sufficient to show that $\eta_N \le \text{const.}\,|A_N|$. The inequality $\gamma_N \le \text{const.}\,|A_N|$ will follow by the same principle we used to prove (5.13). First, we notice that

$$\eta_N \le \sup_{A_N} \left|\frac{\psi_N''}{(\psi_N')^2}\right| |A_N|$$

(by definition of § 1, $\mathscr{A}_N$ has only one element). Let us fix n_1 such that (4.10) hold for $n \ge n_1$. We can apply (5.18) for $\alpha \in A_N$ and $n = n_1, n_1+1, \ldots, N$. For these n we have $k=1$ and $\Phi_n(T_{(1)}) \le h^{-1}(R_2H^2 + 2R_2H + R_2) = C_{11}$. If $\Lambda_0^{-1}\theta^2 < 1$, (5.18) yields

$$\sup_{A_N} \left|\frac{\psi_N''}{(\psi_N')^2}\right| \le \frac{C_{11}}{1-\Lambda_0^{-1}\theta^2} \sup_{A_N} \left|\frac{\psi_{n_1}''}{(\psi_{n_1}')^2}\right|. \tag{5.26}$$

Therefore (5.25) yields the Theorem if the spectral radius of $\mathbf{P}$ is <1. We recall that θ is arbitrarily close to 1, as β is sufficiently small. Also, we will see that r is arbitrarily small, as d is sufficiently small.

By Remark 3.2 we can check that $\Phi_n(T_{(k)}) \le \text{const.}\,|S_k'(1)|^{2\tau-1}$ (note: $\nu < 2\tau - 1$). We also have $|I_k| \leqslant \text{const.}\,|S_{k-1}'(1)|^{-\tau}$, where α on both sides may be different (see Corollary 4.2). Also, $\text{const.} \cdot |S_{k-1}'(1)| \ge |S_k'(1)|^{1/(1+\varepsilon)}$ by Lemma 3.1 (indeed, $S_k = T_{(k)} \circ S_{k-1}$ and $|S_k'| = (|T_{(k)}'| \circ S_{k-1})|S_{k-1}'| \le R_0|S_{k-1}'|^{\varepsilon}|S_{k-1}'| = R_0|S_{k-1}'|^{1+\varepsilon}$). So, $|I_k| \le \text{const.}\,|S_k'|^{-\tau/(1+\varepsilon)}$. Hence,

$$\Phi_n(T_{(k)}) \left| P \cap \bigcup_{\alpha \in J} (I_k \backslash I_{k+1}) \right| \le \text{const.}\,|S_k'(1)|^{-\zeta}, \tag{5.27}$$

where $\zeta = \tau/(1+\varepsilon) + 1 - 2\tau > 0$, if β is sufficiently small. We also have $|S_k'(1)|^{-\zeta} \le \Lambda_0^{-k\zeta}$. So, the sum in (5.23) has uniformly exponentially decreasing terms. By fixing sufficiently small d we can make it arbitrarily close to 0.

This proves the theorem, if $4\Lambda_0^{-1} < 1$. When this condition is not satisfied, we pick $p \in \mathbb{Z}^+$ such that $\Lambda_0^p > 4$ and examine the relation between $(\gamma_{n+p}, \eta_{n+p})$ and (γ_n, η_n). The corresponding matrix, as $r \to 0$, looks like

$$\begin{bmatrix} 2M\Lambda_0^{-p} & * \\ 0 & \Lambda_0^{-p} \end{bmatrix}, \tag{5.28}$$

where M is an integer which is the maximal number of $J' \in \mathscr{A}_{n+p}'$ contained in a fixed $J \in \mathscr{A}_n'$ and such that there is $i \in \{n, n+1, \ldots, n+p-1\}$ with the property: there is $k \le k_1$ such that $\psi_i(J') \subset I_k \backslash I_{k+1}$. This number can be made $=2$, if we slightly perturb intervals $I_1, \ldots, I_{k_1}$. □

Corollary 5.1. *We have* $|B_n| \le 2C_{10}a\lambda^{\sqrt{n}}|A_N|$.

Proof. Clearly, $B_n \subset \psi_n^{-1}([-a\lambda^{\sqrt{n}}, a\lambda^{\sqrt{n}}])$. Theorem 5.1 gives

$$|\psi_n^{-1}(E)| \le \left(\sum_{J \in \mathscr{A}_n} \sup_J \frac{1}{|\psi_n'|}\right)|E| \le C_{10}|E|\,|A_N| \tag{5.29}$$

for an arbitrary measurable set $E \subset I$. □

Remark 5.1. The existence of an a.c.i.m. for T follows directly from [**5**], since the function

$$g(x)=\begin{cases}1/|T'(x)|, & \textit{where } T \textit{ continuous},\\ 0 & \textit{on discontinuities}\end{cases} \tag{5.30}$$

verifies $\operatorname{Var} g<+\infty$ and $\sup g<1$ (a sufficient condition for the existence of an a.c.i.m.). One can obtain another proof (and make this paper self-contained) by imitating the proof of Theorem 5.1 for the sequence of mappings $\psi_n(x)=T^n(x)$ ($\mathscr{A}_n$ becomes the partition into the pieces of continuity of ψ_n). These new functions satisfy the recursive formula $\psi_{n+1}(x)=T_{(k)}(\psi_n(x))$ (if $\psi_n(x)\in I_k\backslash I_{k+1}$). This formula is much easier to differentiate, since x is the only variable.

Proof. We have

$$\operatorname{Var} g<2\sum_k \sup|T'_{(k)}|^{-1}+\sum_k \Delta_{xx}(T_{(k)})|I_k|. \tag{5.31}$$

The first sum converges by Remark 3.3. The second sum converges by an argument similar to the proof of (5.27). □

In view of § 1, Proposition 5.1 (iii), Corollary 5.1 and Remark 5.1 the main theorem is proved.

6. *Some generalizations and final remarks*

Our method applies without any changes to families $f(x)=1-\alpha|x|^{\xi}$, where ξ is sufficiently close to 2. For $\xi<2$ we get an example of a family which is not C^2. It is easy to get examples of families with singularities like $|x|^{\xi}$ with any $\xi>1$.

Another class of examples would be families with a finite or infinite number of singularities like

$$f(x)=\begin{cases}1-\alpha x^2 & \text{for } -1\le x\le 0\\ [10/x]-1 & \text{for } 1\ge x\ge 0.\end{cases}$$

Let us discuss a little different result now. Let $f_0(x)=1-2x^2$ and let for every $\eta>0$ $X(\eta)$ be the η-neighborhood of f_0 in the C^2-topology. If η is sufficiently small then there is a C^1 function $c:X(\eta)\to I$ such that $c(f)$ is the only critical point of $f\in X$. Let $M\subset X(\eta)$ be a submanifold of codimension 1 defined as follows:

$$M=\{f\in X(\eta): f^3(c(f))=f^2(c(f))\}. \tag{6.1}$$

For $f\in M$, $x(f)\overset{\text{def}}{=}f^2(c(f))$ is a hyperbolic fixed point.

THEOREM 6.1. *Suppose that a path $\alpha\mapsto f_\alpha$ intersects M transversally for some α_0. Then α_0 is a one-sided Lebesgue density point in the set of those α that f_α has an a.c.i.m.*

COROLLARY 6.1. *There is an open set of C^2-families satisfying Jakobson's Theorem.*

One can consider a transversality condition in higher jets

$$\left.\frac{d^k}{d\alpha^k}\right|_{\alpha=\alpha_0}(f_\alpha(x(f_\alpha))-x(f_\alpha))\neq 0, \tag{6.2}$$

where $k\ge 1$ is arbitrary.

THEOREM 6.2. *Theorem* 6.1 *remains true if transversality to* M *is replaced with condition* (6.2).

The idea of the proof. Let $G: X(\eta)\to\mathbb{R}$ be defined by

$$G(f)=f(x(f))-x(f). \tag{6.3}$$

It is easy to check that $dG(f)\neq 0$ for $f\in X(\eta)$, if η is sufficiently small. Therefore, there is a mapping $p: X(\eta)\to M$ such that $X\ni f\mapsto (p(f), G(f))\in M\times\mathbb{R}$ is a diffeomorphism (one can write down suitable formulas explicitly).

For every $f\in M$ we have a standard family f_γ corresponding to the fiber $\{f\}\times\mathbb{R}$, so that $p(f_\gamma)=f$ and $G(f_\gamma)=\gamma$ for $|\gamma|<\gamma_0$. We repeat our proof simultaneously for all families f_γ. We obtain a family $\mathscr{F}$ of submanifolds $F\subset X(\eta)$ such that if $g\in F\in\mathscr{F}$ then g has an a.c.i.m. Moreover, there is a constant K with the following property

(i) Every $F\in\mathscr{F}$ corresponds to a graph of a Lipschitz function $h_F: M\to\mathbb{R}$ with a Lipschitz constant $\leq K$ ($M\in\mathscr{F}$; $h_M\equiv 0$).

(ii) Let $A_\infty(f)=\{\gamma; f_\gamma\in F$ for some $F\in\mathscr{F}\}$. The family $\mathscr{F}$ defines a measurable mapping $A_\infty(f_1)\to A_\infty(f_2)$. This mapping has a measure theoretic Jacobian $\in [K^{-1}, K]$ a.e.

From our estimates it follows easily that for every $f\in M$ and $h>0$

$$\frac{|A_\infty(f)\cap[0,h]|}{h}\geq 1-c_1\exp\left(-c_2\left(\log\frac{1}{h}\right)^{1/2}\right), \tag{6.4}$$

where $c_1, c_2>0$.

These estimates are sufficient to show that every family g_α such that $G(g_{\alpha_0})=0$ and $(d^k/d\alpha^k)G(g_\alpha)|_{\alpha=\alpha_0}\neq 0$ for some k intersects the leaves of $\mathscr{F}$ for a positive measure set of parameters α. □

COROLLARY 6.2. *Any analytic family which contains* f_0, *satisfies Jakobson's theorem.*

Let us list two more facts concerning our construction.

(1) The density of an a.c.i.m. we constructed $\in L^p(I)$ for every $p\in[1,2)$. This result is analogous to one of Carleson's results.

(2) If α_0 is sufficiently close to 2 and $\inf_{n\geq 1}|f^n_{\alpha_0}(0)|>0$ then α_0 is also a density point for the set of α such that f_α has an a.c.i.m. The condition that α_0 is close to 2 can be replaced with the conditions given in [**3**].

Appendix

In this Appendix $f(x)=1-2x^2$.

THEOREM A.1. *Let* $I_n=[-3^{-n}, 3^{-n}]$ *for* $n\in\mathbb{Z}^+$. *Let* $V_n=f(I_n)$. *Then for every* n *and* $\rho\in(0,1)$:

(i) $|(f^n)'|\,V_n|\geq(3\frac{1}{9})^n$ *and* $\sim 4^n$ *for large* n,

(ii) $|(f^{n+1})'|\,I_n\backslash\rho I_{n+1}|\geq\Lambda=4\rho/3$ *and* $\geq$const. $\rho(\frac{4}{3})^n$,

(iii) $|W_n|=|f^n(V_{n+1})|\leq(\frac{4}{9})^{n-1}|V_1|$.

Proof. (i) We have $V_n = f(I_n) = [1-2\cdot 9^{-n}, 1]$. We have $|f'| \le 4$. By the Mean Value Theorem we have for $k < n$

$$|f^k(V_n)| \le 2\cdot 9^{-n}\cdot 4^k = 4^{-(n-k-1)}\cdot(\tfrac{4}{9})^{n-1}|V_1| \le (\tfrac{4}{9})^{n-1}|V_1|. \tag{A.1}$$

This implies $f^k(V_n) \subset \pm V_1$ for $n \ge 2$. We also have $|f'|\pm V_1| \ge 4(1-|V_1|) = \frac{28}{9} = 3\frac{1}{9}$.

We can write

$$|(f^n)'|V_n| \ge \left|\prod_{k=0}^{n-1} f'|f^k(V_n)\right|. \tag{A.2}$$

Clearly, it is $\ge (3\frac{1}{9})^n$ and also $\sim 4^n$ for large n. Indeed, the product is $\ge$

$$\prod_{k=0}^{n-1} 4(1-|f^k(V_n)|) \ge 4^n \prod_{k=0}^{n-1}(1-(\tfrac{4}{9})^{n-1}|V_1|4^{-(n-k-1)}). \tag{A.3}$$

Hence,

$$|(f^{n+1})'|I_n\backslash\rho I_{n+1}| \ge |(f^n)'|V_n|\cdot 4\cdot\tfrac{1}{2}|\rho I_{n+1}| \ge 3^n 4\rho 3^{-(n+1)} = \tfrac{4}{3}\rho = \Lambda. \tag{A.4}$$

Using $|(f^n)'|V_n| \ge \text{const.}\cdot 4^n$ we also get $|(f^{n+1})'|I_n\backslash\rho I_{n+1}| \ge \text{const.}\ \rho(\frac{4}{3})^n$. □

If $\rho \ge 27/28$ one gets the same inequalities on $\rho^{-1}I_n\backslash\rho I_n$ with $3\frac{1}{9}$ replaced by 3.

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