WEIGHTED QUADRATIC NORMS AND LEGENDRE POLYNOMIALS

I. I. HIRSCHMAN, JR.

1. Introduction. Let $\omega_n(x) = (n + \frac{1}{2})^{\frac{1}{2}} P_n(x)$, $n = 0, 1, \ldots$, be the normalized Legendre polynomials. If $f(x) \in L^1(-1, 1)$ and if

$$a_n = \int_{-1}^{1} \omega_n(x) f(x)\, dx$$

then we write

$$f(x) \sim \sum_{0}^{\infty} a_n \omega_n(x).$$

Let $T = \{t_n\}$ $(0 \leqslant n < \infty)$ be a sequence of real constants. T determines a linear transformation on setting $Tf(x) = g(x)$ if $g(x) \in L^1(-1, 1)$ and if

$$g(x) \sim \sum_{0}^{\infty} t_n a_n \omega_n(x).$$

(In general T will not be defined for every $f \in L^1(-1, 1)$). Let

$$\mathfrak{N}_{\alpha,\beta}[f] = \left[\int_{-1}^{1} (1 + x)^{\alpha} (1 - x)^{\beta} [f(x)]^2 dx \right]^{\frac{1}{2}}$$

where $-1 < \alpha, \beta < 1$. $\mathfrak{N}_{\alpha,\beta}$ will also be used to denote the space of functions $f(x)$ for which $\mathfrak{N}_{\alpha,\beta}[f]$ is finite. Because of the special role played by $x = \pm 1$ in Legendre series such norms are quite natural. Our objective in the present paper is to give a rather general sufficient condition for T to be a bounded linear transformation of $\mathfrak{N}_{\alpha,\beta}$ into itself.

The origin of this problem lies in the fact that T is a multiplier transformation and multiplier transformations for Fourier series have been extensively studied. Let, just as above, $T = \{t_n\}$ $(-\infty < n < \infty)$ be formally defined as that linear transformation which carries

$$f(\theta) \sim \sum_{-\infty}^{\infty} a_n e^{in\theta}$$

into

$$Tf(\theta) \sim \sum_{-\infty}^{\infty} t_n a_n e^{in\theta}.$$

Received December 6, 1954.

This research was supported by the United States Air Force, through the office of Scientific Research and Development Command.

Let

$$||f(\theta)||_p = \left[\int_{-\pi}^{\pi} |f(\theta)|^p d\theta\right]^{1/p}.$$

T is said to be of type $[p, q]$ if there exists a constant A such that $||Tf||_q \leqslant A||f||_p$ for every f for which $||f||_p$ is finite. Many classical investigations in the theory of Fourier series have been concerned with the problem of showing that a particular multiplier transformation is of some prescribed type. Thus the conjugate function theorem of M. Riesz asserts that $\{i \operatorname{sgn} n\}$ is of type $[p, p]$ for $1 < p < \infty$, and the fractional integration theorem of Hardy and Littlewood is equivalent to the assertion that $\{(in)^{-\sigma}\}'$ is of type $[p, p(1 - p\sigma)^{-1}]$ for $1 < p < \sigma^{-1} < \infty$. (Here the prime indicates that the term corresponding to $n = 0$ is omitted.) An investigation of particular relevance to the present paper is that of Marcinkiewicz **(4)** who, by making use of some very difficult researches of Paley, Littlewood, and later Zygmund, was able to prove that T is of type $[p, p]$, $1 < p < \infty$, if

1.1
$$|t_n| \leqslant A \qquad (n = 0, \pm 1, \pm 2, \ldots),$$
$$\sum_{\pm 2^n}^{\pm 2^{n+1}} |t_k - t_{k-1}| \leqslant A \qquad (n = 0, 1, 2, \ldots).$$

Marcinkiewicz's result plays a central role in the theory of multiplier transformations in that many of the other results can be deduced from it. Let

$$||f||_{\alpha,p} = \left[\int_{-\pi}^{\pi} |f(\theta)|^p |\theta|^{\alpha p} d\theta\right]^{1/p}.$$

T will be said to be of type $[(\alpha, p), (\beta, q)]$ if there exists a constant A such that $||Tf||_{\beta,q} \leqslant A||f||_{\alpha,p}$ for all $f(\theta)$ for which $||f||_{\alpha,p}$ is finite. The author has recently proved that if the conditions (1) hold then T is of type $[(\alpha, p), (\alpha, p)]$ for $1 < p < \infty$, $-p^{-1} < \alpha < 1 - p^{-1}$. See **(3)**. The arguments used depend heavily upon the connection between Fourier series and power series, and thus cannot be extended to other orthogonal expansions. However it can be shown that the conditions 1.1 imply that T is of type $[(\alpha, 2), (\alpha, 2)]$ by quite different arguments. This is, of course, the case of weighted quadratic norms. In the present paper we shall apply these arguments to Legendre series. It will be convenient to state our principal result.

DEFINITION. $T = \{t_n\}$ $(0 \leqslant n < \infty)$ *is said to belong to class* $\mathbf{M}(C)$ *if:*

(a)
$$|t_n| \leqslant C \qquad (n = 0, 1, 2, \ldots);$$

(b)
$$\sum_{2^n}^{2^{n+1}} |t_k - t_{k-1}| \leqslant C \qquad (n = 0, 1, 2, \ldots).$$

We shall show that if T belongs to class $\mathbf{M}(C)$ then

$$\mathfrak{N}_{\alpha,\beta}[Tf] \leqslant D(\alpha, \beta) C \mathfrak{N}_{\alpha,\beta}[f] \qquad (-\tfrac{1}{2} < \alpha, \beta < \tfrac{1}{2})$$

where $D(\alpha, \beta)$ depends only upon α and β. The restriction $-\frac{1}{2} < \alpha, \beta < \frac{1}{2}$

https://doi.org/10.4153/CJM-1955-050-x Published online by Cambridge University Press

here is essential and the result is not otherwise true. As an example let $S_N = \{s_{N,n}\}$ where $s_{N,n}$ is 1 for $0 \leqslant n \leqslant N$ and is 0 for $N < n < \infty$. S_N belongs to $\mathbf{M}(1)$. Thus

1.2 $$\mathfrak{N}_{\alpha,\beta}[S_N f] \leqslant D(\alpha, \beta)\mathfrak{N}_{\alpha,\beta}[f] \qquad (-\tfrac{1}{2} < \alpha, \beta < \tfrac{1}{2}).$$

For the sake of comparison we recall that Pollard has shown in **(6)** that

$$||S_N f||_p \leqslant D(p)||f||_p \qquad (\tfrac{4}{3} < p < 4),$$

the result being false for other values of p. See also in this connection the paper of Newman and Rudin **(5)**.

Our method of demonstration depends upon the following inequalities, valid for $0 < \alpha < \frac{1}{2}$,

$$E'(\alpha) \leqslant \mathfrak{N}_{0,\alpha}[f]^{-2} \sum_{k>j} [a_j(j + \tfrac{1}{2})^{-\frac{1}{2}} - a_k(k + \tfrac{1}{2})^{-\frac{1}{2}}]^2 Q(k, j) \leqslant E''(\alpha),$$

$$E'(\alpha) \leqslant \mathfrak{N}_{\alpha,0}[f]^{-2} \sum_{k>j} [(-1)^j a_j(j + \tfrac{1}{2})^{-\frac{1}{2}} - (-1)^k a_k(k + \tfrac{1}{2})^{-\frac{1}{2}}]^2 Q(k, j) \leqslant E''(\alpha),$$

where

$$Q(k, j) = (k + \tfrac{1}{2})(j + \tfrac{1}{2})|k - j|^{-1-2\alpha}(k + j)^{-1}.$$

Here $E'(\alpha)$ and $E''(\alpha)$ are positive constants depending only upon α.

2. Preliminary estimates. The present section is devoted to establishing results which are needed to prove the identities announced at the end of §1. Let us write $\phi(x) \simeq \psi(x)$ $(x \in X)$ if there exist finite positive constants c_1 and c_2 such that $c_1 \leqslant \phi(x)/\psi(x) \leqslant c_2$ $(x \in X)$. Similarly $\phi(x) \leqslant\simeq \psi(x)$ $(x \in X)$ if there exists a positive constant c such that $\phi(x) \leqslant c\psi(x)$ for $(x \in X)$. We define

$$\Omega_\alpha(x) = \sum_{n=1}^{\infty} [1 - P_n(x)]n^{-1-2\alpha}.$$

LEMMA 2a. *For α fixed $0 < \alpha < \frac{1}{2}$ we have*

$$\Omega_\alpha(x) \simeq (1 - x)^\alpha \qquad (-1 \leqslant x \leqslant 1).$$

Proof. Let $g_0 = 1$,

$$g_m = \frac{1.3 \ldots (2m - 1)}{2.4 \ldots 2m};$$

then **(8**, p. 92**)**

$$P_n(\cos\theta) = \sum_{m=0}^{n} g_m g_{n-m} \cos(n - 2m)\theta.$$

Setting $\theta = 0$ we see that

$$1 = \sum_{m=0}^{n} g_m g_{n-m},$$

and thus

$$1 - P_n(\cos\theta) = \sum_{m=0}^{n} g_m g_{n-m}[1 - \cos(n - 2m)\theta].$$

Distinguishing between even and odd, we find

$$\sum_{n=1}^{\infty} [1 - P_{2n}(\cos\theta)](2n)^{-1-2\alpha} = 2\sum_{j=1}^{\infty} [1 - \cos(2j)\theta] \sum_{n \geqslant j} (2n)^{-1-2\alpha} g_{n-j} g_{n+j},$$

$$\sum_{n=0}^{\infty} [1 - P_{2n+1}(\cos\theta)](2n+1)^{-1-2\alpha}$$
$$= 2\sum_{j=0}^{\infty} [1 - \cos(2j+1)\theta] \sum_{n \geqslant j} (2n+1)^{-1-2\alpha} g_{n-j} g_{n+1+j}.$$

Since

$$g_m(\pi m)^{\frac{1}{2}} \to 1 \qquad (m \to \infty)$$

it is easy to establish that

$$\sum_{n \geqslant j} (2n)^{-1-2\alpha} g_{n-j} g_{n+j} \simeq (2j)^{-1-2\alpha},$$
$$\sum_{n \geqslant j} (2n+1)^{-1-2\alpha} g_{n-j} g_{n+j+1} \simeq (2j+1)^{-1-2\alpha},$$

and thus that

$$\sum_{n=1}^{\infty} [1 - P_n(\cos\theta)] n^{-1-2\alpha} \simeq \sum_{j=1}^{\infty} [1 - \cos(j\theta)] j^{-1-2\alpha}.$$

Let

$$p(\theta) = \sum_{j=1}^{\infty} [1 - \cos(j\theta)] j^{-1-2\alpha};$$

then

$$p'(\theta) = \sum_{j=1}^{\infty} [\sin(j\theta)] j^{-2\alpha}.$$

It follows (**10**, pp. 112–116) that $p'(\theta) \simeq \theta^{2\alpha-1}$ $(\theta \to 0+)$ and thus that $p(\theta) \simeq \theta^{2\alpha} (0 \leqslant \theta \leqslant \pi)$. Combining these results we have the conclusion of our lemma.

LEMMA 2b. *If α is fixed, $(0 < \alpha < \frac{1}{2})$, then*

$$\int_{-1}^{1} \Omega_\alpha(x)\, P_k(x)\, P_j(x)\, dx \simeq -(k-j)^{-1-2\alpha}(k+j)^{-1} \qquad (k > j \geqslant 0).$$

Proof. The integral

$$\int_{-1}^{1} P_k(x)\, P_j(x)\, P_r(x)\, dx \qquad (k > j)$$

is zero except for those values of r such that $k - j \leqslant r \leqslant k + j$ and such that the parity of r equals the parity of $j + k$. In these cases the value of the integral is given by the formula (**1**, p. 311)

$$\int_{-1}^{1} P_k(x)\, P_j(x)\, P_{k+j-2s}(x)\, dx = \frac{2 g_{k-s} g_{j-s} g_s}{(2k + 2j - s + 1) g_{k+j-s}}.$$

Thus

$$\int_{-1}^{1} \Omega_\alpha(x)\, P_k(x)\, P_j(x)\, dx = -\ S(k,j),$$

where

$$S(k,j) = \sum_{s=0}^{j} \frac{2g_{k-s}g_{j-s}g_s}{(2k+2j-s+1)g_{k+j-s}} (k+j-2s)^{-1-2\alpha}.$$

We have

$$S(k,j) \simeq k^{-\frac{1}{2}} \sum_{s=0}^{j} (k+1-s)^{-\frac{1}{2}}(j+1-s)^{-\frac{1}{2}}(s+1)^{-\frac{1}{2}}(k+j-2s)^{-1-2\alpha}.$$

If $k \geqslant 3j/2$, $0 \leqslant s \leqslant j$, then $(k+1-s) \simeq k$, $(k+j-2s) \simeq k$, so that

$$S(k,j) \simeq k^{-2-2\alpha} \sum_{s=0}^{j} (j+1-s)^{-\frac{1}{2}}(s+1)^{-\frac{1}{2}} \simeq k^{-2-2\alpha},$$

which can be rewritten as

$$S(k,j) \simeq (k-j)^{-1-2\alpha}(k+j)^{-1} \qquad (k \geqslant 3j/2).$$

On the other hand if $j < k < 3j/2$ then

$$S(k,j) = S_1(k,j) + S_2(k,j) + S_3(k,j) = \sum_{2j-k \leqslant s \leqslant j} + \sum_{\frac{1}{2}j \leqslant s < 2j-k} + \sum_{0 \leqslant s < \frac{1}{2}j}.$$

If $j < k < 3j/2$ and if $2j - k \leqslant s \leqslant j$ then $(k+1-s) \simeq (k-j)$, $(k+j-2s) \simeq (k-j)$, and $(s+1) \simeq k$, so that

$$S_1(k,j) \simeq k^{-1}(k-j)^{-\frac{1}{2}-1-2\alpha} \sum_{2j-k \leqslant s \leqslant j} (j+1-s)^{-\frac{1}{2}}$$
$$\simeq (k-j)^{-1-2\alpha}(k+j)^{-1}.$$

If $j < k < 3j/2$ and $j/2 \leqslant s < 2j - k$ then $(k+1-s) \simeq (k-s)$, $(k+j-2s) \simeq (k-s)$, $(s+1) \simeq k$, and $(j+1-s) \simeq (k-s)$, so that

$$S_2(k,j) \simeq k^{-1} \sum_{\frac{1}{2}j \leqslant s < 2j-k} (k-s)^{-2-2\alpha} \simeq (k-j)^{-1-2\alpha}(k+j).$$

Finally if $j < k < 3j/2$, $0 \leqslant s \leqslant \frac{1}{2}j$, then $(k+1-s) \simeq k$, $(j+1-s) \simeq k$, and $(k+j-2s) \simeq k$, so that

$$S_3(k,j) \simeq k^{-5/2-2\alpha} \sum_{0 \leqslant s < \frac{1}{2}j} (s+1)^{-\frac{1}{2}} < \simeq (k-j)^{-1-2\alpha}(k+j)^{-1}.$$

It follows that

$$S(k,j) \simeq (k-j)^{-1-2\alpha}(k+j)^{-1} \qquad (j < k < 3j/2),$$

and our demonstration is complete.

Let us set

$$\int_{-1}^{1} \Omega_\alpha(x)(k+\tfrac{1}{2})\, P_k(x)(j+\tfrac{1}{2})\, P_j(x)\, dx = \begin{cases} d_k, & k = j, \\ -c_{kj}, & k \neq j. \end{cases}$$

Since $\Omega_\alpha \geqslant 0$ it is clear that $d_k > 0$, while $c_{kj} > 0$ by Lemma 2b.

LEMMA 2c.

$$\sum_{\substack{k=0\\k\neq j}}^{\infty} c_{kj} = d_j.$$

Proof. We have **(9**, p. 332**)**.

$$p(h, x) = \sum_{k=0}^{\infty} h^k(k + \tfrac{1}{2}) P_k(x) = \frac{\tfrac{1}{2}(1 - h^2)}{(1 - 2hx + h^2)^{3/2}}.$$

If $-1 < h < 1$ then

$$\int_{-1}^{1} \Omega_\alpha(x)(j + \tfrac{1}{2}) P_j(x) p(h, x)\, dx = d_j h^j - \sum_{\substack{k=0\\k\neq j}}^{\infty} c_{kj} h^k.$$

It is easily verified, by integrating term by term, that

$$\int_{-1}^{1} p(h, x)\, dx = 1, \qquad -1 < h < 1,$$

and it is easy to see that

$$\lim_{h\to 1-} p(h, x) = 0$$

uniformly for $-1 \leqslant x \leqslant 1 - \epsilon$ for any fixed $\epsilon > 0$. From these relations we have

$$\lim_{h\to 1-} \int_{-1}^{1} \Omega_\alpha(x)(j + \tfrac{1}{2}) P_j(x) p(h, x)\, dx = \Omega_\alpha(1)(j + \tfrac{1}{2}) P_j(1) = 0;$$

that is,

$$\lim_{h\to 1-} \sum_{\substack{k=0\\k\neq j}}^{\infty} c_{kj} h^k = d_j,$$

and since the c_{kj} are positive the series on the left is not only Abel summable but actually convergent.

3. Some inequalities. The following result is demonstrated in **(2)**:

THEOREM 3a. *If* $\phi_n(\theta)$ $n = 0, 1, \ldots$ *is a uniformly bounded orthonormal set on* $0 \leqslant \theta \leqslant 1$ *and if for* $F(\theta) \in L^1(0, 1)$

$$a_n = \int_0^1 F(\theta)\, \phi_n(\theta)\, d\theta,$$

then

$$\sum_0^{\infty} a_k^2 (n_k + 1)^{-2\alpha} \leqslant A(\alpha) \int_0^1 F(\theta)^2 \theta^{2\alpha} d\theta \qquad (0 \leqslant \alpha < \tfrac{1}{2}),$$

where $n_0, n_1, n_2, \ldots$ *is any rearrangement of* $0, 1, 2, \ldots$.

In order to apply this theorem to Legendre polynomials let us define

$$\phi_n(\theta) = \omega_n(\cos \pi\theta)[\pi \sin \pi\theta]^{\frac{1}{2}}$$
$$F(\theta) = f(\cos \pi\theta)[\pi \sin \pi\theta]^{\frac{1}{2}}$$

$(0 \leqslant \theta \leqslant 1)$. Setting $x = \cos \pi\theta$ we see that

$$a_n = \int_{-1}^{1} f(x)\, \omega_n(x)\, dx = \int_0^1 F(\theta)\, \phi_n(\theta)\, d\theta.$$

The functions $\phi_n(\theta)$ are evidently orthonormal, that they are uniformly bounded follows from **(8**; p. 161**)**. Thus

$$\sum_0^\infty a_k{}^2(n_k + 1)^{-2\alpha} \leqslant A(\alpha) \int_0^1 F(\theta)^2\theta^{2\alpha}d\theta.$$

We have

$$\int_0^1 F(\theta)^2\theta^{2\alpha}d\theta = \int_{-1}^{1} f(x)^2\left[\frac{1}{\pi}\operatorname{arc\,cos} x\right]^{2\alpha} dx \simeq \mathfrak{N}_{0,\alpha}[f]^2.$$

Combining these results we have proved **(7)**

THEOREM 3b. *If* $0 \leqslant \alpha < \frac{1}{2}$ *then*

$$\sum_0^\infty a_k{}^2(n_k + 1)^{-2\alpha} \leqslant B(\alpha)\, \mathfrak{N}_{0,\alpha}[f]^2,$$

where $n_0, n_1, n_2, \ldots$ *is any rearrangement of* $0, 1, 2, \ldots$.

We proceed to prove

LEMMA 3c. *If* $0 \leqslant \alpha < \frac{1}{2}$ *and if*

(i) $$f(x) \in \mathfrak{N}_{0,\alpha},$$

(ii) $$f(x) \sim \sum_0^\infty a_n\omega_n(x),$$

then

$$\lim_{N\to\infty} \sum_0^N a_n^2(N - n + 1)^{-2\alpha} = 0.$$

Proof. Given $\epsilon > 0$, let us choose a finite sum

$$g(x) = \sum_0^M b_n\omega_n(x)$$

such that $\mathfrak{N}_{0,\alpha}[f - g]^2 \leqslant \epsilon$. By Theorem 3b,

$$\sum_0^N (a_n - b_n)^2(N - n + 1)^{-2\alpha} \leqslant A\, \mathfrak{N}_{0,\alpha}[f - g]^2 \leqslant A\epsilon$$

where b_n is defined as 0 if $n > M$. Now

$$\sum_0^N a_n^2(N - n + 1)^{-2\alpha} = \sum_{M+1}^N (a_n - b_n)^2(N - n + 1)^{-2\alpha} + \sum_{n=1}^M a_n^2(N - n + 1)^{-2\alpha}.$$

https://doi.org/10.4153/CJM-1955-050-x Published online by Cambridge University Press

Thus

$$\limsup_{N\to\infty} \sum_0^N a_n^2 (N - n + 1)^{-2\alpha} \leqslant A\epsilon.$$

Since ϵ is arbitrary our desired conclusion follows.

4. The basic norm relations. We need the following version of the Riesz Fischer theorem, the proof of which is omitted:

LEMMA 4a. *Let $\alpha(-1 < \alpha < 1)$ be fixed. If constants $\{a_n\}_0^\infty$ are given and if*

$$\liminf_{r\to\infty} \mathfrak{N}_{0,\alpha}\left[\sum_0^r a_n\omega_n(x)\right] \leqslant A,$$

then there exists a (unique) function $f(x)$ such that

$$\mathfrak{N}_{0,\alpha}[f] \leqslant A, \quad f(x) \sim \sum_0^\infty a_n\omega_n(x).$$

THEOREM 4b. *If, for $0 < \alpha < \frac{1}{2}$, α fixed*

(i) $$f(x) \in \mathfrak{N}_{0,\alpha},$$

(ii) $$f(x) \sim \sum_0^\infty a_n\omega_n(x),$$

then

$$\mathfrak{N}_{0,\alpha}[f]^2 \simeq \sum_{k>j} [a_k(k + \tfrac{1}{2})^{-\frac{1}{2}} - a_j(j + \tfrac{1}{2})^{-\frac{1}{2}}]^2 Q(k, j).$$

Proof. Let

$$g(x) = \sum_0^n b_k\omega_k(x);$$

then by Lemma 2a,

$$\mathfrak{N}_{0,\alpha}[g]^2 \simeq \int_{-1}^1 \Omega_\alpha(x)[g(x)]^2 dx.$$

Using Lemma 2c we see that

$$\begin{aligned}\int_{-1}^1 \Omega_\alpha(x)[g(x)]^2 dx &= \sum_{k=0}^n d_k b_k^2 (k + \tfrac{1}{2})^{-1} - \sum_{j,k=0,\, j\neq k}^n c_{kj} b_k (k + \tfrac{1}{2})^{-\frac{1}{2}} b_j (j + \tfrac{1}{2})^{-\frac{1}{2}},\\ &= \sum_{k>j} [b_k(k + \tfrac{1}{2})^{-\frac{1}{2}} - b_j(j + \tfrac{1}{2})^{-\frac{1}{2}}]^2 c_{kj},\end{aligned}$$

where in this sum k and j vary from 0 to ∞, b_k being equal to zero, for $k > n$. Applying Lemma 2b we have

$$\mathfrak{N}_{0,\alpha}[g]^2 \simeq \sum_{k>j} [b_k(k + \tfrac{1}{2})^{-\frac{1}{2}} - b_j(j + \tfrac{1}{2})^{-\frac{1}{2}}]^2 Q(k, j).$$

Thus our theorem is true for finite sums of Legendre polynomials.

https://doi.org/10.4153/CJM-1955-050-x Published online by Cambridge University Press

Let $f(x) \in \mathfrak{N}_{0,\alpha}$ and let

$$h_r(x) = \sum_{k=0}^{r} a_{r,k}\omega_k(x)$$

be a series of polynomials such that

$$\lim_{r\to\infty} \mathfrak{N}_{0,\alpha}[f - h_r] = 0.$$

Note that this implies that

$$\lim_{r\to\infty} a_{r,k} = a_k \qquad (k = 0, 1, \ldots).$$

We have, if N is fixed,

$$\sum_{0\leqslant j<k\leqslant N} [a_{r,k}(k+\tfrac{1}{2})^{-\frac{1}{2}} - a_{r,j}(j+\tfrac{1}{2})^{-\frac{1}{2}}]^2 Q(k,j) \leqslant A\, \mathfrak{N}_{0,\alpha}[h_r]^2.$$

Letting r increase without limit, we obtain

$$\sum_{0\leqslant j<k\leqslant N} [a_k(k+\tfrac{1}{2})^{-\frac{1}{2}} - a_j(j+\tfrac{1}{2})^{-\frac{1}{2}}]^2 Q(k,j) \leqslant A\, \mathfrak{N}_{0,\alpha}[f]^2.$$

and finally, since N is arbitrary,

$$\sum_{k>j} [a_k(k+\tfrac{1}{2})^{-\frac{1}{2}} - a_j(j+\tfrac{1}{2})^{-\frac{1}{2}}]^2 Q(k,j) \leqslant A\, \mathfrak{N}_{0,\alpha}[f]^2.$$

To establish the converse inequality let

$$f_r(x) = \sum_{0}^{r} a_k\omega_k(x).$$

We have

$$\mathfrak{N}_{0,\alpha}[f_r]^2 \simeq \sum_{0\leqslant j<k\leqslant r} [a_k(k+\tfrac{1}{2})^{-\frac{1}{2}} - a_j(j+\tfrac{1}{2})^{-\frac{1}{2}}]^2 Q(k,j) + \sum_{j=0}^{r} a_j^2(j+\tfrac{1}{2})^{-1} \sum_{k>r} Q(k,j).$$

Now

$$\sum_{0<j<k\leqslant r} [a_k(k+\tfrac{1}{2})^{-\frac{1}{2}} - a_j(j+\tfrac{1}{2})^{-\frac{1}{2}}]^2 Q(k,j) \leqslant \sum_{k>j} [a_k(k+\tfrac{1}{2})^{-\frac{1}{2}} - a_j(j+\tfrac{1}{2})^{-\frac{1}{2}}]^2 Q(k,j)$$

and

$$\sum_{j=0}^{r} a_j^2(j+\tfrac{1}{2})^{-1} \sum_{k>r} Q(k,j) \leqslant A \sum_{j=0}^{r} a_j^2(r-j+1)^{-2\alpha}.$$

Applying Lemma 3c, we see that

$$\limsup_{r\to\infty} \mathfrak{N}_{0,\alpha}[f_r]^2 \leqslant A \sum_{k>j} [a_k(k+\tfrac{1}{2})^{-\frac{1}{2}} - a_j(j+\tfrac{1}{2})^{-\frac{1}{2}}]^2 Q(k,j).$$

The desired conclusion now follows from Lemma 4a.

This theorem is equivalent to the first relation announced at the end of §1. The second relation there can be obtained from the first on noting that $\mathfrak{N}_{\alpha,0}[f(x)] = \mathfrak{N}_{0,\alpha}[f(-x)]$.

5. Bounded multiplier transformations. We are now in a position to prove our main theorem in the case $\beta = 0$, $0 \leqslant \alpha < \frac{1}{2}$. Let S_μ be the multiplier transformation which carries

$$f(x) \sim \sum_0^\infty a_n\omega_n(x) \text{ into } S_N f(x) = \sum_0^N a_n\omega_n(x).$$

LEMMA 5a. *If* $0 \leqslant \alpha < \frac{1}{2}$ *then*

$$\mathfrak{N}_{\alpha,0}[S_N f] \leqslant A(\alpha)\mathfrak{N}_{\alpha,0}[f].$$

Proof. We may suppose $\alpha > 0$, since the case $\alpha = 0$ is trivial. By Theorem 4b we have

$$\begin{aligned}\mathfrak{N}_{0,\alpha}[S_N f]^2 \leqslant A \sum_{m,n\leqslant N} [a_n(n+\tfrac{1}{2})^{-\frac{1}{2}} - a_m(m+\tfrac{1}{2})^{-\frac{1}{2}}]^2 Q(m,n)\\ + A\sum_{\substack{m\leqslant N\\ n>N}} a_m^2(m+\tfrac{1}{2})^{-1}Q(m,n).\end{aligned}$$

Using Theorem 4b again we see that

$$\sum_{m,n\leqslant N}[a_n(n+\tfrac{1}{2})^{-\frac{1}{2}} - a_m(m+\tfrac{1}{2})^{-\frac{1}{2}}]^2 Q(m,n) \leqslant A\mathfrak{N}_{0,\alpha}[f]^2.$$

Further

$$\begin{aligned}\sum_{\substack{m\leqslant N\\ n>N}} a_m^2\,(m+\tfrac{1}{2})^{-1}Q(m,n) &\leqslant A\sum_{m\leqslant N} a_m^2 \sum_{n>N}(n-m)^{-1-2\alpha},\\ &\leqslant A\sum_{m\leqslant N} a_m^2\,(N+1-m)^{-2\alpha},\\ &\leqslant A\,\mathfrak{N}_{0,\alpha}[f]^2,\end{aligned}$$

by Theorem 3b. These inequalities imply our desired result.

Let $b_\mu = 3.2^{\mu-2}$, $r_\mu = 2^{\mu-1}$, let σ_μ be the set of integers $b_\mu - r_\mu \leqslant k < b_\mu + r_\mu$, and let

$$\rho_\mu(x) = [1 - r_\mu^{-2}(x-b_\mu)^2].$$

If

$$f(x) \backsim \sum_0^\infty a_n\omega_n(x)$$

then we define

$$E_\mu(x) = \sum_{n\in\sigma_\mu} a_n\rho_\mu(n)\,\omega_n(x).$$

LEMMA 5b. *If* $0 \leqslant \alpha < \frac{1}{2}$ *then*

$$\sum_{\mu=0}^\infty \mathfrak{N}_{0,\alpha}[E_\mu]^2 \leqslant A\,\mathfrak{N}_{0,\alpha}[f]^2.$$

Proof. We may again suppose $\alpha > 0$. By Theorem 4b,

$$\mathfrak{N}_{0,\alpha}[E_\mu]^2 \simeq \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3$$

https://doi.org/10.4153/CJM-1955-050-x Published online by Cambridge University Press

where

$$\sum\nolimits_1 = \sum_{\substack{m,n \in \sigma_\mu \\ n>m}} [\rho_\mu(n)\, a_n(n + \tfrac{1}{2})^{-\frac{1}{2}} - \rho_\mu(m)\, a_m(m + \tfrac{1}{2})^{-\frac{1}{2}}]^2 Q(n, m),$$

$$\sum\nolimits_2 = \sum_{\substack{m \in \sigma_\mu \\ n>\sigma_\mu}} \rho_\mu(m)^2 a_m^2 (m + \tfrac{1}{2})^{-1} Q(n, m),$$

$$\sum\nolimits_3 = \sum_{\substack{n \in \sigma_\mu \\ m<\sigma_\mu}} \rho_\mu(n)\, a_n^2 (n + \tfrac{1}{2})^{-1} Q(n, m).$$

Let us begin with $\sum_2$. For $m \in \sigma_\mu$ we have

$$\sum_{n>\sigma_\mu} Q(n, m) \leqslant Am(b_\mu + r_\mu - m)^{-2\alpha}$$

where A is indepdnent of m and μ. Since

$$\rho_\mu(m)^2 \leqslant 4(b_\mu + r_\mu - m)^2 r_\mu^{-2} \qquad (m \in \sigma_\mu)$$

it follows that

$$\sum_{n>\sigma_\mu} \rho_\mu(m)^2 (m + \tfrac{1}{2})^{-1} Q(n, m) \leqslant A\,(b_\mu + r_\mu - m)^{2-2\alpha} r_\mu^{-2} \qquad (m \in \sigma_\mu).$$

Making use of the inequalities

$$(b_\mu + r_\mu - m)^{2-2\alpha} \leqslant A\,(m + 1)^{2-2\alpha} \qquad (m \in \sigma_\mu),$$

$$m + 1 \leqslant A\, r_\mu \qquad (m \in \sigma_\mu),$$

we obtain

$$\sum_{n>\sigma_\mu} \rho_\mu(m)^2 (m + \tfrac{1}{2})^{-1} Q(n, m) \leqslant A\,(m + 1)^{-2\alpha} \qquad (m \in \sigma_\mu),$$

and thus

$$\sum\nolimits_2 \leqslant A \sum_{m \in \sigma_\mu} |a_m|^2 (m + 1)^{-2\alpha}.$$

We next turn to $\sum_3$, which follows the pattern established for $\sum_2$. For $n \in \sigma_\mu$ we have

$$\sum_{m<\sigma_\mu} Q(n, m) \leqslant A\, n(n - b_\mu + r_\mu)^{-2\alpha},$$

where A is independent of n and μ. Since

$$\rho_\mu(n)^2 \leqslant 4(n - b_\mu + r_\mu)^2 r_\mu^{-2}$$

it follows that

$$\sum_{m<\sigma_\mu} \rho_\mu(n)^2 (n + \tfrac{1}{2})^{-1} Q(n, m) \leqslant A\,(n - b_\mu + r_\mu)^{2-2\alpha} r_\mu^{-2}.$$

Making use of the inequalities

$$(n - b_\mu + r_\mu)^{2-2\alpha} \leqslant A\,(n + 1)^{2-2\alpha} \qquad (n \in \sigma_\mu),$$

$$r_\mu \geqslant A\,(n + 1) \qquad (n \in \sigma_\mu),$$

we obtain

$$\sum_{m<\sigma_\mu} \rho_\mu(n)^2 (n + \tfrac{1}{2})\, Q(n, m) \leqslant A\,(n+1)^{-2\alpha} \qquad (n \in \sigma_\mu),$$

and thus

$$\textstyle\sum_3 \leqslant A \sum_{n\in\sigma_\mu} |a_n|^2 (n+1)^{-2\alpha}.$$

It remains to treat $\sum_1$. Since

$$a_n\rho_\mu(n)(n+\tfrac{1}{2})^{-\frac{1}{2}} - a_m\rho_\mu(m)(m+\tfrac{1}{2})^{-\frac{1}{2}} = [a_n(n+\tfrac{1}{2})^{-\frac{1}{2}} - a_m(m+\tfrac{1}{2})^{-\frac{1}{2}}]\,\rho_\mu(n) + a_m(m+\tfrac{1}{2})^{-\frac{1}{2}}[\rho_\mu(n) - \rho_\mu(m)],$$

and since $0 \leqslant \rho_\mu(n) \leqslant 1$, we have

$$\textstyle\sum_1 \leqslant 2 \sum_{\substack{n,m\in\sigma_\mu\\ n>m}} [a_n(n+\tfrac{1}{2})^{-\frac{1}{2}} - a_m(m+\tfrac{1}{2})^{-\frac{1}{2}}]^2 Q(n,m) + 2\sum_{\substack{n,m\in\sigma_\mu\\ n>m}} a_m^2 (m+\tfrac{1}{2})^{-1}[\rho_\mu(n) - \rho_\mu(m)]^2 Q(n,m).$$

We assert that

$$\sum_{n=m+1}^{b_\mu+r_\mu} (m+\tfrac{1}{2})^{-1}[\rho_\mu(n) - \rho_\mu(m)]^2 Q(n,m) \leqslant A\,(m+1)^{-2\alpha} \qquad (m \in \sigma_\mu).$$

To verify this we note that

$$\rho_\mu(n) - \rho_\mu(m) = -\,(n-m)(n+m-2b_\mu)\, r_\mu^{-2},$$

$$|\rho_\mu(n) - \rho_\mu(m)| \leqslant A\,(n-m)\, r_\mu^{-1}.$$

It follows that

$$\begin{aligned}\sum_{n=m+1}^{b_\mu+r_\mu} (m+\tfrac{1}{2})^{-1}[\rho_\mu(n) - \rho_\mu(m)]^2 Q(n,m) &\leqslant A r_\mu^{-2} \sum_{n=m+1}^{b_\mu+r_\mu} (n-m)^{1-2\alpha},\\ &\leqslant A\, r_\mu^{-2}\,(b_\mu + r_\mu - m)^{2-2\alpha},\\ &\leqslant A\,(m+1)^{-2\alpha},\end{aligned}$$

as desired. Thus

$$\sum_1 \leqslant 2 \sum_{\substack{n,m\in\sigma_\mu\\ n>m}} [a_n(n+\tfrac{1}{2})^{-\frac{1}{2}} - a_m(m+\tfrac{1}{2})^{-\frac{1}{2}}]^2 Q(n,m) + A \sum_{n\in\sigma_\mu} a_n^2 (n+1)^{-2\alpha}.$$

Summing and applying Theorems 3b and 4b, we have

$$\sum_{\mu=2}^{\infty} \mathfrak{N}_{0,\alpha}[E_\mu]^2 \leqslant A\, \mathfrak{N}_{0,\alpha}[f]^2.$$

and our lemma is proved except for the fact that μ starts at 2 instead of 0, which is evidently without significance.

Let S_μ be the set of integers $2^{\mu-1} \leqslant k < 2^\mu$, $\mu = 1, 2, \ldots$; S_0 is the integer $k = 0$.

LEMMA 5c. *If* $0 \leqslant \alpha < \frac{1}{2}$ *and if* $n_\mu \in S_\mu$ *then*

$$\sum_{\mu=0}^{\infty} \sum_{m \in S_\mu} a_m^2(|m - n_\mu| + 1)^{-2\alpha} \leqslant A(\alpha)\mathfrak{N}_{0,\alpha}[f]^2.$$

Proof. By Theorem 3b,

$$\sum_{m \in \sigma_\mu} \rho_\mu(m)^2 a_m^2(|m - n_\mu| + 1)^{-2\alpha} \leqslant A\, \mathfrak{N}_{0,\alpha}[E_\mu]^2.$$

For $m \in S_\mu$ $\rho_\mu(m) \geqslant A$ and thus

$$\sum_{m \in S_\mu} a_m^2(|m - n_\mu| + 1)^{-2\alpha} \leqslant A \sum_{m \in \sigma_\mu} \rho_\mu(m)^2 a_m^2(|m - n_\mu| + 1)^{-2\alpha},$$
$$\leqslant A\, \mathfrak{N}_{0,\alpha}[E_\mu]^2.$$

Summing over μ and using Lemma 5b we obtain our desired result. Note that a similar inequality holds for $\mathfrak{N}_{\alpha,0}$.

THEOREM 5d. *If*

(i) $f(x) \sim \sum_0^\infty a_n \omega_n(x)$ $\qquad f \in \mathfrak{N}_{0,\alpha}, 0 \leqslant \alpha < \frac{1}{2}$,

(ii) $T = \{t_n\}_0^\infty$ belongs to class $\mathbf{M}(C)$,

(iii) $Tf(x) \sim \sum_0^\infty t_n a_n \omega_n(x)$,

then

$$\mathfrak{N}_{0,\alpha}[Tf] \leqslant AC\, \mathfrak{N}_{0,\alpha}[f].$$

Proof. We set

$$\delta_\mu(x) = \sum_{n \in S_\mu} a_n t_n \omega_n(x)$$

for $\mu = 0, 1, 2, \ldots$. If

$$F_M(x) = \sum_{\mu=0}^{M} \delta_\mu(x),$$

then, by Lemma 4a, it is enough to prove that

$$\mathfrak{N}_{0,\alpha}[F_M] \leqslant AC\, \mathfrak{N}_{0,\alpha}[f] \qquad (M = 1, 2, \ldots).$$

We have

$$\mathfrak{N}_{0,\alpha}[F_M]^2 \simeq \int_{-1}^{1} \Omega_\alpha(x)[F_M(x)]^2 dx;$$

and since

$$\int_{-1}^{1} \Omega_\alpha(x)[F_M(x)]^2 dx = \sum_{\mu=0}^{M} \int_{-1}^{1} \Omega_\alpha(x)[\delta_\mu(x)]^2 dx$$
$$+ \sum_{\mu,\nu=0,\mu\neq\nu}^{M} \int_{-1}^{1} \Omega_\alpha(x)\, \delta_\mu(x)\, \delta_\nu(x)\, dx,$$

it is sufficient to show that

5.1 $$\sum_{\mu,\,\nu=0,\,\mu\neq\nu}^{\infty}\left|\int_{-1}^{1}\Omega_\alpha(x)\,\delta_\mu(x)\,\delta_\nu(x)\,dx\right| \leqslant AC^2\mathfrak{N}_{0,\alpha}[f]^2,$$

and

5.2 $$\sum_{\mu=0}^{\infty}\int_{-1}^{1}\Omega_\alpha(x)[\delta_\mu(x)]^2dx \leqslant AC^2\mathfrak{N}_{0,\alpha}[f]^2.$$

Let us set

$$I_{\nu,\mu} = \left|\int_{-1}^{1}\Omega_\alpha(x)\,\delta_\mu(x)\,\delta_\nu(x)\,dx\right| \qquad (\mu\neq\nu);$$

then

$$I_{\nu,\mu} = \left|\sum_{n\in S_\nu,\,m\in S_\mu} -\,a_n a_m t_n t_m (n+\tfrac{1}{2})^{-\frac{1}{2}}(m+\tfrac{1}{2})^{-\frac{1}{2}}c_{m,n}\right|,$$

and thus by Lemma 2b,

$$I_{\nu,\mu} \leqslant AC^2\sum_{n\in S_\nu,\,m\in S_\mu}|a_n a_m|(n+\tfrac{1}{2})^{-\frac{1}{2}}(m+\tfrac{1}{2})^{-\frac{1}{2}}Q(n,m),$$

$$\leqslant AC^2\sum_{n\in S_\nu,\,m\in S_\mu}[|a_n|^2+|a_m|^2](n+\tfrac{1}{2})^{-\frac{1}{2}}(m+\tfrac{1}{2})^{-\frac{1}{2}}Q(n,m).$$

From this we obtain

$$\sum_{\mu,\,\nu=0,\,\mu\neq\nu}^{\infty} I_{\mu,\nu} \leqslant AC^2\sum_{\mu=0}^{\infty}\sum_{m\in S_\mu}|a_m|^2\sum_{\substack{\nu=0\\ \nu\neq\mu}}^{\infty}\sum_{n\in S}(m+\tfrac{1}{2})^{-\frac{1}{2}}(n+\tfrac{1}{2})^{-\frac{1}{2}}Q(m,n).$$

Now

$$(n+\tfrac{1}{2})^{-\frac{1}{2}}(m+\tfrac{1}{2})^{-\frac{1}{2}}Q(n,m) \leqslant A|n-m|^{-1-2\alpha},$$

and if $m\in S_\mu$ then

$$\sum_{\nu=0,\,\nu\neq\mu}^{\infty}|n-m|^{-1-2\alpha} \leqslant A[(m-n_\mu'+1)^{-2\alpha}+(n_\mu''-m+1)^{-2\alpha}],$$

where n_μ' and n_μ'' are the first and last points of S_μ. Applying Lemma 5a it follows that

$$\sum_{\substack{\mu,\,\nu=0\\ \mu\neq\nu}}^{\infty} I_{\mu,\nu} \leqslant AC^2\sum_{\mu=0}^{\infty}\sum_{m\in S_\mu}|a_m|^2[(|m-n_\mu'|+1)^{-2\alpha}+(|m-n_\mu''|+1)^{-2\alpha}],$$

$$\leqslant AC^2\mathfrak{N}_{0,\alpha}[f]^2,$$

and 5.1 is seen to hold.

Let us next consider

$$\int_{-1}^{1}\Omega_\alpha(x)[\delta_\mu(x)]^2dx.$$

For μ fixed we set

$$s(n,x) = \sum_{b_\mu-r_\mu}^{n} a_m\rho_\mu(m)\,\omega_m(x).$$

https://doi.org/10.4153/CJM-1955-050-x Published online by Cambridge University Press

It follows from Lemma 5a that

5.3 $$\mathfrak{N}_{0,\alpha}[s(n, x)] \leqslant A\,\mathfrak{N}_{0,\alpha}[E_\mu].$$

We have, if $u(n) = t_n/\rho_\mu(n)$,

$$\delta_\mu(x) = \sum_{n \in S_\mu} u(n)[s(n, x) - s(n-1, x)].$$

Summing by parts we find that

$$\delta_\mu(x) = \sum_{n \in S_\mu} s(n, x)\,[u(n) - u(n+1)] + u(2^\mu)\,s(2^\mu - 1, x) - u(2^{\mu-1})\,s(2^{\mu-1} - 1, x),$$

from which using 5.3 it follows that

$$\mathfrak{N}_{0,\alpha}[\delta_\mu] \leqslant A\,\mathfrak{N}_{0,\alpha}[E_\mu]\{\sum_{n \in S_\mu} |u(n) - u(n+1)| + |u(2^\mu)| + |u(2^{\mu-1})|\}.$$

Now it is easily verified that

$$\sum_{n \in S_\mu} |u(n) - u(n+1)| + |u(2^\mu)| + |u(2^{\mu-1})| \leqslant AC$$

and thus

$$\mathfrak{N}_{0,\alpha}[\delta_\mu] \leqslant AC\,\mathfrak{N}_{0,\alpha}[E_\mu].$$

Squaring and summing over μ we see using Lemma 5b that 5.2 holds.

6. Multiplier transformations, continued. Let $P(\beta, \alpha)$ stand for the proposition that if $T \in \mathbf{M}(C)$ then $\mathfrak{N}_{\beta,\alpha}[Tf] \leqslant AC\,\mathfrak{N}_{\beta,\alpha}[f]$ where A depends only on α and β. Theorem 5d shows that $P(0, \alpha)$ is valid if $0 \leqslant \alpha < \frac{1}{2}$. We wish to show that $P(\beta, \alpha)$ is valid for $(-\frac{1}{2} < \beta, \alpha < \frac{1}{2})$. We begin with two general principles.

LEMMA 6a. *If $P(\beta, \alpha)$ is valid so is $P(\alpha, \beta)$.*

Proof. Let $f(x) \in \mathfrak{N}_{\alpha,\beta}$; then, if $F(x) = f(-x)$,

$$\mathfrak{N}_{\alpha,\beta}[f] = \mathfrak{N}_{\beta,\alpha}[F], \qquad \mathfrak{N}_{\alpha,\beta}[Tf] = \mathfrak{N}_{\beta,\alpha}[TF].$$

LEMMA 6b. *If $P(\beta, \alpha)$ is valid so is $P(-\beta, -\alpha)$.*

Proof. Let $f(x) \in \mathfrak{N}_{\beta,\alpha}$. Let

$$g(x) = \sum_0^r b_k \omega_k(x).$$

We have

$$\left| \int_{-1}^1 [Tf(x)]\,g(x)\,dx \right| \leqslant \mathfrak{N}_{\beta,\alpha}[Tf]\,\mathfrak{N}_{-\beta,-\alpha}[g] \leqslant AC\,\mathfrak{N}_{\beta,\alpha}[f]\,\mathfrak{N}_{-\beta,-\alpha}[g],$$

and since

$$\int_{-1}^1 [Tf(x)]\,g(x)\,dx = \int_{-1}^1 f(x)[Tg(x)]\,dx,$$

https://doi.org/10.4153/CJM-1955-050-x Published online by Cambridge University Press

this can be rewritten as

$$\left| \int_{-1}^{1} f(x)[Tg(x)]\, dx \right| \leqslant CA\, \mathfrak{N}_{\beta,\alpha}[f]\, \mathfrak{N}_{-\beta,-\alpha}[g],$$

which, since it is true for every $f \in \mathfrak{N}_{\beta,\alpha}$, implies that $\mathfrak{N}_{-\beta,-\alpha}[Tg] \leqslant CA\, \mathfrak{N}_{-\beta,-\alpha}[g]$. An evident approximation argument enables us to remove the restriction that $g(x)$ be a finite sum of Legendre polynomials.

Lemmas 6a and 6b together imply that $P(0, \alpha)$ and $P(\beta, 0)$ are true for $-\frac{1}{2} < \alpha, \beta < \frac{1}{2}$.

LEMMA 6c. *$P(\beta, \alpha)$ is valid if $-\frac{1}{2} < \alpha, \beta \leqslant 0$.*

Proof. Let $f \in \mathfrak{N}_{\beta,\alpha}$. We have

$$\begin{aligned} \mathfrak{N}_{\beta,\alpha}[Tf] &\leqslant A\, \mathfrak{N}_{0,\alpha}[Tf] + A\, \mathfrak{N}_{\beta,0}[Tf], \\ &\leqslant CA\, \mathfrak{N}_{0,\alpha}[f] + CA\, \mathfrak{N}_{\beta,0}[f], \end{aligned}$$

where we have used the fact that $P(0, \alpha)$, $P(\beta, 0)$ are true for $-\frac{1}{2} < \alpha, \beta \leqslant 0$. Since

$$\mathfrak{N}_{0,\alpha}[f] \leqslant A\, \mathfrak{N}_{\beta,\alpha}[f], \qquad \mathfrak{N}_{\beta,0}[f] \leqslant A\, \mathfrak{N}_{\beta,\alpha}[f],$$

we have

$$\mathfrak{N}_{\beta,\alpha}[Tf] \leqslant CA\, \mathfrak{N}_{\beta,\alpha}[f],$$

as desired.

Lemma 6c in conjunction with our previous results shows that $P(\beta, \alpha)$ is valid if $-\frac{1}{2} < \alpha, \beta < \frac{1}{2}$, and if $\alpha\beta \geqslant 0$; that is if α and β are of the same sign. The case where α and β are of different signs is slightly more difficult.

LEMMA 6d. *If $0 \leqslant \alpha, \beta < \frac{1}{2}$, if $T \in \mathbf{M}(C)$, and if*

$$F(x) = (1 - x)\, T[f(x)] - T[(1 - x) f(x)]$$

then

$$\mathfrak{N}_{-\beta,0}[F] \leqslant AC\, \mathfrak{N}_{0,\alpha}[f]$$

where A depends only on α and β.

Proof. It is enough to prove this in the case where $f(x)$ is a finite sum of Legendre polynomials. We have

$$f(x) = \sum_{0}^{\infty} a_k \omega_k(x) = \sum_{0}^{\infty} R_k P_k(x), \qquad R_k = a_k(k + \tfrac{1}{2})^{\frac{1}{2}}.$$

Since (**9**, p. 308),

$$(1 - x)\, P_n(x) = -\, (n + 1)(2n + 1)^{-1} P_{n+1}(x) + P_n(x) - n(2n + 1)^{-1} P_{n-1}(x),$$

we find, after a short computation, that $F(x) = F_1(x) + F_2(x)$ where

$$\begin{aligned} F_1(x) &= \sum_{0}^{\infty} (n + 1)(2n + 1)^{-1} R_n[-t_n + t_{n+1}]\, P_{n+1}(x), \\ F_2(x) &= \sum_{0}^{\infty} n(2n + 1)^{-1} R_n[-t_n + t_{n-1}]\, P_{n-1}(x). \end{aligned}$$

Note that these series are only formally infinite. Let $g(x) \in \mathfrak{N}_{\beta,0}$ and let

$$g(x) \sim \sum_0^\infty b_k \omega_k(x) = \sum_0^\infty R_k' P_k(x) R_k' = b_k(k + \tfrac{1}{2})^{\frac{1}{2}}.$$

We have

$$\int_{-1}^{1} F_1(x)\, g(x)\, dx = 2\sum_0^\infty (n+1)(2n+1)^{-1}(2n+1)^{-1} R_n R_{n+1}'[-t_n + t_{n+1}],$$

$$\left| \int_{-1}^{1} F(x)\, g(x)\, dx \right| \leqslant A \sum_{\mu=0}^{\infty} \sum_{n \in S_\mu} |a_n b_{n+1}| |t_n - t_{n+1}|.$$

If $\alpha_\mu =$ l.u.b. $|a_n|$ for $n \in S_\mu$ and $\beta_\mu =$ l.u.b. $|b_n|$ for $n \in S_\mu$ then (see the remark following the proof of Lemma 5c)

$$\begin{aligned} \left| \int_{-1}^{1} F_1(x)\, g(x)\, dx \right| &\leqslant A \sum_{\mu=0} \alpha_\mu(\beta_\mu + \beta_{\mu+1}) \sum_{n \in S_\mu} |t_n - t_{n+1}| \\ &\leqslant AC \sum_{\mu=0}^{\infty} \alpha_\mu(\beta_\mu + \beta_{\mu+1}) \leqslant AC\left(\sum_{\mu=0}^{\infty} \alpha_\mu^2\right)^{\frac{1}{2}} \left(\sum_{\mu=0}^{\infty} \beta_\mu^2\right)^{\frac{1}{2}} \\ &\leqslant AC\, \mathfrak{N}_{0,\alpha}[f]\, \mathfrak{N}_{\beta,0}[g]. \end{aligned}$$

Since this is true for every $g \in \mathfrak{N}_{\beta,0}$ it implies that $\mathfrak{N}_{-\beta,0}[F_1] \leqslant AC\, \mathfrak{N}_{0,\alpha}[f]$. Similarly we can show that $\mathfrak{N}_{-\beta,0}[F_2] \leqslant AC\, \mathfrak{N}_{0,\alpha}[f]$.

Lemma 6e. *$P(\beta, \alpha)$ is valid if $-\frac{1}{2} < \beta \leqslant 0 \leqslant \alpha < \frac{1}{2}$.*

Proof. We have

$$\mathfrak{N}_{\beta,\alpha}[Tf] \leqslant A\, \mathfrak{N}_{0,\alpha}[Tf] + A\, \mathfrak{N}_{\beta,0}[(1-x)\, Tf].$$

Since $P(0, \alpha)$ is valid

$$\mathfrak{N}_{0,\alpha}[Tf] \leqslant AC\, \mathfrak{N}_{0,\alpha}[f].$$

If $F(x)$ is defined as in Lemma 6d, then

$$\begin{aligned} \mathfrak{N}_{\beta,0}[(1-x)\, Tf] &= \mathfrak{N}_{\beta,0}[T\{(1-x) f(x)\} + F(x)] \\ &\leqslant \mathfrak{N}_{\beta,0}[T\{(1-x) f(x)\}] + \mathfrak{N}_{\beta,0}[F(x)]. \end{aligned}$$

By $P(\beta, 0)$

$$\begin{aligned} \mathfrak{N}_{\beta,0}[T\{(1-x) f(x)\}] &\leqslant AC\, \mathfrak{N}_{\beta,0}[(1-x) f(x)] \\ &\leqslant AC\, \mathfrak{N}_{\beta,\alpha}[f(x)]. \end{aligned}$$

Since $f \in \mathfrak{N}_{0,\alpha}$, Lemma 6d implies that

$$\mathfrak{N}_{\beta,0}[F(x)] \leqslant AC\, \mathfrak{N}_{0,\alpha}[f] \leqslant AC\, \mathfrak{N}_{\beta,\alpha}[f].$$

Thus combining these results

$$\mathfrak{N}_{\beta,\alpha}[Tf] \leqslant AC\, \mathfrak{N}_{\beta,\alpha}[f],$$

as desired.

Our lemmas yield

Theorem 6f. *$P(\alpha, \beta)$ is valid for $-\frac{1}{2} < \alpha, \beta < \frac{1}{2}$.*

It is to be noted that there is another general principle which we have not invoked. It is easily shown by arguments like those used in the Riesz-Thorin convexity theorem that if $P(\alpha', \beta')$ and $P(\alpha'', \beta'')$ are valid then so is $P(\alpha, \beta)$ where $\alpha = \alpha'(1 - \theta) + \alpha''\theta$, $\beta = \beta'(1 - \theta) + \beta''\theta$, $0 \leqslant \theta \leqslant 1$. However this would not have shortened our arguments.

We shall now indicate briefly and without going into detail how it can be seen that the restriction $-\frac{1}{2} < \alpha, \beta < \frac{1}{2}$ is essential in Theorem 6f. If Theorem 6f were valid for α and β not satisfying the above restriction then 1.2 of §1 would hold. That this is impossible can be seen using the methods of Newman and Rudin **(5)**.

7. Fractional integration. Let $f(x) \in \mathfrak{N}_{\alpha,\beta}$ where $-\frac{1}{2} < \alpha, \beta < \frac{1}{2}$ and let $f(x)$ have mean value zero so that

$$f(x) \sim \sum_{1}^{\infty} a_n \omega_n(x).$$

We set

$$I(\sigma) f(x) \sim \sum_{1}^{\infty} [n(n+1)]^{-\sigma/2} a_n \omega_n(x).$$

$I(\sigma)$ is a fractional integration operator if σ is positive and a fractional differentiation operator if σ is negative. Note that

$$I(-2) f(x) = [(x^2 - 1) f'(x)]'.$$

In the present section we shall apply our results to the study of $I(\sigma)$.

THEOREM 7a. *Let $f(x)$ have mean value zero. If*

(i) $-\frac{1}{2} < \alpha_1, \beta_1 < \frac{1}{2}$, $-\frac{1}{2} < \alpha_2, \beta_2 < \frac{1}{2}$,
(ii) $0 < \theta < 1$,
(iii) $\sigma = \sigma_2\theta$, $\alpha = (1 - \theta)\alpha_1 + \theta\alpha_2$, $\beta = (1 - \theta)\beta_1 + \theta\beta_2$,

then

$$\mathfrak{N}_{\alpha,\beta}[I(\sigma) f] \leqslant A\, \mathfrak{N}_{\alpha_1,\beta_1}[f]^{1-\theta}\, \mathfrak{N}_{\alpha_2,\beta_2}[I(\sigma_2) f]^{\theta}$$

where A depends only upon $\alpha_1, \beta_1, \alpha_2, \beta_2, \sigma_2$ and θ, but not upon f.

Using Theorem 6f we can show that, if $f(x)$ has mean value zero and if $-\frac{1}{2} < \alpha, \beta < \frac{1}{2}$, then

$$\mathfrak{N}_{\alpha,\beta}[I(i\tau) f] \leqslant A(\tau)\, \mathfrak{N}_{\alpha,\beta}[f]$$

where $A(\tau) = 0(|\tau|)$ as $\tau \to \pm\infty$. Note that although Theorem 6f is stated for real multipliers, it evidently holds for complex multipliers as well. This can be seen by decomposing into real and imaginary parts. The inequality we have just established asserts that "fractional integration" of purely imaginary order is a bounded transformation of $\mathfrak{N}_{\alpha,\beta}$ into itself. It now follows from a general theorem on groups of multiplier transformations that this property of fractional integration of purely imaginary order implies **(3; 11)** the conclusion of Theorem 7a.

https://doi.org/10.4153/CJM-1955-050-x Published online by Cambridge University Press

We shall now use Theorem 7a to prove an analogue of a classical theorem of Hardy and Littlewood **(10**, p. 227**)**. The method employed here could also be used to give a new demonstration of their result.

THEOREM 7b. *Let $f(x)$ have mean value zero. If*

(i) $\alpha_1 = \alpha - \sigma, \beta_1 = \beta - \sigma \qquad (\sigma > 0),$

(ii) $\alpha, \beta < \frac{1}{2}, -\frac{1}{2} < \alpha_1, \beta_1,$

then

$$\mathfrak{N}_{\beta_1\alpha_1,}[I(\sigma) f] \leqslant A\ \mathfrak{N}_{\beta,\alpha}[f],$$

where A depends only upon α, β and σ but not upon f.

Proof. It is enough to prove this for $f(x)$ a finite sum of Legendre polynomials. We first assert that if $-\frac{1}{2} < \alpha < \frac{1}{2}$ then

7.1 $$\mathfrak{N}_{0,\alpha_1}[I(\sigma) f] \leqslant A\ \mathfrak{N}_{0,\alpha}[f],$$

where $\sigma > 0$, $\alpha_1 = \alpha - \sigma$, $\alpha_1 > -\frac{1}{2}$. Suppose first that $\alpha_1 \leqslant 0 \leqslant \alpha$. Let $\mathfrak{N}_{0,-\alpha_1}[g] \leqslant 1$, where

$$g(x) \sim \sum_0^\infty b_n \omega_n(x).$$

We have

$$\begin{aligned}\left| \int_{-1}^1 [I(\sigma) f(x)]\, g(x)\, dx \right| &= \left| \sum_1^\infty a_n [n(n+1)]^{-\frac{1}{2}\sigma} b_n \right| \\ &\leqslant A \left[\sum_1^\infty a_n^2 (n+1)^{-2\alpha} \right]^{\frac{1}{2}} \left[\sum_1^\infty b_n^2 (n+1)^{+2\alpha_1} \right]^{\frac{1}{2}} \\ &\leqslant A\ \mathfrak{N}_{0,\alpha}[f]\, \mathfrak{N}_{0,-\alpha_1}[g] \\ &\leqslant A\ \mathfrak{N}_{0,\alpha}[f].\end{aligned}$$

Here we have used Theorem 3b. Since this is true for an arbitrary g such that $\mathfrak{N}_{0,-\alpha_1}[g] \leqslant 1$, it implies 7.1 for $-\frac{1}{2} < \alpha_1 \leqslant 0 \leqslant \alpha < \frac{1}{2}$.

Next assume that $0 \leqslant \alpha_1 < \alpha < \frac{1}{2}$, where $\alpha_1 = \alpha - \sigma$. We have from the case already considered,

$$\mathfrak{N}_{0,0}[I(\alpha) f] \leqslant A\ \mathfrak{N}_{0,\alpha}[f].$$

By Theorem 7a, if $\sigma = \theta\alpha$

$$\mathfrak{N}_{0,\alpha_1}[I(\sigma) f] \leqslant A\ \mathfrak{N}_{0,\alpha}[f]^{1-\theta} \mathfrak{N}_{0,0}[I(\alpha) f]^\theta \leqslant A\ \mathfrak{N}_{0,\alpha}[f].$$

Thus 7.1 is true if $0 \leqslant \alpha_1 < \alpha < \frac{1}{2}$. If $-\frac{1}{2} < \alpha_1 < \alpha \leqslant 0$, let $\mathfrak{N}_{0,-\alpha_1}[g] \leqslant 1$. We have

$$\left| \int_{-1}^1 [I(\sigma) f(x)]\, g(x)\, dx \right| = \left| \int_{-1}^1 f(x) [I(\sigma)\, g(x)]\, dx \right| \leqslant \mathfrak{N}_{0,\alpha}[f]\, \mathfrak{N}_{0,-\alpha}[I(\sigma)\, g(x)].$$

https://doi.org/10.4153/CJM-1955-050-x Published online by Cambridge University Press

But

$$\mathfrak{N}_{0,-\alpha}[I(\sigma)\, g(x)] \leqslant A\, \mathfrak{N}_{0,-\alpha_1}[g(x)] \leqslant A$$

(because $-\alpha = -\alpha_1 - \sigma$, $0 \leqslant -\alpha < -\alpha_1 < \frac{1}{2}$), so that

$$\left| \int_{-1}^{1} [I(\sigma) f(x)]\, g(x)\, dx \right| \leqslant A\, \mathfrak{N}_{0,\alpha}[f].$$

Since this is true for every g with $\mathfrak{N}_{0,-\alpha_1}[g] \leqslant 1$, it implies 7.1 for $-\frac{1}{2} < \alpha_1 < \alpha \leqslant 0$. Thus, finally, 7.1 is true if $-\frac{1}{2} < \alpha_1 < \alpha < \frac{1}{2}$. Similarly we can show that if $-\frac{1}{2} < \beta_1 < \beta < \frac{1}{2}$, $\beta_1 = \beta - \sigma$ $(\sigma > 0)$, then

7.2 $$\mathfrak{N}_{\beta_1,0}[I(\sigma) f] \leqslant A\, \mathfrak{N}_{\beta,0}[f].$$

Let us set

$$f_1(x) = \tfrac{1}{2}[1 - \operatorname{sgn} x] f(x),$$
$$f_2(x) = \tfrac{1}{2}[1 + \operatorname{sgn} x] f(x).$$

We have from 7.2 that

7.3 $$\mathfrak{N}_{\beta_1,0}[I(\sigma) f_1] \leqslant A\, \mathfrak{N}_{\beta,0}(f_1] \leqslant A\, \mathfrak{N}_{\beta,\alpha}[f].$$

Since $I(\sigma) \in \mathbf{M}(C)$ (for some C) Theorem 6f gives

7.4 $$\mathfrak{N}_{\beta,\alpha_1}[I(\sigma) f_1] \leqslant A\, \mathfrak{N}_{\beta,\alpha_1}[f_1] \leqslant A\, \mathfrak{N}_{\beta,\alpha}[f].$$

The inequalities (3) and (4) together yield

7.5 $$\begin{aligned}\mathfrak{N}_{\beta_1,\alpha_1}[I(\sigma) f_1] &\leqslant A\, \mathfrak{N}_{\beta_1,0}[I(\sigma) f_1] + A\, \mathfrak{N}_{\beta,\alpha_1}[I(\sigma) f_1] \\ &\leqslant A\, \mathfrak{N}_{\beta,\alpha}[f].\end{aligned}$$

Similarly from 7.1 we have that

7.6 $$\mathfrak{N}_{0,\alpha_1}[I(\sigma) f_2] \leqslant A\, \mathfrak{N}_{0,\alpha}[f_2] \leqslant A\, \mathfrak{N}_{\beta,\alpha}[f].$$

Again, since $I(\sigma) \in \mathbf{M}(C)$ (for some C),

7.7 $$\mathfrak{N}_{\beta_1,\alpha}[I(\sigma) f_2] \leqslant A\, \mathfrak{N}_{\beta_1,\alpha}[f_2] \leqslant A\, \mathfrak{N}_{\beta,\alpha}[f],$$

and hence

7.8 $$\begin{aligned}\mathfrak{N}_{\beta_1,\alpha_1}[I(\sigma) f_2] &\leqslant A\, \mathfrak{N}_{0,\alpha_1}[I(\sigma) f_2] + A\, \mathfrak{N}_{\beta_1\alpha}[I(\sigma) f_2] \\ &\leqslant A\, \mathfrak{N}_{\beta,\alpha}[f].\end{aligned}$$

Making use of 7.5 and 7.8, we obtain

$$\begin{aligned}\mathfrak{N}_{\beta_1,\alpha_1}[I(\sigma) f] &\leqslant \mathfrak{N}_{\beta_1,\alpha_1}[I(\sigma) f_1] + \mathfrak{N}_{\beta_1,\alpha_1}[I(\sigma) f_2] \\ &\leqslant A\, \mathfrak{N}_{\beta,\alpha}[f],\end{aligned}$$

as desired.

References

1. E. T. Copson, *An introduction to the theory of functions of a complex variable* (Oxford, 1935).
2. I. I. Hirschman, Jr., *A note on orthogonal functions* (to appear in Pacific J. Math.).
3. ———, *The decomposition of Walsh and Fourier series*, No. 15 in the Memoirs of the American Mathematical Society.
4. J. Marcinkiewicz, *Sur les multiplicateurs dés series de Fourier*, Studia Math., *8* (1939), 78–91.
5. Jerome Newman and Walter Rudin, *Mean convergence of orthogonal series*, Proc. Amer. Math. Soc., *3* (1952), 219–222.
6. H. Pollard, *The mean convergence of orthogonal series*, Trans. Amer. Math. Soc., *6* (1947), 387–403.
7. M. F. Rosskopf, *Some inequalities for non-uniformly bounded orthonormal polynomials*, Trans. Amer. Math. Soc., *36* (1934), 853–867.
8. G. Szegö, *Orthogonal Polynomials* (New York, 1939).
9. E. T. Whittaker and G. N. Watson, *A Course of modern analysis* (Cambridge, 1952).
10. A. Zygmund, *Trigonometrical series* (Warsaw-Lwow, 1935).
11. I. I. Hirschman, Jr., *A convexity theorem for certain groups of transformations*, J. d'Analyse Math., *2* (1952), 209–218.

Washington University, St. Louis