

On common fixed point theorems

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The aim of this paper is to extend a recent result of S.A. Husain and V.M. Sehgal [*Bull. Austral. Math. Soc.* 13 (1975), 261-267]. The condition that the function should be continuous in Husain and Sehgal, *op. cit.*, is replaced by a semicontinuity condition. Moreover, a different proof is given, the last part of which requires no continuity or semicontinuity condition whatsoever.

In a recent paper Husain and Sehgal [1] proved some results on fixed and common fixed points under the condition of continuity of the function. In this paper the theorem has been proved under considerably weaker conditions.

Let R^+ denote the set of nonnegative reals. Let ψ denote a family of mappings such that each $\phi \in \psi$, $\phi : (R^+)^5 \rightarrow R^+$, and ϕ is upper semicontinuous and nondecreasing in each coordinate variable. Also let $\gamma(t) = \phi(t, t, a_1 t, a_2 t, t)$ where γ is a function $\gamma : R^+ \rightarrow R^+$ where $a_1 + a_2 = 3$.

LEMMA 1. For every $t > 0$, $\gamma(t) < t$ if and only if

$$\lim_{n \rightarrow \infty} \gamma^n(t) = 0.$$

Proof (Necessity). Since ϕ is upper semicontinuous, then γ is upper semicontinuous. Assume $\lim_{n \rightarrow \infty} \gamma^n(t) = A$ where $A \neq 0$. Then

$$A = \lim_{n \rightarrow \infty} \gamma^{n+1}(t) \leq \gamma \lim_{n \rightarrow \infty} \gamma^n(t) = \gamma(A) < A;$$

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that is, $A < A$, a contradiction. Therefore $A = 0$.

(Sufficiency). Since ϕ is nondecreasing, then γ is nondecreasing. Given $\lim_{n \rightarrow \infty} \gamma^n(t) = 0$, assume $\gamma(t) > t$ for some $t > 0$. Then $\gamma^n(t) > t$ for some $t > 0$ for $n = 1, 2, 3, \dots$. Thus $\lim_{n \rightarrow \infty} \gamma^n(t) \not\rightarrow 0$, a contradiction. Also if $\gamma(t) = t$ for some $t > 0$, then $\lim_{n \rightarrow \infty} \gamma^n(t) \not\rightarrow 0$. Hence, for all $t > 0$, $\gamma(t) < t$.

THEOREM 1. *Let (X, d) be a complete metric space and let S and T be selfmappings of X . Suppose there exists a $\phi \in \Psi$ such that for all $x, y \in X$,*

$$(1) \quad d(S(x), T(y)) \leq \phi(d(x, y), d(x, S(x)), d(x, T(y)), d(y, S(x)), d(y, T(y)))$$

where ϕ satisfies the condition: for any $t > 0$,

$$(2) \quad \phi(t, t, a_1t, a_2t, t) < t \text{ where } a_1 + a_2 = 3.$$

Then there exists a $z \in X$ such that z is a unique common fixed point of S and T .

Proof. Since $\gamma(t) = \phi(t, t, t, 2t, t) < t$, then, by Lemma 1, $\lim_{n \rightarrow \infty} \gamma^n(t) = 0$. Now let $x_0 \in X$ be any point. Then define a sequence of iterates $\{x_n\}$ in the following way:

$$x_1 = S(x_0), x_2 = T(x_1), x_3 = S(x_2) \dots x_{2n} = T(x_{2n-1}), x_{2n+1} = S(x_{2n}), \dots$$

Claim. $d(x_1, x_2) \leq d(x_0, x_1)$.

Assume $d(x_0, x_1) < d(x_1, x_2)$. Then using the triangular inequality, $d(x_0, x_2) < 2d(x_1, x_2)$. Let $r = d(x_1, x_2)$. Then

$$r \leq \phi(d(x_0, x_1), d(x_1, x_2), d(x_1, x_1), d(x_0, x_2), d(x_0, x_1)) < \phi(r, r, r, 2r, r).$$

By (2), $\phi(r, r, r, 2r, r) < r$ and thus $r < r$, a contradiction: therefore, $d(x_1, x_2) \leq d(x_0, x_1)$ and

$$d(x_1, x_2) \leq \phi(d(x_0, x_1), d(x_0, x_1), 0, 2d(x_0, x_1), d(x_0, x_1)) \leq \gamma d(x_0, x_1) .$$

Similarly, $d(x_2, x_3) \leq \gamma(d(x_1, x_2)) \leq \gamma^2(d(x_0, x_1))$ and in general,

$$d(x_n, x_{n+1}) \leq \gamma^n(d(x_0, x_1)) . \text{ Since } \lim_{n \rightarrow \infty} \gamma^n(t) = 0 \text{ for } t > 0 ;$$

therefore,

$$(3) \quad \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0 .$$

In order to show $\{x_n\}$ is Cauchy, it is sufficient to show that $\{x_{2n}\}$ is Cauchy. Suppose that $\{x_{2n}\}$ is not a Cauchy sequence. Then there is an $\epsilon > 0$ such that for each integer $2k, k \in I^+$, there exist integers $2n(k)$ and $2m(k)$ with $2k \leq 2n(k) < 2m(k)$ such that

$$(4) \quad d(x_{2n(k)}, x_{2m(k)}) < \epsilon .$$

Let, for each integer $2k, k \in I^+$, $2m(k)$ be the least integer exceeding $2n(k)$ satisfying (4); that is, $d(x_{2n(k)}, x_{2m(k)-2}) \leq \epsilon$ and

$$(5) \quad d(x_{2n(k)}, x_{2m(k)}) > \epsilon .$$

Let $d_n = d(x_n, x_{n+1})$. Then for each integer $2k, k \in I^+$,

$$\epsilon < d(x_{2n(k)}, x_{2m(k)}) \leq d(x_{2n(k)}, x_{2m(k)-2}) + d_{2m(k)-2} + d_{2m(k)-1} .$$

By (3) and (5),

$$(6) \quad d(x_{2n(k)}, x_{2m(k)}) \rightarrow \epsilon \text{ as } k \rightarrow \infty .$$

It follows from the triangular inequality that

$$|d(x_{2n(k)}, x_{2m(k)-1}) - d(x_{2n(k)}, x_{2m(k)})| \leq d_{2m(k)-1}$$

and

$$|d(x_{2n(k)+1}, x_{2m(k)-1}) - d(x_{2n(k)}, x_{2m(k)})| \leq d_{2m(k)-1} + d_{2n(k)} .$$

By (6), as $k \rightarrow \infty$,

$$d(x_{2n(k)}, x_{2m(k)-1}) \rightarrow \varepsilon$$

and

$$d(x_{2n(k)+1}, x_{2m(k)-1}) \rightarrow \varepsilon .$$

Now let

$$p(2k) = d(x_{2n(k)}, x_{2m(k)}) ,$$

$$q(2k) = d(x_{2n(k)}, x_{2m(k)-1}) ,$$

and

$$r(2k) = d(x_{2n(k)+1}, x_{2m(k)-1}) .$$

Then

$$p(2k) \leq d_{2n(k)} + d(S(x_{2n(k)}), T(x_{2m(k)-1})) .$$

By (1),

$$p(2k) < d_{2n(k)} + \phi(q(2k), d_{2n(k)}, p(2k), r(2k), d_{2m(k)-1}) .$$

Since ϕ is upper semicontinuous, as $n \rightarrow \infty$, it follows that $\varepsilon \leq \phi(\varepsilon, 0, \varepsilon, \varepsilon, 0) \leq \phi(\varepsilon, \varepsilon, \varepsilon, 2\varepsilon, \varepsilon) < \varepsilon$, a contradiction. Therefore $\{x_n\}$ is a Cauchy sequence and hence by completeness, there is a $z \in X$ such that $x_n \rightarrow z$. We show that z is a common fixed point of S and T .

Since $\{x_n\}$ converges to z , therefore $\{x_{2n}\}$ and $\{x_{2n+1}\}$ both converge to z . Let $d(S(z), z) = \varepsilon > 0$. Thus we have, for $n \geq N$, $d(x_{2n}, z) < \frac{\varepsilon - \gamma(\varepsilon)}{4}$ and $d(x_{2n}, x_{2n+1}) \leq \frac{\varepsilon - \gamma(\varepsilon)}{4}$. Therefore using the triangular inequality,

$$d(z, x_{2n-1}) \leq d(z, x_{2n}) + d(x_{2n}, x_{2n-1}) < \frac{\varepsilon - \gamma(\varepsilon)}{4} + \frac{\varepsilon - \gamma(\varepsilon)}{4} = \frac{\varepsilon - \gamma(\varepsilon)}{2} ;$$

that is,

$$(7) \quad d(z, x_{2n-1}) < \varepsilon .$$

Using the triangular inequality and (7) we have

$$(8) \quad d(x_{2n-1}, S(z)) \leq d(x_{2n-1}, z) + d(z, S(z)) < \varepsilon + \varepsilon ;$$

that is, $d(x_{2n-1}, S(z)) < 2\varepsilon$.

Now

$$\varepsilon = d(S(z), z) \leq d(S(z), T(x_{2n-1})) + d_{2n} + d(x_{2n+1}, z) .$$

By (1),

$$\varepsilon \leq \phi(d(z, x_{2n-1}), d(z, S(z)), d(z, x_{2n}), d(x_{2n-1}, S(z)), d_{2n-1}) + \\ + d_{2n} + d(x_{2n+1}, z) .$$

By (7) and (8), $\varepsilon \leq \phi(\varepsilon, \varepsilon, \varepsilon, 2\varepsilon, \varepsilon) + \frac{\varepsilon - \gamma(\varepsilon)}{4} + \frac{\varepsilon - \gamma(\varepsilon)}{4}$; that is,

$\varepsilon \leq \gamma(\varepsilon) + \frac{\varepsilon - \gamma(\varepsilon)}{2} = \frac{\gamma(\varepsilon) + \varepsilon}{2}$. Since $\gamma(\varepsilon) < \varepsilon$, then $\frac{\gamma(\varepsilon) + \varepsilon}{2} < \varepsilon$; that is, $\varepsilon < \varepsilon$, a contradiction. Therefore $z = S(z)$. Similarly $z = T(z)$.

It remains to show that z is a unique common fixed point. Let $z \neq y$ be two common fixed points of S and T . Then

$$d(z, y) = d(S(z), T(y)) \leq \phi(d(z, y), d(z, S(z)), d(z, T(y)), d(y, S(z)), \\ d(y, T(y))) \\ = \phi(d(z, y), 0, d(z, y), d(x, y), 0) < \gamma(d(z, y)) < d(z, y) .$$

Therefore $z = y$ and S and T have a unique common fixed point.

Taking ϕ to be continuous we get a slightly revised version of the result of Husain and Sehgal [1] as a corollary.

References

- [1] S.A. Husain and V.M. Sehgal, "On common fixed points for a family of mappings", *Bull. Austral. Math. Soc.* 13 (1975), 261-267.

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