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ON THE MAIN INVARIANT OF ELEMENTS ALGEBRAIC OVER A HENSELIAN VALUED FIELD

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Abstract Let v be a henselian valuation of a field K with value group G, let $\bar{v}$ be the (unique) extension of v to a fixed algebraic closure $\bar{K}$ of K and let $(\tilde{K}, \tilde{v})$ be a completion of (K, v). For $\alpha \in \bar{K} \setminus K$, let $M(\alpha, K)$ denote the set $\{\bar{v}(\alpha - \beta) : \beta \in \bar{K},\ [K(\beta) : K] < [K(\alpha) : K]\}$. It is known that $M(\alpha, K)$ has an upper bound in $\bar{G}$ if and only if $[K(\alpha) : K] = [\tilde{K}(\alpha) : \tilde{K}]$, and that the supremum of $M(\alpha, K)$, which is denoted by $\delta_K(\alpha)$ (usually referred to as the main invariant of α), satisfies a principle similar to the Krasner principle. Moreover, each complete discrete rank 1 valued field (K, v) has the property that $\delta_K(\alpha) \in M(\alpha, K)$ for every $\alpha \in \bar{K} \setminus K$. In this paper the authors give a characterization of all those henselian valued fields (K, v) which have the property mentioned above.

Keywords: valued fields; valuations and their generalizations; non-Archimedean valued fields

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1. Introduction

Let v be a valuation of a field K with value group G and let $\bar{v}$ be a fixed prolongation of v to an algebraic closure $\bar{K}$ of K with value group $\bar{G}$. In 1936, MacLane [**9**] gave an iterative method of describing all extensions of v to a simple transcendental extension $K(x)$ of K when v is a discrete valuation of rank 1. In the general case using some ideas of MacLane, Alexandru *et al.* [**1**, **2**] gave a description of all extensions of v to $K(x)$ by means of 'minimal pairs'. A pair $(\alpha, \delta) \in \bar{K} \times \bar{G}$ is said to be minimal (with respect to K and $\bar{v}$) if, whenever $\beta \in \bar{K}$ satisfies $\bar{v}(\alpha - \beta) \geqslant \delta$, then $[K(\alpha) : K] \leqslant [K(\beta) : K]$. It is clear that when $\alpha \in K$, then (α, δ) is a minimal pair for each $\delta \in \bar{G}$ and that a pair (α, δ) in $(\bar{K} \setminus K) \times \bar{G}$ is minimal if and only if δ is greater than each element of the set $M(\alpha, K)$ defined by

$$M(\alpha, K) = \{\bar{v}(\alpha - \beta) : \beta \in \bar{K},\ [K(\beta) : K] < [K(\alpha) : K]\}. \tag{1.1}$$

This led to the invariant $\delta_K(\alpha)$ defined for those $\alpha \in \bar{K} \setminus K$ for which $M(\alpha, K)$ has an upper bound in $\bar{G}$, by

$$\delta_K(\alpha) = \sup\{\bar{v}(\alpha - \beta) : \beta \in \bar{K},\ [K(\beta) : K] < [K(\alpha) : K]\}, \tag{1.2}$$

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where, for the sake of definition of supremum, $\bar{G}$ may be viewed as a subset of its Dedekind order completion. Alexandru *et al*. proved that if (K, v) is a complete discrete rank 1 valued field, then $M(\alpha, K)$ has an upper bound in $\bar{G}$ and, moreover, $\delta_K(\alpha) \in M(\alpha, K)$ for each $\alpha \in \bar{K} \setminus K$ (see [**2**, Theorem 3.9] and [**11**, p. 74]). They also proved that $\delta_K(\alpha)$ satisfies a fundamental principle [**10**, Remark 3.3] stated below which is similar to the well-known Krasner principle [**4**, 16.8] satisfied by the Krasner constant.

Fundamental Principle. Let (K, v) be a complete discrete rank 1 valued field and $(\bar{K}, \bar{v})$ be as in the foregoing text. If $\alpha, \beta \in \bar{K}$ are such that $\bar{v}(\alpha - \beta) > \delta_K(\alpha)$, then $\bar{v}(K(\alpha)) \subseteq \bar{v}(K(\beta))$ and $R(K(\alpha)) \subseteq R(K(\beta))$, where $R(L)$ denotes the residue field of the valuation obtained by restricting $\bar{v}$ to a subfield L of $\bar{K}$.

In 1999 it was proved that $\delta_K(\alpha)$ satisfies the above principle when (K, v) is a henselian valued field of any rank (see [**7**]). However, unlike in the discrete rank 1 case, there are instances when $\delta_K(\alpha) \in \bar{G}$ but fails to belong to $M(\alpha, K)$ (see Example 2.1). This has led us to consider the following problem.

> *How can we characterize those henselian valued fields (K, v) for which to each $\alpha \in \bar{K} \setminus K$, there corresponds $\beta \in \bar{K}$ satisfying $[K(\beta) : K] < [K(\alpha) : K]$ and $\delta_K(\alpha) = \bar{v}(\alpha - \beta)$?*

In the present paper, we solve this problem by proving the following theorem.

Theorem 1.1. *Let v be a henselian valuation of any rank of a field K and let $(\bar{K}, \bar{v})$ be as above. The following two statements are equivalent.*

(i) *To each $\alpha \in \bar{K} \setminus K$ there corresponds $\beta \in \bar{K}$ with $[K(\beta) : K] < [K(\alpha) : K]$ such that $\delta_K(\alpha) = \bar{v}(\alpha - \beta)$.*

(ii) *For each $\theta \in \bar{K}$, $K(\theta)/K$ is a defectless extension with respect to the valuation obtained by restricting $\bar{v}$.*

Recall that a finite extension (K', v') of a henselian valued field (K, v) is said to be defectless if $[K' : K] = ef$, where e and f are, respectively, the index of ramification and the residual degree of v'/v.

The above theorem has, in turn, given rise to the following problem.

> *For a henselian field (K, v), if $K(\theta)/K$ is defectless for each $\theta \in \bar{K}$, then is it true that every finite extension of (K, v) is defectless?*

An example has been given in the last section to show that the answer to the above question is 'no' in general.

2. Definitions, notation and some preliminary results

In what follows in this paper, (K, v) is a henselian valued field of any rank with value group G and $\bar{v}$ is a (unique) extension of v to a fixed algebraic closure $\bar{K}$ of K with value group $\bar{G}$. For an overfield L of K contained in $\bar{K}$, $R(L)$ and $G(L)$ will, respectively, stand for the residue field and the value group of the valuation of L obtained by restricting $\bar{v}$. For a finite extension L/K, $\operatorname{def}(L/K)$ will stand for the defect of the valued field extension L/K with respect to the valuation v_L obtained by restricting $\bar{v}$ to L, i.e.

$$\operatorname{def}(L/K) = [L : K]/ef,$$

where e and f are the index of ramification and the residual degree of v_L/v. By the degree (over K) of an element $\alpha \in \bar{K}$ we shall mean the degree of the extension $K(\alpha)/K$. For any ξ in the valuation ring of $\bar{v}$, ξ^* will denote its $\bar{v}$-residue, i.e. the image of ξ under the canonical homomorphism from the valuation ring of $\bar{v}$ onto its residue field. We need the following theorem, which is already known (see [**7**]); its proof is omitted.

Theorem A. Let (K, v) be a henselian valued field of any rank. Let $\alpha, \beta \in \bar{K}$ be such that $\bar{v}(\alpha - \beta) > \bar{v}(\alpha - \gamma)$ for any $\gamma \in \bar{K}$ satisfying $[K(\gamma) : K] < [K(\alpha) : K]$, then

(i) $G(K(\alpha)) \subseteq G(K(\beta))$,

(ii) $R(K(\alpha)) \subseteq R(K(\beta))$, and

(iii) $\operatorname{def}(K(\alpha)/K)$ divides $\operatorname{def}(K(\beta)/K)$.

If $f(x)$ is a fixed non-zero polynomial in $K[x]$, then using Euclidean algorithm, each $F(x) \in K[x]$ can be uniquely represented as a finite sum $\sum_{i\geqslant 0} F_i(x) f(x)^i$, $\deg F_i(x) < \deg f(x)$, called the f-expansion of F. Let $(\alpha, \delta) \in \bar{K} \times \bar{G}$ be a minimal pair. The valuation $\bar{w}_{\alpha,\delta}$ of $\bar{K}(x)$, defined on $\bar{K}[x]$ by

$$\bar{w}_{\alpha,\delta}\left(\sum_i c_i (x - \alpha)^i\right) = \min\{\bar{v}(c_i) + i\delta\}, \quad c_i \in \bar{K}, \tag{2.1}$$

will be referred to as the valuation defined by the pair (α, δ). The description of $\bar{w}_{\alpha,\delta}$ on $K[x]$ is given by the already known theorem [**6**, Theorem 1.4] stated below.

Theorem B. Let $\bar{w}_{\alpha,\delta}$ be the valuation of $\bar{K}(x)$ defined by a minimal pair (α, δ). If $f(x)$ is the minimal polynomial of α over K, then for any $F(x) \in K[x]$ with f-expansion $\sum_i F_i(x) f(x)^i$, we have

$$\bar{w}_{\alpha,\delta}(F(x)) = \min\{\bar{v}(F_i(\alpha)) + i\bar{w}_{\alpha,\delta}(f(x))\}. \tag{2.2}$$

The following theorem will be used in the sequel. It is an immediate consequence of the well-known fact that completion of a henselian valued field is henselian and of Corollary 3.10 in [**2**].

Theorem C. Let $(\tilde{K}, \tilde{v})$ be a completion of a henselian valued field (K, v), and let α be an element of $\bar{K} \setminus K$. Then $M(\alpha, K)$ has an upper bound in $\bar{G}$, if and only if $[K(\alpha) : K] = [\tilde{K}(\alpha) : \tilde{K}]$.

It may be pointed out that the supremum of $M(\alpha, K)$ being in G does not necessarily imply that it belongs to $M(\alpha, K)$. Here is an example to support this assertion.

Example 2.1. Let k_0 be the algebraic closure of the finite field F_2 of two elements, and let $K_0 = k_0((T))$ be the field of Laurent series in T with valuation v_0 given by $v_0(T) = 1$. Let $\bar{v}_0$ be the extension of v_0 to an algebraic closure $\bar{K}_0$ of K_0. Let K be the inseparable closure of K_0 in $\bar{K}_0$, with valuation v which is the restriction of $\bar{v}_0$. Then (K, v) being an algebraic extension of a complete rank 1 valued field is henselian. Let α be a root of the polynomial $x^2 - x - T^{-1} = 0$. As shown in [**5**], there does not exist any $c \in K$ such that $\bar{v}(\alpha - c) \geqslant 0$, whereas $\delta_K(\alpha) = 0$ by virtue of [**3**, Lemma 6] and the fact that K is a perfect field.

We prove two lemmas; the first one is well known [**6**, Lemma 2.1(ii)]. For the sake of completion, we prove it here.

Lemma 2.2. *Let (α, δ) be a minimal pair (with respect to K and $\bar{v}$) and θ be an element of $\bar{K}$ with $\bar{v}(\theta - \alpha) \geqslant \delta$. Let $h(x) \in K[x]$ be a polynomial such that for each root β of $h(x)$, $\bar{v}(\alpha - \beta) < \delta$. Then $\bar{v}(h(\theta) - h(\alpha)) > \bar{v}(h(\alpha))$.*

Proof. Write $h(x) = c\prod_j (x - \beta_j)$. Then

$$\frac{h(\theta)}{h(\alpha)} = \prod_j \left(\frac{\theta - \beta_j}{\alpha - \beta_j}\right) = \prod_j \left(1 + \frac{\theta - \alpha}{\alpha - \beta_j}\right).$$

By hypothesis, we have

$$\bar{v}\left(\frac{\theta - \alpha}{\alpha - \beta_j}\right) \geqslant \delta - \bar{v}(\alpha - \beta_j) > 0.$$

Therefore

$$\bar{v}\left(\frac{h(\theta)}{h(\alpha)} - 1\right) > 0$$

as desired. □

Lemma 2.3. *Let (K, v) be henselian and θ be an element of $\bar{K} \setminus K$ such that $\delta_K(\theta)$ defined by (1.2) belongs to $M(\theta, K)$. If $\alpha \in \bar{K}$ is an element of smallest degree over K such that $\bar{v}(\theta - \alpha) = \delta_K(\theta)$, then*

(a) *$(\alpha, \delta_K(\theta))$ is a minimal pair, and*

(b) *$\bar{w}_{\alpha,\delta}(G(x)) = \bar{v}(G(\theta))$, for any polynomial $G(x) \in K[x]$ of degree less than the degree of θ over K, where the valuation $\bar{w}_{\alpha,\delta}$ is as defined by (2.1) with $\delta = \delta_K(\theta)$.*

Proof. (a) We show that for every $\gamma \in \bar{K}$ with $\deg \gamma < \deg \alpha$, the inequality $\bar{v}(\alpha - \gamma) < \delta_K(\theta)$ holds. For such an element γ, the choice of α gives $\bar{v}(\theta - \gamma) < \bar{v}(\theta - \alpha)$, which by virtue of the strong triangle law implies that

$$\bar{v}(\alpha - \gamma) = \min\{\bar{v}(\alpha - \theta), \bar{v}(\theta - \gamma)\} = \bar{v}(\theta - \gamma).$$

Consequently, $\bar{v}(\alpha - \gamma) < \delta_K(\theta)$ as desired.

(b) Write $G(x) = c\prod(x-\beta_i)$. By virtue of (2.1),

$$\bar{w}_{\alpha,\delta}(G(x)) = \bar{v}(c) + \sum_i \min\{\bar{v}(\alpha-\beta_i), \delta\}.$$

Clearly it is enough to prove that for each root β_i of $G(x)$,

$$\min\{\bar{v}(\alpha-\beta_i), \delta\} = \bar{v}(\theta-\beta_i). \tag{2.3}$$

Since $\deg G(x)$ is less than $[K(\theta) : K]$, it follows that

$$\bar{v}(\theta-\beta_i) \leqslant \delta_K(\theta) = \delta.$$

Keeping in mind the above inequality and the fact that $\bar{v}(\theta-\alpha) = \delta$, one can quickly verify (2.3). □

3. Proof of (i) implies (ii) in Theorem 1.1

Assuming (i) we prove assertion (ii) of Theorem 1.1 by induction on the degree of the extension $K(\theta)/K$. Clearly it is enough to prove that to each $\theta \in \bar{K} \setminus K$, there corresponds $\alpha \in \bar{K}$ such that $[K(\alpha) : K] < [K(\theta) : K]$ and $\mathrm{def}(K(\alpha)/K) = \mathrm{def}(K(\theta)/K)$. Fix an element $\theta \in \bar{K}$ of degree $m \geqslant 2$. Let $\alpha \in \bar{K}$ be of smallest degree over K such that $\delta_K(\theta) = \bar{v}(\theta-\alpha)$. Let $f(x)$ denote the minimal polynomial of α over K of degree $n \geqslant 1$. For the sake of simplicity we shall denote $\delta_K(\theta)$ by δ. By Lemma 2.3 (a), (α, δ) is a minimal pair. Let $\bar{w}_{\alpha,\delta}$ be the valuation of $\bar{K}(x)$ as given by (2.1). Observe that when $\gamma \in \bar{K}$ and $\deg\gamma < \deg\alpha = n$, then $\bar{v}(\theta-\gamma) < \bar{v}(\theta-\alpha)$ and, consequently, $\bar{v}(\alpha-\gamma) < \bar{v}(\alpha-\theta) = \delta$. Therefore by Theorem A, $G(K(\alpha)) \subseteq G(K(\theta))$, $R(K(\alpha)) \subseteq R(K(\theta))$ and $\mathrm{def}(K(\alpha)/K)$ divides $\mathrm{def}(K(\theta)/K)$. If e denotes the smallest positive integer such that $e\bar{v}(f(\theta)) \in G(K(\alpha))$, then, by Lagrange's theorem, e divides $[G(K(\theta)) : G(K(\alpha))]$. Thus we conclude that en divides m. Let us denote m/en by l. Then

$$l = \left(\frac{[G(K(\theta)) : G(K(\alpha))]}{e}\right)[R(K(\theta)) : R(K(\alpha))]\left(\frac{\mathrm{def}(K(\theta)/K)}{\mathrm{def}(K(\alpha)/K)}\right). \tag{3.1}$$

We shall prove that

$$[R(K(\theta)) : R(K(\alpha))] = l. \tag{3.2}$$

Clearly (3.1) and (3.2) immediately yield $\mathrm{def}(K(\theta)/K) = \mathrm{def}(K(\alpha)/K)$ as desired.

Choose a polynomial $h(x) \in K[x]$ of degree less than n such that $e\bar{v}(f(\theta)) = -\bar{v}(h(\alpha))$. The equality (3.2) is proved if we show that $(f(\theta)^e h(\alpha))^*$ is algebraic over $R(K(\alpha))$ of degree l. Suppose to the contrary that $(f(\theta)^e h(\alpha))^*$ is algebraic over $R(K(\alpha))$ of degree $q < l$. Then there exist polynomials $A_i(x) \in K[x]$ each of degree less than n and $A_0(\alpha)^* \neq 0$ such that

$$((f(\theta)^e h(\alpha))^*)^q + A_{q-1}(\alpha)^*((f(\theta)^e h(\alpha))^*)^{q-1} + \cdots + A_0(\alpha)^* = 0. \tag{3.3}$$

For $0 \leqslant i \leqslant q-1$ we write $h(\alpha)^i A_i(\alpha)$ as $B_i(\alpha)$ and $h(\alpha)^q$ as $B_q(\alpha)$, where each $B_i(x) \in K[x]$ is of degree less than n. So (3.3) can be rewritten as

$$(B_q(\alpha)f(\theta)^{eq})^* + (B_{q-1}(\alpha)f(\theta)^{e(q-1)})^* + \cdots + (B_0(\alpha))^* = 0, \tag{3.4}$$

with $(B_0(\alpha))^* = (A_0(\alpha))^* \neq 0$. Recall that (α, δ) is a minimal pair and $\bar{v}(\theta - \alpha) = \delta$. So, by Lemma 2.2, $(B_i(\alpha)/B_i(\theta))^* = 1$. Therefore (3.4) shows that

$$\bar{v}(B_q(\theta)f(\theta)^{eq} + B_{q-1}(\theta)f(\theta)^{e(q-1)} + \cdots + B_0(\theta)) > 0. \tag{3.5}$$

Set

$$G(x) = B_q(x)f(x)^{eq} + B_{q-1}(x)f(x)^{e(q-1)} + \cdots + B_0(x). \tag{3.6}$$

Observe that

$$\deg G(x) < eqn + n \leqslant e(l-1)n + n = m - en + n \leqslant m.$$

As the expansion of $G(x)$ given by (3.6) is its f-expansion, it follows from Theorem B that

$$\bar{w}_{\alpha,\delta}(G(x)) = \min_{0 \leqslant i \leqslant q}\{\bar{v}(B_i(\alpha)) + ie\bar{w}_{\alpha,\delta}(f(x))\} \leqslant \bar{v}(B_0(\alpha)) = 0. \tag{3.7}$$

Keeping in view that $\deg G(x) < m$, we have by Lemma 2.3 (b) and (3.5)

$$\bar{w}_{\alpha,\delta}(G(x)) = \bar{v}(G(\theta)) > 0,$$

which contradicts (3.7). This contradiction proves (3.2), and hence (ii) follows.

4. Proof of (ii) implies (i) in Theorem 1.1

Suppose that (ii) holds. Since $K(\theta)/K$ is defectless for $\theta \in \bar{K}$, it follows that $[K(\theta) : K] = [\tilde{K}(\theta) : \tilde{K}]$, where $(\tilde{K}, \tilde{v})$ is a completion of (K, v). Therefore by virtue of Theorem C, $M(\theta, K)$ has an upper bound in $\bar{G}$; consequently $\delta_K(\theta)$ is defined in the Dedekind order completion of $\bar{G}$. Assume that (i) does not hold. Choose an element $\alpha \in \bar{K} \setminus K$ of degree, say n, over K for which $\delta_K(\alpha) \notin M(\alpha, K)$. We shall obtain the desired contradiction by showing that $K(\alpha)/K$ is not defectless. Since $M(\alpha, K)$ is totally ordered without last element, it contains a well-ordered cofinal subset. So we can choose a net $\{\delta_i\}_{i \in I}$ in $M(\alpha, K)$ satisfying

(1) $\{\delta_i\}_{i \in I}$ is cofinal in $M(\alpha, K)$ and $\delta_i < \delta_j$, for $i < j$, $i, j \in I$; and

(2) $\delta_i = \bar{v}(\alpha - \beta_i)$, $\beta_i \in \bar{K}$ is such that $\deg \beta_i < n$ and whenever $\gamma \in \bar{K}$ has degree less than $\deg \beta_i$, then $\bar{v}(\alpha - \gamma) < \delta_i$.

If necessary on replacing $\{\delta_i\}_{i \in I}$ by a subnet, we may assume that all β_i are of the same degree (say s) over K. Keeping in mind that $\delta_i < \delta_j$ for $i < j$, we have

$$\bar{v}(\beta_i - \beta_j) \geqslant \min\{\bar{v}(\beta_i - \alpha), \bar{v}(\alpha - \beta_j)\} = \delta_i,$$

and for any $\gamma \in \bar{K}$ with $\deg \gamma < s$

$$\bar{v}(\beta_i - \gamma) = \bar{v}(\beta_i - \alpha + \alpha - \gamma) = \bar{v}(\alpha - \gamma) < \delta_i.$$

Consequently it follows from Theorem A, that

$$G(K(\beta_i)) \subseteq G(K(\beta_j)), \quad i < j, \tag{4.1}$$

$$R(K(\beta_i)) \subseteq R(K(\beta_j)), \quad i < j, \quad i, j \in I. \tag{4.2}$$

As all the extensions $K(\beta_i)/K$ are defectless and of the same degree $s < n$, it is clear that equality holds in (4.1) and (4.2). Thus for any $j \in I$,

$$\bigcup_{i \in I} G(K(\beta_i)) = G(K(\beta_j)), \qquad \bigcup_{i \in I} R(K(\beta_i)) = R(K(\beta_j)). \tag{4.3}$$

We are going to prove that

$$G(K(\alpha)) = \bigcup_{i \in I} G(K(\beta_i)), \qquad R(K(\alpha)) = \bigcup_{i \in I} R(K(\beta_i)). \tag{4.4}$$

As the extension $K(\alpha)/K$ is of degree $n > s$, (4.3) and (4.4) immediately imply that $K(\alpha)/K$ is not defectless, leading to the desired contradiction.

To prove (4.4), let $F(x) \in K[x]$ be any polynomial of degree less than n. It is enough to prove that there exists $k \in I$ such that $\bar{v}(F(\alpha) - F(\beta_k)) > \bar{v}(F(\beta_k))$. Let γ be a root of $F(x)$. Since $\bar{v}(\alpha - \gamma) \in M(\alpha, K)$, it follows from property (1) of the net $\{\delta_i\}_{i \in I}$ that there exists $k \in I$ such that $\bar{v}(\alpha - \gamma) < \delta_k$. Choosing k sufficiently large, we may assume that

$$\bar{v}(\alpha - \gamma_t) < \delta_k \tag{4.5}$$

for each root γ_t of $F(x)$. Write $F(x) = c \prod (x - \gamma_t)$. Then

$$\frac{F(\alpha)}{F(\beta_k)} = \prod_t \left(\frac{\alpha - \gamma_t}{\beta_k - \gamma_t} \right) = \prod_t \left(1 + \frac{\alpha - \beta_k}{\beta_k - \gamma_t} \right). \tag{4.6}$$

Since $\bar{v}(\alpha - \gamma_t) < \delta_k$ by (4.5) and $\bar{v}(\alpha - \beta_k) = \delta_k$ by choice of δ_k, we have, on using the strong triangle law,

$$\bar{v}(\beta_k - \gamma_t) = \min\{\bar{v}(\beta_k - \alpha), \bar{v}(\alpha - \gamma_t)\} = \bar{v}(\alpha - \gamma_t);$$

consequently (4.6) shows that

$$\bar{v}\left(\frac{F(\alpha)}{F(\beta_k)} - 1 \right) > 0,$$

which proves (4.4) and completes the proof of the theorem.

5. An example

We give an example to show that the assumption '$K(\alpha)/K$ defectless for each $\alpha \in \bar{K}$' does not imply in general that every finite extension of a henselian valued field (K, v) is defectless. The construction of the field (K, v) given below appears in a different context in [**8**, Chapter 8.4].

Let $F_p((t))$ be the field of Laurent series in an indeterminate t with coefficients from the finite field F_p of p elements, p prime, with valuation v_t given by $v_t(t) = 1$. Fix $x, y \in F_p((t))$ both of v_t-valuation 1, which are algebraically independent over $F_p(t)$. Set $s = x^p + ty^p$ and $L = F_p(s, t)$. Then s and t are algebraically independent over F_p, because $F_p(s^{1/p}, t^{1/p}, x) = F_p(t^{1/p}, x, y)$ is of transcendence degree 3 over F_p. Let K be the algebraic closure of L in $F_p((t))$ with valuation v, which is the restriction of v_t. We claim that K/L is a separable extension; this will imply that K being the separable closure of L in a complete discrete rank 1 valued field is henselian (see [**4**, 17.18]). Since K/L is a normal extension, the claim is proved once we show that whenever an element α of $F_p((t))$ belongs to $L^{1/p} = F_p(s^{1/p}, t^{1/p})$, then $\alpha \in L$. Write $\alpha = P/Q$, with P, Q in $F_p[s^{1/p}, t^{1/p}]$. Since $Q^p \in F_p[s, t] \subseteq L$, on replacing α by αQ^p we may assume that $\alpha \in F_p[s^{1/p}, t^{1/p}]$. Write

$$\alpha = \sum_{i,j} a_{ij} s^{i/p} t^{j/p}, \quad a_{ij} \in F_p, \quad a_{ij} \neq 0. \tag{5.1}$$

It is to be shown that p divides each i and j. Suppose this is false. On replacing α by $\alpha - c$ for some $c \in F_p[s, t]$, we may assume that for each pair (i, j) appearing in (5.1), either p does not divide i or p does not divide j. Let $a_{mn} s^{m/p} t^{n/p}$ be the smallest degree monomial in the variables $s^{1/p}$, $t^{1/p}$ occurring in (5.1) in which the exponent of $s^{1/p}$ is also the smallest. On dividing α by a suitable integral power of s, we may further assume that $0 \leqslant m \leqslant p - 1$. Keeping in mind that $v_t(s) = p$, it can easily be seen that

$$v_t(\alpha) = \bar{v}_t(a_{mn} s^{m/p} t^{n/p}) = m + (n/p).$$

As $\alpha \in F_p((t))$, $v_t(\alpha)$ is an integer and hence p divides n. So our supposition gives that p does not divide m, i.e. $0 < m < p$. On recalling that $s^{1/p} = x + t^{1/p} y$, we can rewrite α as

$$\alpha = A_0(x, y, t) + A_1(x, y, t) t^{1/p} + \cdots + A_{p-1}(x, y, t) t^{(p-1)/p}, \tag{5.2}$$

where each $A_j(x, y, t) \in F_p[x, y, t]$. It is clear that all the non-zero summands on the right-hand side of (5.2) have different $\bar{v}_t$-valuation. It may be pointed out that $A_m(x, y, t) \neq 0$; in fact one can easily verify that $a_{mn} y^m t^{n/p}$ is the monomial of smallest degree (in y, t) among those monomials of $A_m(x, y, t)$ which are free from x. Keeping in mind that $m \geqslant 1$, we conclude that $v_t(\alpha - A_0(x, y, t))$ is not a rational integer, which is not so. This contradiction proves that K/L is a separable extension.

We next show that for each α algebraic over K, $K(\alpha)/K$ is a defectless extension. Since a complete discrete valued field is defectless [**4**, 18.8], the above assertion is proved as soon as it is shown that

$$[K(\alpha) : K] = [\tilde{K}(\alpha) : \tilde{K}], \tag{5.3}$$

$\tilde{K} = F_p((t))$ being the completion of K. To verify (5.3), observe that if $g(x)$ is the minimal polynomial of α over $\tilde{K}$, then the coefficients of $g(x)$ are algebraic over K and these coefficients being in $\tilde{K}$ must belong to K, for K is algebraically closed in $\tilde{K} = F_p((t))$.

Finally it may be pointed out that $K(s^{1/p}, t^{1/p})/K$ has defect p. Since K/L is a separable extension, we have

$$[K(s^{1/p}, t^{1/p}) : K] = [L(s^{1/p}, t^{1/p}) : L] = p^2.$$

Keeping in mind that $K(s^{1/p}, t^{1/p}) \subseteq F_p((t^{1/p}))$, it now follows that $K(s^{1/p}, t^{1/p})/K$ has index of ramification p, residual degree 1 and defect p.

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