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A smooth surface in P^4 not of general type has degree at most 66

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Abstract. We use geometric information obtained from the generic initial ideal of a hyperplane section of a surface in $\mathbb{P}^4$ not of general type to bound the degree of such a surface.

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This is a continuation of the papers of Braun and Fløystad [BF] and Cook [C] on the bound on the degree of smooth surfaces not of general type in P^4.

We will prove the following:

THEOREM 1. *Let S be a smooth surface of degree d in P^4 not of general type. Then $d \leqslant 66$.*

The new idea in this paper is to bound the degree of the sporadic zeros of a generic hyperplane section of the surface by considering the geometric implications of having sporadic zeros in high degree. First, we need to collect various theorems and formulae from earlier sources.

1. If S is a smooth surface in P^4 then it satisfies the *double point formula* ([H, pg. 434])

$$d^2 - 5d - 10(\pi - 1) + 2(6\chi\mathcal{O}_S - K^2) = 0. \tag{1}$$

2. The following result of Ellingsrud and Peskine gives us some initial bounds.

PROPOSITION 2 ([EP]). *If S is a smooth surface not of general type in P^4, then for $\sigma = 5, 6$ or 7, either* $\deg S \leqslant 5(\sigma+1)(\sigma-2)/(\sigma-4)$ *or S lies on a hypersurface, V_σ, of degree σ.*

In particular, $\deg S \leqslant 66$ or $S \subset V_7$, so we may assume that S lies on a hypersurface of degree 7. Furthermore it is known ([K]), that if S lies on a hypersurface of degree 3 then $\deg S \leqslant 8$. Therefore we may assume that S lies on a hypersurface of minimal degree $s = 4, 5, 6$ or 7.

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3. If S is a surface not of general type and the degree of $S > 5$ then $K^2 \leqslant 9$. ([BPV])

4. In [BF], $\chi\mathcal{O}_S$ is bounded from below, using generic initial ideal theory, in terms of invariants arising from a generic hyperplane section C of S:

Let $\mathbf{C}[x_0, x_1, x_2, x_3]$ be the ring of polynomials of P^3 under the reverse lexicographical ordering. Let C be a curve in P^3, then the generic initial ideal of C, $\mathrm{gin}(I_C)$, is generated by elements of the form $x_0^i x_1^j x_2^k$.

DEFINITION. A monomial $x_0^a x_1^b x_2^c$ is a *sporadic zero* of C if $x_0^a x_1^b x_2^c \notin \mathrm{gin}(I_C)$, but there exists $c' > c$ such that $x_0^a x_1^b x_2^{c'} \in \mathrm{gin}(I_C)$.

Let α_t is the number of sporadic zeros in degree t and assume $\alpha_t = 0$ for $t > m$. Let Γ be a generic hyperplane section of C. Then

$$\mathrm{gin}(I_\Gamma) = \mathrm{gin}(I_C)^{\mathrm{sat}}_{x_3=0},$$

where the saturation is with respect to x_2. The generic initial ideal of Γ is of the form

$$\mathrm{gin}(I_\Gamma) = (x_0^s,\ x_0^{s-1}x_1^{\lambda_{s-1}}, \ldots, x_1^{\lambda_0}),$$

where $\sum \lambda_i = d$ and $\lambda_{i+1} + 2 \geqslant \lambda_i \geqslant \lambda_{i+1} + 1$. The $\lambda_0 > \lambda_1 > \cdots > \lambda_{s-1} > 0$ are called the *connected invariants* of Γ.

In [BF] they show that

$$\chi\mathcal{O}_S \geqslant \sum_{t=0}^{s-1}\left(\binom{\lambda_t + t - 1}{3} - \binom{t-1}{3}\right) - \sum_{t=0}^{m}\alpha_t(t-1). \tag{2}$$

5. If π is the genus of C, then

$$\pi = 1 + \sum_{i=0}^{s-1}\left(\binom{\lambda_i}{2} + (i-1)\lambda_i\right) - \sum_{t=0}^{m}\alpha_t. \tag{3}$$

Combining all these facts we obtain

$$\begin{aligned} 18 \geqslant 2K^2 &= d^2 - 5d - 10(\pi - 1) + 12\chi\mathcal{O}_S \\ &\geqslant d^2 - 5d - 10\left(\sum_{i=0}^{s-1}\left(\binom{\lambda_i}{2} + (i-1)\lambda_i\right) - \sum_{t=0}^{m}\alpha_t\right) \\ &\quad +12\left(\sum_{t=0}^{s-1}\left(\binom{\lambda_t + t - 1}{3} - \binom{t-1}{3}\right) - \sum_{t=0}^{m}\alpha_t(t-1)\right). \end{aligned} \tag{4}$$

6. By the work of Gruson and Peskine ([GP]) on the numerical invariants of points in P^2 we have, for $d > (s-1)^2 + 1$

$$1 + \sum_{i=0}^{s-1}\left(\binom{\lambda_i}{2} + (i-1)\lambda_i\right) \leqslant \frac{d^2}{2s} + (s-4)\frac{d}{2} + 1 = G(d, s). \tag{5}$$

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7. Braun and Floystad ([BF]) show that if $s \geqslant 2$ and $d > (s-1)^2 + 1$

$$\sum_{t=0}^{s-1}\left(\binom{\lambda_t + t - 1}{3} - \binom{t-1}{3}\right) \geqslant s\binom{\frac{d}{s} + \frac{s-3}{2}}{3} + 1 - \binom{s-1}{4}. \quad (6)$$

Thus

$$\begin{aligned} 18 \;\geqslant\; & d^2 - 5d - 10\left(\frac{d^2}{2s} + (s-4)\frac{d}{2} - \sum_{t=0}^{m}\alpha_t\right) \\ & +12\left(s\binom{\frac{d}{s} + \frac{s-3}{2}}{3} + 1 - \binom{s-1}{4} - \sum_{t=0}^{m}\alpha_t(t-1)\right) \\ =\; & d^2 - 5d - 10\left(\frac{d^2}{2s} + (s-4)\frac{d}{2}\right) + 12s\binom{\frac{d}{s} + \frac{s-3}{2}}{3} \\ & +12\left(1 - \binom{s-1}{4}\right) - \sum_{t=0}^{m}\alpha_t(12t-22). \end{aligned} \quad (7)$$

We will use Equation (7) to get an initial bound on the degree and then, using Mathematica© and Equation (4), we will improve the bound.

Thus, we need to find the smallest possible upper bound for $\sum_{t=0}^{m}\alpha_t(12t-22)$ or equivalently $A = \sum_{t=0}^{m}\alpha_t t$.

We will bound A by bounding the number of sporadic zeros and then by bounding the degree of the sporadic zeros by geometric considerations.

The bound on the number of sporadic zeros

Let $\gamma = G(d,s) - \pi$. Any bound on γ will also bound the number of sporadic zeros (see Equations (3) and (5)). By [EP], $\gamma \leqslant d(s-1)^2/2s$. Furthermore, if S is a surface not of general type (of degree > 5) we have $K^2 < 6\chi$, substituting this into the double point formula (1), we get $\pi \geqslant (d^2 - 5d + 10)/10$ and thus

$$\sum \alpha_t \leqslant 1 + \sum_{i=0}^{s-1}\left(\binom{\lambda_i}{2} + (i-1)\lambda_i\right) - \frac{d^2 - 5d + 10}{10}, \quad (8)$$

or

$$\gamma \leqslant \frac{d^2}{2s} + (s-4)\frac{d}{2} - \frac{d^2 - 5d}{10}.$$

Taking the minimum of the bounds for γ, we get, for $s = 4, \gamma \leqslant 9d/8$, for $s = 5, \gamma \leqslant d$, for $s = 6, \gamma \leqslant d(90-d)/60$ and for $s = 7, \gamma \leqslant d(70-d)/35$.

The bound on the degree of sporadic zeros

For Equations (7) and (4) to hold for large degree, A will need to be large. If we have sporadic zeros in large degree this would improve our chances of making A

large enough. Furthermore, every generator of gin(I_C) of the form $x_0^a x_1^b x_2^c$ with $c > 0$ gives us a sporadic zero in each degree i for $a + b \leqslant i \leqslant a + b + c - 1$. Thus we could obtain the largest upper bound on A by assuming that there were one generator of gin(I_C) of the form $x_1^{\lambda_0} x_2^z$ where z is the maximum number of sporadic zeros. Then $A \leqslant \sum_{\lambda_0}^{\lambda_0+z-1} t$. But (as we saw in [C]) this bound is much too big.

Let us consider the following situation. Suppose $C \subset P^3$ is a smooth curve such that gin(I_C) has at least three generators, one of which, M, is of degree r and all the others are of degree $\leqslant r - 2$.

LEMMA 3. *C has a secant line of order r.*

Proof. Every minimal generator of gin(I_C) either arises from a minimal generator of I_C or from a generator of gin(I_C) in one degree lower. (See [B])

Let J be the ideal generated by elements of I_C in degree $\leqslant r-1$. By considering the Hilbert function associated to J, we see that degree $(V(J))$ = degree $(C) + 1$. Hence, $V(J) = C \cup X$ and $X \supset L$ a line.

Let f be the generator of I_C in degree r corresponding to M and let $F = \{f = 0\}$. By Bezout's Theorem $F \cap L$ in r points (up to multiplicity) and all these points must lie on C. Let $F \cap L = \sum m_i p_i$ where p_i are the points of C.

Claim. L meets C at p_i with multiplicity m_i.

Proof of Claim. C is locally cut out at p_i by polynomials F_1 and F_2 of degree r. The line $L = \{l_1 = l_2 = 0\}$ meets $V(F_1)$ and $V(F_2)$ at p_i with multiplicity m_i and thus

$$\text{length}\left(\frac{\mathcal{O}_{P^3,p_i}}{l_1, l_2, F_j}\right) = m_i,$$

for $j = 1, 2$.

However

$$\frac{\mathcal{O}_{P^3,p_i}}{l_1, l_2, F_j} = \frac{\mathcal{O}_{P^1,p_i}}{F_j|_{l_1=l_2=0}},$$

and $F_j|_{l_1=l_2=0} = t^{m_i}$ where t is the local defining equation of p_i in L.

Hence

$$\text{length}\left(\frac{\mathcal{O}_{P^3,p_i}}{l_1, l_2, F_1, F_2}\right) = m_i,$$

and this is the intersection multiplicity of L and C at p_i. $\square$

Let us now return to the case where $S \subset P^4$ is a smooth surface not of general type. Suppose that for generic hyperplanes $H = \{h = 0\}$, the generic hyperplane

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section C_h of S is such that $\mathrm{gin}(I_{C_h})$ has at least three generators, one in degree $r > d/2$ and all others in degree $\leqslant r-2$ and hence, by Lemma 3, C_h has an r-secant line, L_h.

LEMMA 4. *Generically, these secant lines are secant lines of S.*

Proof. Suppose for a generic h, $L_h \subset S$. Then for a generic hyperplane H, $S \cap H \supset L_h$. But generically $S \cap H$ is a smooth irreducible curve, and hence must be L_h. But then $S = P^2$. □

Thus we are in the following situation. For a generic hyperplane $H = \{h = 0\}$ there exists $L_h \subset H$ such that L_h is a secant line of S of order $r > d/2$. Let $B \subset G(1, 4)$ parametrize these secant lines in the Grassmannian of lines in P^4. Let $V = \cup_{b\in B} L_b$ be the union of these lines in P^4.

PROPOSITION 5. *S contains a plane curve of degree $\geqslant r$.*

Proof. As any line in P^4 is contained in a 2-dimensional family of hyperplanes, the dimension of $B \geqslant 2$.

$V \cap S$ is at most a 2-dimensional space, so if the dimension of $B \geqslant 3$, two lines must meet. Let L_1 and L_2 be two of these intersecting lines. Let P be the plane containing the lines. Then S intersects P in at least $2r - 1 > d$ points and hence $S \cap P \supset C$ a plane curve. As the secant lines L_i will meet C with multiplicity at least r, the degree of $C \geqslant r > d/2$.

Now suppose the dimension of $B = 2$ and no two lines from B meet. Let

$$\Phi\colon B \times B \dashrightarrow \mathrm{P}^{4^*},$$

send the pair $(a, b) \in B \times B$ to the hyperplane containing l_a and l_b. As all lines from B are skew this map is well defined away from the diagonal.

If the dimension of the image of Φ is 4, then there exists a generic hyperplane which contains two r-secant lines. However this contradicts the generic hyperplane section having only one r-secant line.

If the dimension of a fiber is $\geqslant 2$, then S would be contained in the hyperplane of the image. However S is non-degenerate.

Thus the generic fiber is 1-dimensional and the image of Φ is 3-dimensional. That is, there is a 3-dimensional space of hyperplanes in P^4 each containing a 1-dimensional family of skew r-secant lines of S.

Let $H = \{h = 0\}$ be a hyperplane in the image of Φ containing a 1-dimensional family of skew lines. Let $S_h \subset H$ be the surface which is the union of these lines. $I_{S_h} = (f_h, h)$.

$$S_h \cap S = C = \cup_{b\in B, l_b \subset H} (l_b \cap S).$$

All points of C lie on an r-secant line, hence if $g \in I_C$ is a polynomial of degree $\leqslant r - 1$, g must vanish on S_h. Therefore the only generator of I_C in degree $\leqslant r - 1$ is f_h (and h).

$S_h \subset H$ and $S \cap H = C_h$ is a hyperplane section of S containing C and hence $I_{C_h} \subset I_C$ and so all generators of I_{C_h} in degree $\leqslant r-1$ are divisible by f_h

Now S is contained in a hypersurface $V_\sigma \subset P^4$ of degree $\sigma \leqslant 7 < r-1$, where $V_\sigma = \{f_\sigma = 0\}$. Hence $V_\sigma|_{h=0}$ contains S_h for generic H in the image of Φ and thus V_σ contains the union of all these surfaces, which must form a three-dimensional space. As V_σ is irreducible, any polynomial in the ideal of S of degree $\leqslant r-1$ must be divisible by f_σ and so I_S has only the one generator in degree $\leqslant r-1$. But this is impossible.

Hence there must be two secant lines meeting and as in the case of $\dim(B) = 3$ or 4, we get a plane curve of degree $\geqslant r > d/2$. □

LEMMA 6. *If S is a smooth surface not of general type in* P^4 *of* $d > 50$, S *cannot contain a plane curve of degree* $r > d/2$.

Proof. Let $C \subseteq P$ be a plane curve of degree $r > d/2$ contained in S. Let H be a hyperplane containing P.

Then $S \cap H = C_h = C \cup C_{\text{res}}$.

We have

$$0 \to \mathcal{O}_{C\cup C_{\text{res}}} \to \mathcal{O}_C \oplus \mathcal{O}_{C_{\text{res}}} \to \mathcal{O}_{C\cap C_{\text{res}}} \to 0,$$

therefore

$$h^1(\mathcal{O}_{C_h}) \geqslant h^1(\mathcal{O}_C) + h^1(\mathcal{O}_{C_{\text{res}}})$$

and hence

$$g(C_h) \geqslant g(C) + g(C_{\text{res}}) \geqslant g(C).$$

C is a plane curve of degree $d_C \geqslant d/2$ and so

$$g(C) = \frac{(d_C-1)(d_C-2)}{2} - \delta \geqslant \frac{(\frac{d}{2}-1)(\frac{d}{2}-2)}{2}.$$

On the other hand, by the Gruson–Peskine ([GP]) bound

$$g(C_h) \leqslant \frac{d^2}{2s} + (s-4)\frac{d}{2} + 1.$$

(The inequality is true for general hyperplane sections and as the projective genus will stay constant it is true for all hyperplane sections.) Hence

$$\frac{d^2}{2s} + (s-4)\frac{d}{2} + 1 \geqslant \frac{(\frac{d}{2}-1)(\frac{d}{2}-2)}{2}.$$

This means for $s = 7$, degree $\leqslant 42$, for $s = 6$, degree $\leqslant 42$ and for $s = 5$, degree $\leqslant 50$.

For $s = 4$ the inequality holds. However the Gruson–Peskine inequality assumes there are no sporadic zeros. If we suppose that the number of sporadic zeros is $\leqslant 3d/4$ then naively we have $A \leqslant \sum_{\lambda_0}^{\lambda_0+(3d/4)-1} t$. By connectedness $\lambda_0 \leqslant d/4+3$ and hence $A \leqslant (5/32)d^2 + (13/8)d - 3$. Substituting back into Equation (7) we get

$$0 \geqslant \frac{d^3}{8} - \frac{23}{8}d^2 - \frac{17}{2}d + 33,$$

and hence $d \leqslant 25$. Therefore we may assume that the number of sporadic zeros is $> 3d/4$, then $g(C_h) < d^2/8 + 1 - 3d/4$ and we in fact get a contradiction. $\square$

Thus if the degree of $S \geqslant 50$ and S lies on a hypersurface of degree 4, 5, 6 or 7, then a generic hyperplane section C of S cannot have an r-secant line with $r > d/2$. In terms of the generic initial ideal of C, this means that, either

(1) All generators of $\mathrm{gin}(I_C)$ are in degree $\leqslant d/2$
or
(2) If there exists a generator of $\mathrm{gin}(I_C)$ in maximal degree $r > d/2$ then there must exist a second generator in degree $r - 1$.

We want to maximize $A = \sum_{t=0}^{m} \alpha_t t$ subject to conditions (1) and (2). Let z be the maximum number of sporadic zeros.

(i) If $\lambda_0 + z - 1 \leqslant d/2$ then $A \leqslant \sum_{t=\lambda_0}^{\lambda_0+z-1} t$.
(ii) If $\lambda_0 + z - 1 > d/2$ but $\lambda_0 + \lambda_1 + z - 1 \leqslant d$ then $A \leqslant \sum_{t=\lambda_0}^{\lfloor d/2 \rfloor} t + \sum_{t=\lambda_1+1}^{z-\lfloor d/2 \rfloor+\lambda_0+\lambda_1-1} t$.
(iii) If $\lambda_0+\lambda_1+z-1 > d$ let $r = \lceil(\lambda_0+\lambda_1+z)/2\rceil$ then $A \leqslant \sum_{t=\lambda_0}^{r} t+\sum_{t=\lambda_1+1}^{r-1} t$.

To get a first estimate of A we have, by the connectedness of the invariants, $\lambda_0 \leqslant d/s + s - 1$ and $\lambda_1 \leqslant d/s + s - 2$.

Using the bounds on γ, we get

$$\text{for } s = 4, \quad A \leqslant \tfrac{153}{256}d^2 + \tfrac{45}{16}d + \tfrac{1}{4},$$
$$\text{for } s = 5, \quad A \leqslant \tfrac{9}{20}d^2 + \tfrac{7}{2}d + \tfrac{1}{4}.$$

Substituting back into the original Equation (7) above, we get

$$\text{for } s = 4, \quad 0 \geqslant \tfrac{1}{8}d^3 - \tfrac{523}{64}d^2 - \tfrac{29}{2}d - 6 \quad \text{and hence } d \leqslant 67,$$
$$\text{for } s = 5, \quad 0 \geqslant \tfrac{2}{25}d^3 - \tfrac{27}{5}d^2 - 32d - 21 \quad \text{and hence } d \leqslant 71.$$

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We now need to consider Equation (4) which is much more accurate than Equation (7). From point 2, we know that either deg $S \leqslant 90$ or S is contained in a hypersurface of degree 5. Furthermore, if S is contained in a hypersurface of degree 5 then $d \leqslant 71$ and if S is contained in a hypersurface of degree 4 then $d \leqslant 67$. Therefore we can write down all possible configurations of the connected invariants $\lambda_0 > \lambda_1 > \cdots > \lambda_{s-1}$ for high degree.

For example if $s = 5$ and $d = 71$ the possible invariants are

$$\begin{gathered} 18 > 16 > 14 > 12 > 11 \\ 17 > 16 > 14 > 13 > 11 \\ 17 > 15 > 14 > 13 > 12. \end{gathered}$$

(To obtain the list we used a program of Rich Liebling.)

We then obtain an upper bound on the number of sporadic zeros, z using

$$z \leqslant \frac{d^2}{8} - \sum \left(\binom{\lambda_i}{2} - (i-1)\lambda_i \right) + \frac{9d}{8} \quad \text{if } s = 4,$$

and Equation (8) if $s \geqslant 5$.

We again get an upper bound on A using (i), (ii), or (iii). Substituting everything into Equation (4) and see when the inequality holds. (We checked the inequalities on Mathematica©.)

We get $s \leqslant 7$ and

$$\begin{array}{lll} \text{for} & s = 7, & d \leqslant 43 \\ \text{for} & s = 6, & d \leqslant 44 \\ \text{for} & s = 5, & d \leqslant 66 \\ \text{and for} & s = 4, & d \leqslant 65. \end{array}$$

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