REGIONS OF STABILITY OF ASTEROIDS

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ABSTRACT. Using Hill's modified stability criterium, regions of orbital elements are established for conditions of stability. The model of the three-dimensional restricted problem of three bodies is used with the Sun and Jupiter as the primaries. Four different cases are studied: direct and retrograde, outside and inside asteroidal orbits. The directions of the asteroidal orbits refer to the synodical reference frame and the positions refer to Jupiter's orbit. The orbital parameters of the asteroids are the semi-major axis (a), the eccentricity (e), and the inclination from Jupiter's orbital plane (i). The effects of the other orbital elements are not investigated in this paper. The argument of the perihelion and the longitude of the ascending node are fixed at $\Omega = \omega = 90^\circ$ and the time of perihelion passage is T = 0 for all orbits.

The aim of this paper is to give quantitative evaluation of the stability of asteroids, the results being also applicable to comets and meteor streams. The evolution of the solar system may be studied using planets, satellites or smaller bodies like asteroids. The unquestionable advantage of approaching the problem via the investigation of asteroids is that there are a very large number of asteroids with well defined orbital elements while the number of planets and natural satellites in the solar system is much smaller.

Establishing regions of stability enhances the location and discovery of additional minor planets. On the other hand, bodies with unstable orbits might be, under certain conditions, available for capture or for significant orbital changes without large artificial perturbations. Furthermore, changes in the observed orbital parameters may change the character of the motion from stability to instability and various evolutionary trends could be observed concerning the solar system.

ANALYSIS

For simplicity's sake the first results are derived for the

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two-dimensional case to clarify the underlying ideas. This corresponds to the physical simplification of assuming that the asteroids' orbits are coplanar with Jupiter's orbit. Our final results do include three-dimensional effects.

The key quantity in the analysis is known as the Jacobian constant, given by

$$C = 2\Omega - v^2, \tag{1}$$

where Ω is the dimensionless potential function of the restricted problem in the synodic system and v is the dimensionless velocity of the third particle relative to this system. The potential function is a combination of the gravitational and centrifugal effects and is given by

$$\Omega = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\mu(1-\mu), \tag{2}$$

where μ is the mass parameter obtained from the masses of the primaries (m_1 and m_2) as follows

$$\mu = \frac{m_2}{m_1 + m_2}, \text{ with } m_2 \leqq m_1 . \tag{3}$$

The distances between the primaries and the third body are r_1 and r_2. The primaries are located on the axis of syzygies which rotates with the unit angular velocity around the center of mass. The unit of length is the distance between the primaries and the dimensionless time is $t = t^*n$ where t^* the actual time and n is the mean motion of the primaries.

The above-mentioned distances are computed from

$$r_1^2 = (x-\mu)^2 + y^2$$

and

$$r_2^2 = (x-\mu+1)^2 + y^2 .$$

When computing Jupiter's effect on an asteroid, we have

$$\mu = \frac{m_j}{m_s + m_j} = 9.53875 \times 10^{-4} ,$$

using for m_j and m_s the recent values given by Circular No. 163 of the

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U.S. Naval Observatory.

The basic idea of establishing regions of stability is to find regions in the orbital plane where motion may take place. For a given asteroid, the value of C is computed using Eqn (1). This requires knowledge of its position and velocity, which might be obtained from its orbital elements. Once the actual value of the Jacobian constant C_{ac} of the asteroid is known, this is compared with the critical value C_{cr} at which the forbidden regions of motion change. These critical values are described in considerable detail in the literature (see for instance, Szebehely 1967) and tabulated values are available. If the actual value of the Jacobian constant is much higher than the critical value, then the motion is confined to a well defined region of the plane, that is, no exchanges or escapes are possible. The forbidden and allowable regions of motion are separated by the so-called curves of zero velocity which close or open up at the critical values of the Jacobian constant. These curves intersect the axis of syzygies at the collinear equilibrium points located on both sides of Jupiter at a dimensionless distance of approximately $(\mu/3)^{1/3} \simeq 0.06825$. The third collinear equilibrium point is located approximately at a unit distance from the sun on the opposite side from Jupiter. For large values of C_{ac} we have three distinct and separate regions for possible motion. One is outside the Sun-Jupiter system which region can not be penetrated by asteroids originally moving in this outside regions. Such asteroids cannot be captured by either the Sun or by Jupiter. The second region encloses the Sun and an asteroid in this region can not be captured by Jupiter nor can it leave the system. The third region is around Jupiter and an asteroid (or a satellite of Jupiter) can not leave Jupiter's neighborhood, cannot become a minor planet governed by the Sun and cannot escape the system. This is the case when $C_{ac} > C^1_{cr} \simeq 3.03971$. If $C_{ac} < C^1_{cr}$ the inside two regions (around the Sun and around Jupiter join forming a region where the body might join or leave the Sun and/or Jupiter. This happens when $C^1_{cr} > C_{ac} > C^2_{cr} \simeq 3.03844$. The outer region still remains separated from the inner region and neither penetration nor escape is possible. When $C^2_{cr} > C_{ac}$ the outer region joins the inner region and outside particles may penetrate and inside particles may escape. Further reduction of the value of $C_{ac} \leq 3$ eliminates all forbidden regions and in Hill's sense there is no stability. Since the values of C^1_{cr} and C^2_{cr} are close and anytime $C_{ac} < C^2_{cr}$ we have the possibility of exchange, of communication, of penetration or of escape we select $C_{cr} = C^2_{cr}$ as our critical value to determine stability and introduce a measure of stability by the equation

$$S = \frac{C_{ac} - C_{cr}}{C_{cr}} , \qquad (4)$$

where $C_{cr} = 3.03844$ corresponds to the collinear equilibrium point usually denoted by L_1 and located close to Jupiter on the opposite side from the Sun.

The next step is to evaluate the actual value of the Jacobian constant for an asteroid. Given the semi-major axis and the eccentricity of the orbit, the relative velocities are computed at the perigee-distance, $a(1-e)$ or at the apogee-distance, $a(1+e)$ with respect to the Sun, assuming, once again for simplicity's sake that these points are on the axis of syzygies. The Jacobian constant becomes

$$C = \frac{1}{a} \pm 2\,[a(1-e^2)]^{\frac{1}{2}} \quad , \tag{5}$$

once the appropriate substitutions are made into Eqn (1). Note that the above equation applies at the perigee and uses two-body approximations because of the small value of μ. The + sign refers to direct and the − sign to retrograde orbits. For the straightforward but tedious derivation see for instance, Szebehely (1967). For circular orbits the Jacobian constant is

$$C = \frac{1}{a} \pm 2a^{\frac{1}{2}} \quad . \tag{6}$$

If the limiting critical value, C_{cr} is substituted for C and the resulting, essentially cubic equation is solved for a we obtain the limiting value(s) of a for direct and for retrograde orbits. These values are $a = 0.80438$ for direct and $a = 0.24786$ for retrograde orbits. If $C_{ac} > C_{cr}$ the orbit is stable, consequently, the next step is to investigate the effect of changes in the quantity a on the Jacobian constant. Since

$$\frac{dC}{da} = -\frac{1}{a^2} \pm \frac{1}{a^{\frac{1}{2}}} \tag{7}$$

we see that for retrograde orbits $dC/da < 0$, for any value of a. Therefore if a is increased above the previously given value ($a = 0.24786$) C_{ac} will be smaller than C_{cr} and instability will set in. For example, if $a = 0.3$, the actual value of the Jacobian constant becomes $C_{ac} = 2.2379$ which results in a negative value for the measure of stability and according to Eqn (4) it becomes $S = -0.2635$, indicating instability. Similar analysis may be performed for direct orbits.

Taking the partial derivative of Eqn (5) with respect to the eccentricity we may once again establish its role. This is left to the reader since our figures and tables given in the next chapter, clearly demonstrate the effects of changes in the eccentricity. The only remark to be made is that $\partial C/\partial e > 0$ for retrograde and negative for direct orbits.

The three-dimensional effects are brought into the picture by using the zero-velocity surfaces instead of the previously mentioned zero-velocity curves of the restricted problem of three bodies. The Jacobian integral is identical in form in the two and three-dimensional cases, that is, Eqn (1) is still applicable with

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 ,$$

$$r_1^2 = (x-\mu)^2 + y^2 + z^2 ,$$

and

$$r_2^2 = (x-\mu+1)^2 + y^2 + z^2 .$$

The method used is identical to the two-dimensional case excepting the complications of the topology of the three-dimensional limiting surfaces.

RESULTS

The results of the computations are represented in Figures 1 to 3 and in the corresponding Tables 1 to 3. Figure 1 represents direct inner orbits. The parameter is the semi-major axis, the value of which is shown on the curves, using astronomical units (A.U.). Note that the unit of distance in the discussion of the previous chapter was the Sun-Jupiter distance, but the Figures and Tables give results in A.U.-s to comply with astronomical conventions.

Consider Figure 1 and the corresponding Table 1. For zero eccentricity and inclination, the previous section gave for the limiting semi-major axis $a = 0.80438$. Since we now use A.U.-s we multiply this value by the Sun-Jupiter distance or by 5.2028 A.U. and obtain 4.185 A.U., corresponding to the origin of Figure 1, i.e., to the point $e = i = 0$. The same value is shown as the first entry in Table 1.

As an example consider asteroid No. 25 (Phocaea) with $i = 21\overset{\circ}{.}59$, $e = 0.254$ and $a = 2.4$ A.U. Locating the point corresponding to the given i and e values our Table 1 or Figure 1 gives $a \simeq 3.2$ (see point A). Consequently, if Phocaea's semi-major axis would be larger than 3.2 its orbit would show instability. In fact, its semi-major axis is 2.4, consequently, its orbit is stable, and its measure of stability as defined by Eqn (4) is $S = 0.119$. The charts, of course may be used for finding limiting values of i and e. Moving to the curve corresponding to $a = 2.4$ on Fig. 1 we see that if the eccentricity of Phocaea would be as high as 0.5 (see point B), then its limiting inclination should be less than $42\overset{\circ}{.}50$. In other words, for a given semi-major axis higher eccentricity and higher inclination reduce the stability, as expected. In fact, the non-existing asteroid mentioned above and marked by B on Fig. 1 has a measure of stability $S = 0$, while asteroid #25 is stable.

In general, therefore, the use of Fig. 1 (or Table 1) is to locate the point for a given (e,i) set and read off the corresponding value of the limiting semi-major axis, a^*. If the minor planet's actual

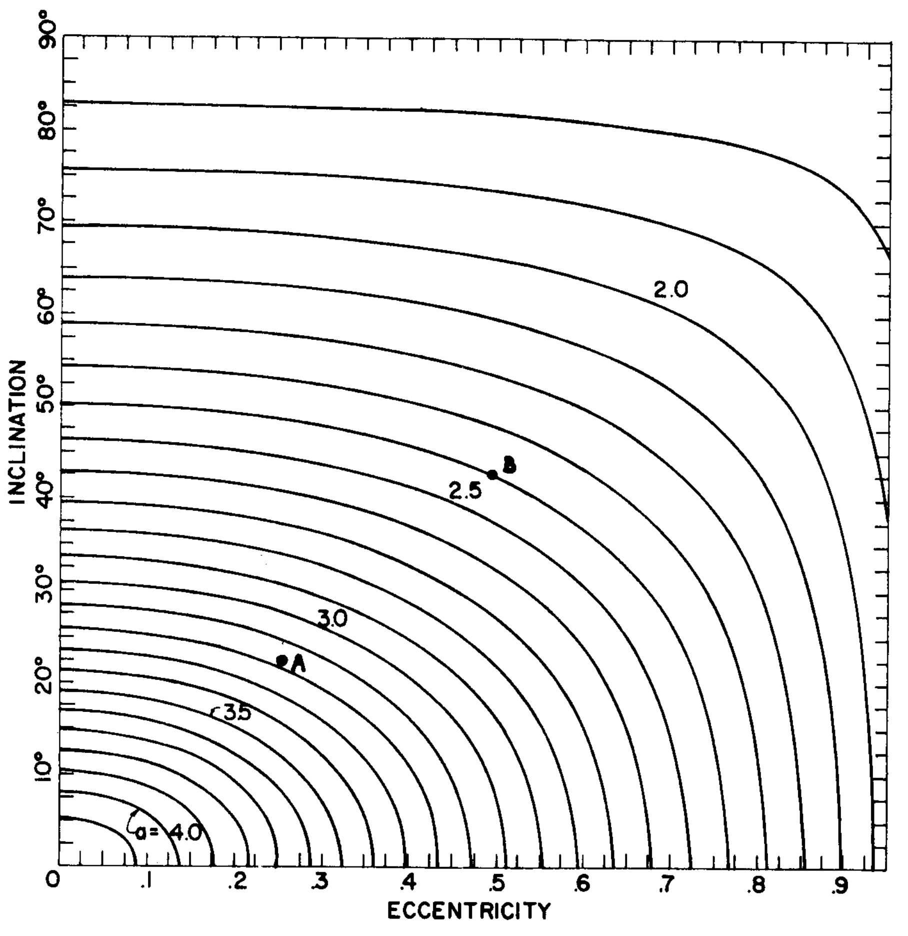

FIGURE 1. DIRECT INSIDE ORBITS

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TABLE 1. LIMITING VALUES OF SEMI-MAJOR AXES FOR DIRECT INNER ORBITS

	i° = 0	10	20	30	40	50	60	70	80	90
e = 0.0	4.185	3.919	3.455	3.034	2.686	2.404	2.176	1.989	1.837	1.712
.1	4.083	3.850	3.420	3.015	2.675	2.398	2.172	1.988	1.836	1.712
.2	3.849	3.677	3.322	2.960	2.644	2.380	2.162	1.982	1.834	1.712
.3	3.572	3.453	3.180	2.875	2.594	2.350	2.145	1.973	1.831	1.712
.4	3.296	3.214	3.013	2.767	2.526	2.309	2.121	1.961	1.826	1.712
.5	3.033	2.978	2.832	2.643	2.444	2.257	2.090	1.944	1.819	1.712
.6	2.787	2.749	2.647	2.506	2.350	2.195	2.052	1.923	1.810	1.712
.7	2.554	2.529	2.460	2.359	2.243	2.122	2.005	1.896	1.799	1.712
.8	2.327	2.312	2.268	2.203	2.123	2.036	1.948	1.863	1.784	1.712
.9	2.094	2.086	2.063	2.027	1.981	1.929	1.874	1.818	1.763	1.712

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semi-major axis, a is smaller we have stability and if $a > a^*$ we have instability. A new measure of stability could be established by evaluating the ratio $S' = (a^* - a)/a(0,0)$, where $a(0,0) = 4.185$. This measure for Phocaea becomes $(3.2 - 2.4)/4.185 = 0.167$. It is not recommended that S' should replace S as given by Eqn (4) since this latter has more general applications.

Another observation of general validity may be made from Fig. 1 (or from Table 1). Considering a fixed value for the semi-major axis we see that small eccentricity allows higher inclination, while high eccentricity requires low inclination for stability. Furthermore, as the value of the semi-major axis decreases larger eccentricities and larger inclinations are allowed.

Finally, we note that some of the constant semi-major axis curves may be approximated by elliptic arcs, especially, in the middle range of the chart.

Figure 2 and Table 2 represent retrograde orbits inside the limiting zero-velocity surface. The topology of the zero velocity surfaces is identical to the case discussed in connection with direct orbits but the evaluation of the actual values of the Jacobian constants is significantly different, as shown by the $\pm$ signs in Eqn (5). The origin of Figure 2 corresponding to $e = i = 0$ is $a = 0.24786 \times 5.2028$ A.U. $= 1.2896$ A.U. and the semi-major axis increases with increasing eccentricity and inclination. The reason for this is that, as mentioned before, $\partial C/\partial e > 0$ for retrograde orbits.

Comparing Figs. 1 and 2 we observe the low values of the semi-major axes for retrograde orbits indicating that asteroids in retrograde orbits must be closer to the Sun than those in direct orbits. This is the consequence of using Hill's criterion for stability and it should be kept in mind that this criterion is a necessary but not sufficient condition. In previous papers the difference between linearized stability investigations and Hill's method have been subjected to analysis (Szebehely, 1978). The linear analysis shows considerably higher stability for retrograde orbits than Hill's method.

Figure 3 and Table 3 refer to outer direct orbits. These are asteroids or meteoroids outside of Jupiter's orbit and the limiting value for the semi-major axis may be computed from Eqn (6). We use the plus sign and find the solution to the equation

$$C_{cr} = \frac{1}{a} + 2a^{\frac{1}{2}} \qquad (8)$$

for $a > 1$. Note that the solution of this equation for $a < 1$ was already established before ($a = 0.80438$ or $a = 4.185$ A.U.). The solution of Eqn (8), using once again, $C_{cr} = 3.03844$ is $a = 1.25$ or $a = 6.504$ A.U. If the semi-major axis (for $i = e = 0$) is larger than this value we have $C_{ac} > C_{cr}$ and stability occurs. It is important to

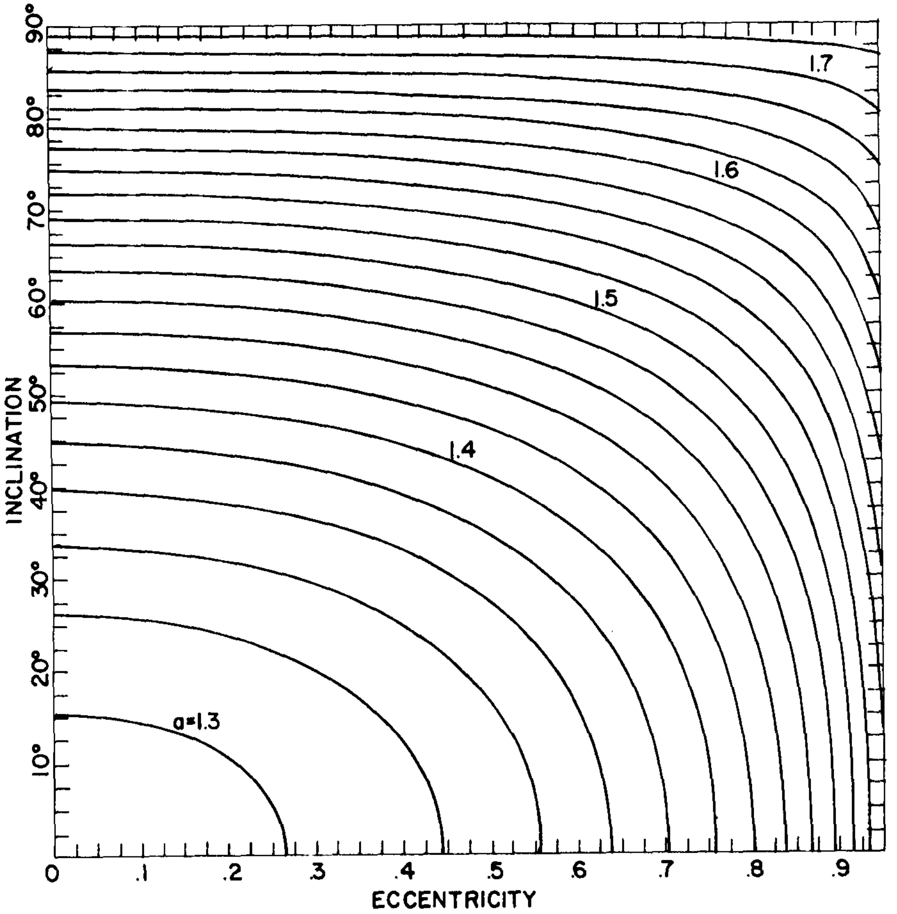

FIGURE 2. RETROGRADE INSIDE ORBITS

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TABLE 2. LIMITING VALUES OF SEMI-MAJOR AXES FOR RETROGRADE INNER ORBITS

	i° = 0	10	20	30	40	50	60	70	80	90
e = 0.0	1.290	1.294	1.307	1.329	1.361	1.404	1.458	1.526	1.610	1.712
.1	1.291	1.295	1.308	1.331	1.362	1.405	1.459	1.527	1.610	1.712
.2	1.295	1.300	1.313	1.335	1.366	1.408	1.462	1.529	1.612	1.712
.3	1.303	1.307	1.320	1.342	1.373	1.414	1.468	1.533	1.614	1.712
.4	1.314	1.318	1.331	1.352	1.383	1.423	1.475	1.539	1.618	1.712
.5	1.329	1.333	1.346	1.366	1.396	1.436	1.486	1.548	1.623	1.712
.6	1.350	1.354	1.366	1.386	1.415	1.452	1.500	1.559	1.629	1.712
.7	1.379	1.382	1.394	1.413	1.440	1.475	1.519	1.573	1.637	1.712
.8	1.419	1.423	1.433	1.450	1.475	1.506	1.546	1.593	1.649	1.712
.9	1.485	1.487	1.496	1.510	1.530	1.555	1.587	1.623	1.665	1.712

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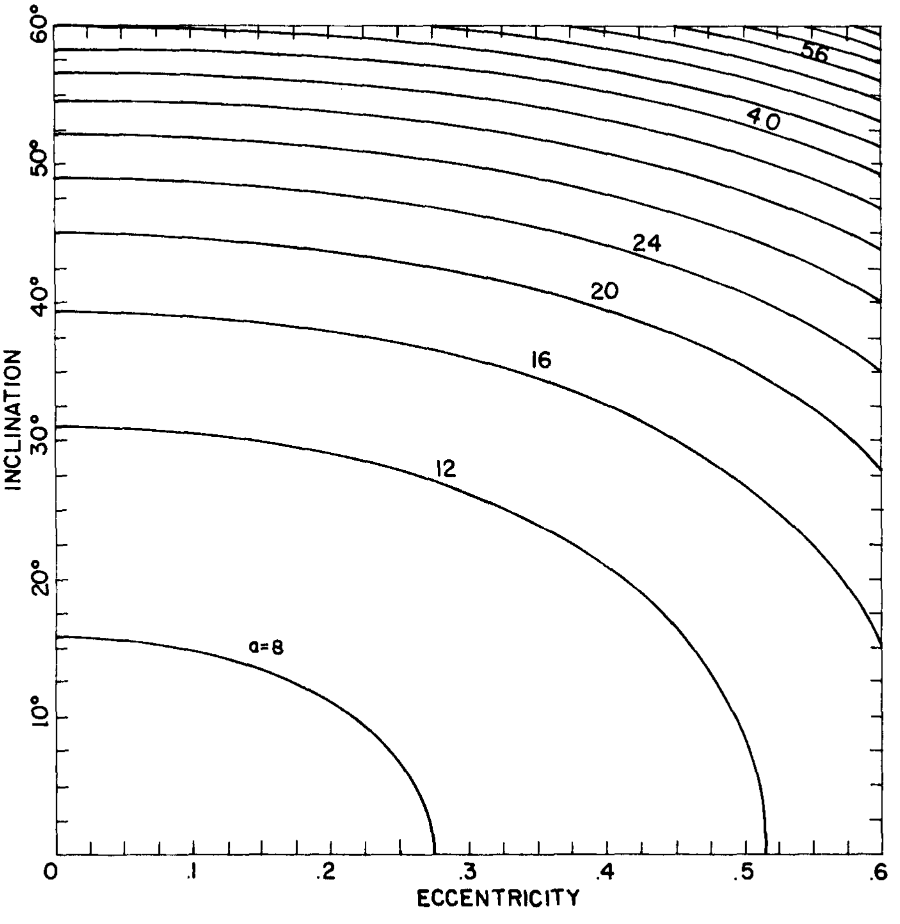

FIGURE 3. DIRECT OUTSIDE ORBITS

TABLE 3. LIMITING VALUES OF SEMI-MAJOR AXES FOR DIRECT OUTER ORBITS

i° =	0	10	20	30	40	50	60
e = 0.0	6.504	7.138	8.800	11.601	16.347	25.137	44.142
.1	6.727	7.332	8.974	11.784	16.568	25.445	44.650
.2	7.338	7.893	9.498	12.342	17.245	26.382	46.186
.3	8.294	8.818	10.411	13.336	18.461	28.071	48.952
.4	9.658	10.179	11.813	14.899	20.391	30.763	53.365
.5	11.619	12.167	13.918	17.294	23.379	34.949	60.236
.6	14.592	15.207	17.195	21.074	28.136	41.642	71.243
.7	19.571	20.323	22.762	27.560	36.353	53.244	90.352

clarify the meaning of Hill's stability criteria for inside and for outside orbits. If a direct inside orbit has larger semi-major axis than 4.185 A.U. its Jacobian constant is below C_{cr} and the separation of the outside and inside regions ceases to exist. In other words, the asteroid will not be confined to a closed region including the Sun and Jupiter. On the other hand, if an outside direct orbit has smaller semi-major axis than 6.504 A.U. then, once again the inside and outside regions may communicate and the orbit of the outside asteroid may enter the region including the Sun and Jupiter, in fact it may be captured by Jupiter. So Hill's criteria for "inside" and "outside" are quite different. Stability for inside asteroids means they cannot leave a region including the Sun and Jupiter, while stability for outside asteroid means they cannot enter the same region. This explains the different trends shown on Figs. (1) and (3). The larger the semi-major axis is of an outside asteroid, the larger its inclination and eccentricity may be without instability setting in (see Fig. 3). The opposite is the situation for inside orbits as we have seen on Fig. 1. (Note that other perturbations such as Saturn's for outside and Mars' for inside orbits are not included in these considerations.)

Finally, a discussion is offered of retrograde outside orbits. These orbits are all unstable according to Hill's definition since Eqn (6) with a negative sign does not have real solutions for $a > 1$. In other words, all outside retrograde orbits may enter the inner region since their Jacobian constant is smaller than C_{cr}. (Note that this does not apply to inside retrograde orbits since Eqn (6) has a real solution (using the negative sign) for $a < 1$.

Further consequences (and in some respect disadvantages) of Hill's method are that stability for inside orbits means that they cannot escape the Sun-Jupiter system. Outside orbits may escape but are not allowed to enter the inside regions. In terms of instability this means that if an inside orbit is unstable it may in fact escape the system and an unstable outside orbit may enter the Jupiter-Sun region. Stable inside bodies may have chaotic orbits, may be captured by the Sun or by Jupiter, may have figure-eight orbits, etc., but cannot leave the region. Stable outside orbits, once again may take any shape (including escape) as long as they do not enter the region of the inside orbits.

CONCLUSIONS

Well defined regions of stability and instability of minor planetary orbits are established using Hill's stability criterion. For direct inside orbits the maximum value of the semi-major axis is 4.185 A.U. when the eccentricity and inclination are zero. For retrograde inside orbits the minimum value of the semi-major axis is 1.290 A.U., again for $e = i = 0$. For outside direct orbits, the minimum value of the semi-major axis is 6.504 A.U. for $e = i = 0$. There are no stable outside retrograde orbits.

For given orbital parameters (a,e,i) and direction of motion the Hill-type stability of minor planets or of meteor-streams can be determined by the tables and charts given.

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