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ON THE DEGREE OF AN ANALYTIC MAP GERM

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ABSTRACT. Let $f = (f_1, \ldots, f_n)\colon (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ be a real analytic mapping and 0 is isolated in $f^{-1}(0)$. The aim of this paper is to describe the degree $\deg_0 f$ in terms of parametrizations of irreducible components of the real analytic curve given by the equations $f_1(x) = \cdots = f_{n-1}(x) = 0$ near $0 \in \mathbb{R}^n$.

1. **Introduction.** If $f = (f_1, \ldots, f_n)\colon (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ is a smooth mapping and 0 is isolated in $f^{-1}(0)$, then the degree of f at 0 is defined as follows: choose a ball B_r about $0 \in \mathbb{R}^n$ with radius $r > 0$, so small that $f^{-1}(0) \cap \bar{B}_r = \{0\}$ and let S_r be its boundary $(n-1)$-dimensional sphere. Choose an orientation of each copy of $\mathbb{R}^n$. The degree of f at 0 is the degree of the mapping $(f/\|f\|)\colon S_r \longrightarrow S_1$, where the spheres are oriented as $(n-1)$-spheres in $\mathbb{R}^n$ (cf. [2],[8]). One can introduce the degree of the continuous mapping in this way (cf. [5]). In our paper we consider only analytic mapping, but every smooth mapping can be replaced by the analytic one with the same degree (cf. [4], Proposition 4.1). The advantage of our approach is, that we can make use of the structure of a real analytic curve. We need well-known results about decomposition of the analytic curve into irreducible components and their parametrizations around the point, which we present in the following proposition:

PROPOSITION 1.1. *Let A be a real analytic curve in the neighbourhood of $0 \in \mathbb{R}^n$, $0 \in A$. There exist an arbitrary, small enough neighbourhood Ω of $0 \in \mathbb{R}^n$ and a positive integer k such that:*

(a) $A \cap \Omega = A_1 \cup \cdots \cup A_k$, A_i is an analytic curve, irreducible in Ω, $A_i \cap A_j = \{0\}$ for $i \neq j$, $i,j \in \{1, \ldots, k\}$,

(b) there exist a real number $\delta > 0$ and parametrizations, i.e. one-to-one, analytic homeomorphisms, $p_i\colon I \longrightarrow A_i$, $p_i(0) = 0$, $I = \{t \in \mathbb{R} : |t| < \delta\}$, of every irreducible component A_i of A, $i = 1, \ldots, k$.

The main result of this paper is the following theorem:

THEOREM 1.2. *Let $f = (f_1, \ldots, f_n)\colon (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ be an analytic mapping such that 0 is isolated in $f^{-1}(0)$. Let $A = \{x \in \Omega : f_1(x) = \cdots = f_{n-1}(x) = 0\}$, Ω is a neighbourhood of $0 \in \mathbb{R}^n$ and let us suppose the following conditions are fulfilled:*

(i) 0 is not isolated in A,

(ii) for every $x \in A \setminus \{0\}$ the differentials: $df_1(x), \ldots, df_{n-1}(x)$ are linearly independent.

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Let $p_i\colon I \longrightarrow A_i$ be a parametrization of the irreducible component A_i of the analytic curve A, $i = 1, \ldots, k$. Then the following formula holds:

$$\deg_0 f = \sum_{i=1}^{k} \deg_0[(f_n \circ p_i)\det(df_1 \circ p_i, \ldots, df_{n-1} \circ p_i, p_i')].$$

We illustrate the above theorem by two simple examples. Let us notice that if $f\colon(\mathbb{R},0) \longrightarrow (\mathbb{R},0)$ is an analytic function, 0 is isolated in $f^{-1}(0)$, then $\deg_0 f = (1/2)[\operatorname{sgn} f(t^+) - \operatorname{sgn} f(t^-)]$, $t^- < 0 < t^+$ are enough close to 0 ($\operatorname{sgn} a$ means the signum of a real number $a \neq 0$).

Example. Let $f = (f_1, f_2)\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $f_1 = x_1^2 - x_2^3$, $f_2 = x_1^3 + x_2^5$. Then $A = \{x_1^2 - x_2^3 = 0\}$, $p(t) = (t^3, t^2)$, $df_2 = (3x_1^2, 5x_2^4)$ and by Theorem 1.2 we obtain

$$\deg_0 f = deg_0(t^9 + t^{10}) \det \begin{bmatrix} 3t^6, 5t^8 \\ 3t^2, 2t \end{bmatrix} = 1.$$

Example. Let $f = (f_1, \ldots, f_n)\colon (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$, $f_i = x_n^2 - x_i^2$, $i = 1, \ldots, n-1$, $f_n = x_i \ldots x_n$. By easy calculation we obtain, that A is a collection of 2^{n-1} lines, we find their parametrizations and finally, using Theorem 1.2 we have $\deg_0 f = (-1)^{n-1}2^{n-1}$.

Proof of Theorem 1.2 is given in Section 2 of this paper. In Section 3 we consider the situation, where A is smooth at 0. In Section 4 we compare $\deg_0 f$ with the *Teissier's number* $T_0(f_{\mathbb{C}})$ of the complexification $f_{\mathbb{C}}$ of f, which is defined as follows: for every l in some Zariski open subset of $\mathbb{P}^{n-1}(\mathbb{C})$, $T_0(f_{\mathbb{C}})$ means the multiplicity of the curve $f_{\mathbb{C}}^{-1}(l\mathbb{C})$ at 0 (cf. [13],[14],[15]). We prove the following theorem.

Theorem 1.3. *If $f = (f_1, \ldots, f_n)\colon (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ is a real analytic mapping, and $0 \in \mathbb{C}^n$ is isolated in $(f_{\mathbb{C}})^{-1}(0)$, then $|\deg_0 f| \leq T_0(f_{\mathbb{C}})$.*

As a consequence of the above theorem we obtain the Eisenbud-Levine-Teissier inequalities (compare [4]).

At the end of this part we would like to mention two applications of Theorem 1.2. The proofs are simple consequence our main theorem and Proposition 2.3 of this paper. The first one concerns results obtained by Arnold and Khovansky (cf. [1],[7]).

Example. Let us assume, that the components of the mapping $f = (f_1, \ldots, f_n)\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are homogeneous forms of the degrees $m_1, \ldots, m_n$ and $f^{-1}(0) = 0$. If $\Sigma_{i=1}^n(m_i - 1)$ is odd, then $\deg_0 f = 0$.

The second one concerns polynomial equations and can be obtained also by Bezout's theorem. For any polynomial F of n complex variables let F^+ be the sum of its monomials of the greatest degrees.

Example. Let $F = (F_1, \ldots, F_n)\colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a polynomial mapping with real coefficients and odd degrees and let us put $F^+ = (F_1^+, \ldots, F_n^+)$. If $(F^+)^{-1}(0) = 0$, then there exists a real solution of the system of equations $F_1(z) = \cdots = F_n(z) = 0$.

The problem of calculating the degree by reduction to $n-1$ dimensional case was investigated by Bliznyakov (cf. [3]).

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2. **The main theorem.** The proof of the main theorem we will precede by two auxiliary lemmas. By Ω we mean some open, connected neighbourhood of the point $0 \in \mathbb{R}^n$. If $f_1, \ldots, f_{n-1} \colon \Omega \longrightarrow \mathbb{R}$ are analytic functions in $\Omega, f_1(0) = \cdots = f_{n-1}(0) = 0$, then by A we always mean the set $\{x \in \Omega : f_1(x) = \cdots = f_{n-1}(x) = 0\}$. If A is an analytic curve, then by A, $i = 1, \ldots, k$ we mean its irreducible component and by $p_i : I \longrightarrow A_i$ we mean a parametrization of A_i.

LEMMA 2.1. *Let $f_1, \ldots, f_{n-1} : \Omega \longrightarrow \mathbb{R}$ be analytic functions in Ω, $f_1(0) = \cdots = f_{n-1}(0) = 0$. If 0 is not an isolated point of A, and the differentials $df_1(x), \ldots, df_{n-1}(x)$ are linearly independent for $x \in A \setminus \{0\}$, then there exists a real number $\epsilon_0 > 0$ such that for every positive real number $\epsilon < \epsilon_0$ and for every i, $i = 1, \ldots, k$, the following conditions hold:*

(a) $A_i \cap S_\epsilon = \{a_i^-, a_i^+\} = \{p_i(t_i^-), p_i(t_i^+)\}$ where $t_i^- < 0 < t_i^+$ and the intersection is transversal,

(b) for every of set linearly independent vectors $v_1, \ldots, v_{n-1}$ of the tangent space $T_{a_i^-}S_\epsilon$ ($T_{a_i^+}S_\epsilon$ resp.), such that $\det\big(p_i'(t_i^-), v_1, \ldots, v_{n-1}\big) > 0$ [$\det(p_i'(t_i^+), v_1, \ldots, v_{n-1}) > 0$ resp.] we have $\det\big(p_i(t_i^-), v_1, \ldots, v_{n-1}\big) < 0$, [$\det(p_i(t_i^+), v_1, \ldots, v_{n-1}) > 0$ resp.].

PROOF. It is sufficient to prove the above lemma for any irreducible component A_i of the analytic curve A. Then $A_i \cap S_\epsilon = \{x \in \Omega : x = p_i(t), \|p_i(t)\|^2 = \epsilon^2, t \in (-\delta, \delta)\}$. Since p_i is analytic, then we obtain

$$\|p_i(t)\|^2 = \langle p_i(t), p_i(t)\rangle = a_{l_i} t^{2l_i} + \cdots + a_{l_i} > 0, \quad l_i \in \mathbb{N} \tag{1}$$

and $\|p_i(t)\|^2 = 0$ iff $t = 0$. By the condition $a_{l_i} > 0$ we have $A_i \cap S_\epsilon = \{p_i(t_i^-), p_i(t_i^+)\}$ for $t_i^- < 0 < t_i^+$, enough close to 0, $p_i(t_i^-) \neq p_i(t_i^+)$. After differentiating (1) we obtain

$$\langle p_i'(t), p_i(t)\rangle = l_i a_{l_i} t^{2l_i - 1} + \cdots + a_{l_i} > 0, \quad l_i \in \mathbb{N}, \tag{2}$$

which implies transversality of the intersection $A_i \cap S_\epsilon$ for every $t \neq 0$, $t \in (-\delta, \delta)$. The condition (a) has been proved.

One can check, that the equalities $\langle p_i(t), v_i\rangle = 0, \ldots, \langle p_i(t), v_{n-1}\rangle = 0$, properties of Gram's determinant G and (2) imply

$$\begin{aligned} &\operatorname{sgn}\det\big(p_i'(t), v_1, \ldots, v_{n-1}\big) \\ &\quad = \operatorname{sgn}\Big[\det\big(p_i'(t), v_1, \ldots, v_{n-1}\big)^T \det\big(p_i(t), v_1, \ldots, v_{n-1}\big)\Big] \\ &\quad = \operatorname{sgn}[G(v_1, \ldots, v_{n-1})\langle p_i(t), p_i'(t)\rangle] \\ &\quad = \operatorname{sgn} t, \quad t \neq 0. \end{aligned} \tag{3}$$

It ends the proof.

Moreover, by equality (2), we have

LEMMA 2.2. *For every* $i \in \{1, \ldots, k\}$, $\deg_0\langle p_i, p_i'\rangle = 1$.

We can pass now to the proof of our theorem.

PROOF OF THEOREM 1.2. We proceed in two steps. In the first step we find some regular value of the mapping $f_\epsilon \colon S_\epsilon \longrightarrow S_1$ and in the second step we calculate the degree of the mapping f_ϵ (cf. [2],[8]).

First step. We can choose $\epsilon_0 > 0$ such that, for every $\epsilon > 0$ with $\epsilon < \epsilon_0, f^{-1}(0) \cap \bar{B}(0, \epsilon) = \{0\}$ and Lemma 2.1 holds. So we have

$$A \cap S_\epsilon = \bigcup_{i=1}^{k} \{a_i^-, a_i^+\}$$

If we denote $n = (0, \ldots, 0, 1) \in S_1$, $s = (0, \ldots, 0, -1) \in S_1$, we check, that $f_\epsilon^{-1}(n) \cup f_\epsilon^{-1}(s) = \{a_1^-, a_1^+, \ldots, a_k^-, a_k^+\}$. We shall show that $a_i^-, a_i^+, \ldots, a_k^-, a_k^+$ are pairwise different regular points of f_ϵ, which implies, that $n, s \in S_1$ are regular values of f_ϵ.

One sees easily, that for every $a \in A \cap S_\epsilon = \{a_1^-, a_1^+, \ldots, a_k^-, a_k^+\}$ the following holds

$$d(f/\|f\|)(a) = \big(df_1(a)/|f_n(a)|, \ldots, df_{n-1}(a)/|f_n(a)|, 0\big). \tag{1}$$

The point a is a regular point of A, so (1) implies that the kernel $\ker d(f/\|f\|)(a)$ is equal to T_aA. Since intersection $S_\epsilon \cap A$ is transversal at the point a, then $T_aS_\epsilon \cap T_aA = \{0\}$ and $T_aS_\epsilon \oplus T_aA = \mathbb{R}^n$. One can show that $d(f/\|f\|)(a)$ is an isomorphism of $\mathbb{R}^n/\ker d(f/\|f\|)(a)$ onto $T_{f(a)}S_1$. Moreover T_aS_ϵ is isomorphic to $\mathbb{R}^n/\ker d(f/\|f\|)(a)$, then $d(f/\|f\|)(a)\big|T_aS_\epsilon$ is an isomorphism of T_aS_ϵ onto $T_{f(a)}S_1$. Then (1) implies, that $df_\epsilon(a)$ is an isomorphism of T_aS_ϵ onto $T_{f(a)}S_1$, which means that a is a regular point of f_ϵ.

Second step. Assume, that the points $n, s \in S_1$ are regular values of f_ϵ, then

$$2\deg_0 f = \sum_{i=1}^{k}\big(\operatorname{sgn} df_\epsilon(a_i^-) + \operatorname{sgn} df(a_i^+)\big). \tag{2}$$

Since $v_1, \ldots, v_{n-1} \in T_aS_\epsilon$, $a = p_i(t)$, $i = 1, \ldots, k$, and $\det\big(p_i(t_i), v_1, \ldots, v_{n-1}\big) > 0$ then formula (1) implies, for $j \in \{1, \ldots, n-1\}$,

$$(df_\epsilon)(a)(v_j) = \big(1/|f_n(a)|\big)\big(\langle df_1(a), v_j\rangle, \ldots, \langle df_{n-1}(a), v_j\rangle, 0\big). \tag{3}$$

Since $f_1(a) = \cdots = f_{n-1}(a) = 0$, we have

$$f_\epsilon(a) = \big(0, \ldots, 0, f_n(a)\big)/|f(a)|. \tag{4}$$

Using (3), (4) and the equalities $\langle df_1(a), p_i'(t)\rangle = 0, \ldots, \langle df_{n-1}(a), p_i'(t)\rangle = 0$ we calculate

$$\begin{aligned} &\|p_i'(t)\|^2 |f_n(a)|^n \det\big(f_\epsilon(a), df_\epsilon(a)(v_1), \ldots, df_\epsilon(a)(v_{n-1})\big) \\ &\quad = f_n(a)\det\big(df_1(a), \ldots, df_{n-1}(a), p_i'(t)\big)^T \det\big(p_i'(t), v_1, \ldots, v_{n-1}\big). \end{aligned} \tag{5}$$

By the definition of $\operatorname{sgn} df_\epsilon(a)$ and (5) we obtain

$$\begin{aligned}
\operatorname{sgn} df_\epsilon(a) &= \operatorname{sgn}\det\bigl(f_\epsilon(a), df_\epsilon(a)(v_1), \dots, df_\epsilon(a)(v_{n-1})\bigr) \\
&= \operatorname{sgn}\bigl[f_n(a)\det\bigl(f_\epsilon(a), df_\epsilon(a)(v_1), \dots, df_\epsilon(a)(v_{n-1})\bigr) \\
&\qquad\qquad \times \operatorname{sgn}\det\bigl(p_i'(t), v_1, \dots, v_{n-1}\bigr)\bigr].
\end{aligned} \tag{6}$$

Then, Lemma 2.1 (b) and (6) give

$$\begin{aligned}
\operatorname{sgn} df_\epsilon(a_i^+) &= \operatorname{sgn}\bigl[f_n(a_i^+)\det\bigl(df_1(a_i^+), \dots, df_{n-1}(a_i^+), p_i'(t_i^+)\bigr)\bigr] \\
\operatorname{sgn} df_\epsilon(a_i^-) &= -\operatorname{sgn}\bigl[f_n(a_i^-)\det\bigl(df_1(a_i^-), \dots, df_{n-1}(a_i^-), p_i'(t_i^-)\bigr)\bigr]
\end{aligned}$$

The last equalities and (2) end the proof of Theorem 1.2.

PROPOSITION 2.3. *Let* $f = (f_1, \dots, f_n)\colon (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0)$ *be an analytic mapping such that 0 is isolated in* $f^{-1}(0)$. *If the condition (i) of Theorem 1.2 is not fulfilled, then* $\deg_0 f = 0$. *If the condition (i) is fulfilled, then there exists a linear automorphism* $L\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *such that* $\deg_0(L \circ f) = 0$ *or the mapping* $L \circ f$ *fulfills the assumptions of Theorem 1.2.*

PROOF. If the condition (i) of Theorem 1.2 is not fulfilled, then there exists $\epsilon_0 > 0$, such that for every $\epsilon > 0$, $\epsilon < \epsilon_0$ the mapping f_ϵ is not surjective $\bigl(f_\epsilon^{-1}(0, \dots, 0, 1) = \emptyset\bigr)$. It means, that $\deg_0 f = 0$.

If the condition (i) of Theorem 1.2 is fulfilled we can define the mapping

$$\Omega \backslash f_n^{-1}(0) \ni x \longrightarrow f(x) = \bigl((f_1/f_n)(x), \dots, (f_{n-1}/f_n)(x)\bigr) \in \mathbb{R}^{n-1}.$$

Let, by Sard's theorem, $y = (y_1, \dots, y_{n-1}) \in \mathbb{R}^{n-1}$ be any regular value of f. The analytic mapping $L \circ f = (f_1 - y_1 f_n, \dots, f_{n-1} - y_{n-1} f_n, f_n)\colon (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ has isolated zero in $f^{-1}(0)$ and $\deg_0(L \circ f) = \deg_0 f$.

Let A_y be the following, analytic subset of Ω:

$$A_y = \{x \in \Omega : (f_1 - y_1 f_n)(x) = \dots = (f_{n-1} - y_{n-1} f_n)(x) = 0\}.$$

If $(f^{-1})(y) = \emptyset$, then $\deg_0(L \circ f) = 0$.

We may then assume that $f^{-1}(y) \neq \emptyset$ in Ω. One can check the following: $x \in f^{-1}(y)$ iff $x \in A_y \backslash \{0\}$. It implies, that $L \circ f$ fulfills condition (i) of Theorem 1.2.

Since y is a regular value of f, then $d(f_1/f_n)(x), \dots, d(f_{n-1}/f_n)(x)$ are linearly independent. The differentials $d(f_1 - y_1 f_n)(x), \dots, d(f_{n-1} - y_{n-1} f_n)(x)$ are linearly independent, since for every $i \in \{1, \dots, n-1\}$, $d(f_i/f_n)(x) = \bigl(1/f_n(x)\bigr) d(f_i - y_i f_n)(x)$ if $x \in A_y \backslash \{0\}$. The mapping $L \circ f$ fulfills condition (ii) of Theorem 1.2 , so it completes the proof of Proposition 2.3.

COROLLARY 2.4. *In the statement of Theorem 1.2 we may replace the function* f_n *by any function* f_i, $i = 1, \dots, n-1$.

At the end of this section we will compare $|\deg_0 f|$ with the number of the irreducible components of A.

Let $f_1,\dots,f_{n-1}(\mathbb{R}^n,0)\longrightarrow(\mathbb{R},0)$ be analytic functions and let their differentials be linearly independent at non-empty set $A\setminus\{0\}$. Then A has $k\geq 1$ irreducible components. Let $f_n\colon(\mathbb{R}^n,0)\longrightarrow(\mathbb{R}^n,0)$ be any analytic function such, that $A\cap\{x\in\Omega: f_n(x)=0\}=\{0\}$. Then the mapping $f=(f_1,\dots,f_n)$ fulfils the assumptions of Theorem 1.2 and we obtain:

THEOREM 2.5. $|ind_0 f|\leq k$.

PROPOSITION 2.6. *If* $f_1,\dots,f_{n-1}$ *are as above and*

$$f_n(x)=\operatorname{Jac}\big(f_1(x),\dots,f_{n-1}(x),1/2\|x\|^2\big),$$

then $\deg_0 f=k$.

PROOF. One can check, that f fulfills the assumptions of Theorem 1.2. The equalities $\langle df_1\big(p_i(t)\big),p_i(t)\rangle=\cdots=\langle df_{n-1}\big(p_i(t)\big),p_i(t)\rangle=0$, $i=1,\dots,k$, properties of Gram's determinant G and Lemma 2.2 imply

$$\begin{aligned}\deg_0 f&=\sum_{i=1}^{k}\deg_0\Big[\operatorname{Jac}(f_1,\dots,f_{n-1},1/2\|x\|^2)\big(p_i(t)\big)\\&\qquad\det\Big(df_1\big(p_i(t)\big),\dots,df_{n-1}\big(p_i(t)\big),p_i'(t)\Big)\Big]\\&=\sum_{i=1}^{k}\deg_0 G\Big(df_1\big(p_i(t)\big),\dots,df_{n-1}\big(p_i(t)\big)\langle p_i(t),p_i'(t)\rangle\Big)\\&=\sum_{i=1}^{k}\deg_0\langle p_i(t),p_i'(t)\rangle=k.\end{aligned}$$

It ends the proof.

3. **One-dimensional smooth case.**

It is natural to ask about the particular case of the formula obtained in Theorem 1.2 , if A is a one-dimensional manifold around 0. We find a generalization of the formulas, known in $\mathbb{R}^2$ (cf. [7]). Let us put $D(x)=\partial(f_1,\dots,f_{n-1})/\partial(x_1,\dots,x_{n-1})(x)$.

THEOREM 3.1. *Let* $f=(f_1,\dots,f_n)\colon(\mathbb{R}^n,0)\longrightarrow(\mathbb{R}^n,0)$ *be a real analytic mapping, 0 is isolated in* $f^{-1}(0)$ *and* $D(0)\neq 0$. *Then A a is one-dimensional manifold in* Ω *with a parametrization* $p\colon I\longrightarrow A$ *and* $\deg_0 f=\operatorname{sgn} D(0)\deg_0(f_n\circ p)$.

PROOF. The first part of the thesis is a consequence of the implicit mapping theorem. One can check, that Theorem 1.2 gives

$$\deg_0 f=\deg_0[(f_n\circ p)\det(df_1\circ p,\dots,df_{n-1}\circ p,p')]\tag{1}$$

If we differentiate the system of equations $f_1\big(p(t)\big)=0,\dots,f_{n-1}\big(p(t)\big)=0$ fulfill for every $t\in I$, then we will obtain

$$\det\Big(df_1\big(p(t)\big),\dots,df_{n-1}\big(p(t)\big),p'(t)\Big)=D\big(p(t)\big)\|p'(t)\|^2\tag{2}$$

for every $t\in I$. By (1) and (2) we end the proof.

Let us introduce the following notation (cf. [15]). For every C^1 mapping $f=(f_1,\dots,f_n)$ let $J_f^0=f_n$ and $J_f^{k+1}=\operatorname{Jac}(f_1,\dots,J_f^k)$ for $k\geq 0$.

THEOREM 3.2. *If* $f\colon(\mathbb{R}^n,0)\longrightarrow(\mathbb{R}^n,0)$ *is an analytic mapping and* $J_f^1(0)=\cdots=J_f^{k-1}(0)=0$, $J_f^k(0)\neq 0$, *then 0 is isolated in* $f^{-1}(0)$, *and*

$$\deg_0 f=[\bigl(1-(-1)^k\bigr)/2]\operatorname{sgn}J_f^k(0).$$

PROOF. Since $J_f^k(0)\neq 0$, then the differentials: $df_1(0),\dots,df_{n-1}(0)$ are linearly independent. There exists an $n-1$ minor of the matrix $\bigl(df_1(0),\dots,df_{n-1}(0)\bigr)$ not equal 0. After some permutation of the coordinates we may assume $D(0)\neq 0$. Then, by the implicit mapping theorem, there exists Ω, in which the set A is a one-dimensional manifold parametrized by $p(t)=\bigl(h_1(t),\dots,h_{n-1}(t),t\bigr)$, $t\in I$. If we differentiate the system of equations $f_1\bigl(p(t)\bigr)=0,\dots,f_{n-1}\bigl(p(t)\bigr)=0$, $t\in I$, then

$$J_f^{i+1}\bigl(p(t)\bigr)=D\bigl(p(t)\bigr)\bigl[J_f^i\bigl(p(t)\bigr)\bigr]' \quad i=0,1,\dots \tag{1}$$

Using (1) we can prove by induction ($f^{(i)}$ means i-th differential)

$$\bigl[f_n\bigl(p(t)\bigr)\bigr]^{(i)}=\bigl[1/D\bigl(p(t)\bigr)\bigr]^i J_f^i\bigl(p(t)\bigr)+A_1(t)J_f^{i-1}\bigl(p(t)\bigr)+\cdots+A_i(t)J_f^0\bigl(p(t)\bigr) \tag{2}$$

where $A_j\ j=1,\dots,i$ are analytic functions of $t\in I$.

The assumption $J_f^1(0)=\cdots=J_f^{k-1}(0)=0$ and (2) follow

$$(f_n\circ p)^{(k)}(0)=[1/D(0)]^k J_f^k(0) \tag{3}$$

By $J_f^k(0)\neq 0$ and (3) we conclude, that $f_n\circ p$ has an isolated zero at $t=0$. It implies, f has isolated zero at $x=0$. Since we can use Theorem 3.1, then we obtain

$$\deg_0 f=\operatorname{sgn}D(0)\deg_0(f_n\circ p) \tag{4}$$

The equality (3) follows

$$\deg_0(f_n\circ p)=\bigl[\bigl(1-(-1)^k\bigr)/2\bigr]\operatorname{sgn}D(0)\operatorname{sgn}J_f^k(0) \tag{5}$$

Finally (4) and (5) complete the proof of Theorem 3.2.

4. Evaluation of the degree by some complex invariants. Eisenbud-Levine-Teissier's inequalities.

Let $g=(g_1,\dots,g_n)\colon(\mathbb{C}^n,0)\longrightarrow(\mathbb{C}^n,0)$ be a finite holomorphic mapping (i.e. 0 is isolated in $g^{-1}(0)$). By $m_0(g)$ we mean the multiplicity of g at 0 (cf. [2],[16]) and $\operatorname{ord}g=\min\{\operatorname{ord}g_i,\dots,\operatorname{ord}g_n\}$, where $\operatorname{ord}g_i$ means the order of the function g_i at 0. For any $l=(l_1;\dots;l_n)\in\mathbb{P}^{n-1}(\mathbb{C})$ the set $g^{-1}(l\mathbb{C})=\{z: g(z)=lt, t\in\mathbb{C}\}$ is an analytic curve in some neighbourhood of $0\in\mathbb{C}^n$. By $\operatorname{mult}_0 S$ we mean the multiplicity of the puredimensional analytic subset S at $0\in S$ (cf. [16]). Let us put $T_0(g)=\operatorname{mult}_0 g^{-1}(l\mathbb{C})$ for any l in some Zariski open subset of $\mathbb{P}^{n-1}(\mathbb{C})$ (cf. [12],[13],[14]). A real analytic mapping is finite if its complexification $f_\mathbb{C}$ is finite.

In this part of the paper we investigate the relations between the following invariants of the finite real analytic mapping f: the degree $\deg_0 f$, the Teissier's number $T_0(f_\mathbb{C})$ and the multiplicity $m_0(f_\mathbb{C})$.

PROPOSITION 4.1. *If* $g = (g_1, \ldots, g_n) : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^n, 0)$ *is a finite holomorphic mapping, then:*

(a) $T_0(g) \le (1/\max_{i=1}^n\{\operatorname{ord} g_i\}) m_0(g)$ *with '=' if* $\bigcap_{i=1}^n (\mathit{in} g_i)^{-1}(0) = \emptyset$ *in* $\mathbb{P}^{n-1}(\mathbb{C})$.

(b) $T_0(g) \ge (1/\max_{i=1}^n\{\operatorname{ord} g_i\}) \prod_{i=1}^n \operatorname{ord} g_i$ *with '=' if there exists* $i_0 \in \{1, \ldots, n\}$ *such that* $\operatorname{ord} g_i \ge \operatorname{ord} g_i$ *for* $i \ne i_0$ *and* $\bigcap_{i=1}^n (\mathit{in} g_i)^{-1}(0)$ *is finite in* $\mathbb{P}^{n-1}(\mathbb{C})$.

PROOF. We can assume that $\operatorname{ord} g_n \ge \operatorname{ord} g_i$, $i = 1, \ldots, n-1$. Let $l = (l_1; \ldots; l_n)$ be in some Zariski open subset of $\mathbb{P}^{n-1}(\mathbb{C})$, $l_n \ne 0$. Then $T_0(g) = \operatorname{mult}_0\{l_n g_1(z) - l_1 g_n(z) = 0, \ldots, l_n g_{n-1}(z) - l_{n-1} g_n(z) = 0\} = \operatorname{mult}_0 g^{-1}(l\mathbb{C})$. If $g^{-1}(l\mathbb{C}) = S_1 \cup \cdots \cup S_r$ is the decomposition of the analytic curve $g^{-1}(l\mathbb{C})$ into irreducible components with parametrisations p_i: $U \longrightarrow S_i$ where U is a small enough disc around 0, then

$$\begin{aligned} m_0(g) &= \operatorname{mult}_0(l_n g_1 - l_1 g_n, \ldots, l_n g_{n-1} - l_{n-1} g_n, g_n) \\ &= \sum_{i=1}^k k_i \operatorname{ord}(g_n \circ p_i) \ge \operatorname{ord} g_n \sum_{i=1}^k \operatorname{ord} p_i \\ &= \operatorname{ord} g_n \operatorname{mult}_0 g^{-1}(l\mathbb{C}), \end{aligned}$$

where k_i are some positive integers (cf. [9]). If we assume additionally that $\bigcap_{i=1}^n (\mathit{in} g_i)^{-1}(0) = \emptyset$ in $\mathbb{P}^{n-1}(\mathbb{C})$, then above inequality we can replace by the equality. This ends the proof of (a).

Let L: $\mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a linear form with sufficient general coefficients. Using well known inequality $m_0 h \ge \operatorname{ord} h_i \cdots \operatorname{ord} h_n$ (if $h = (h_1, \ldots, h_n)$) we have

$$\begin{aligned} T_0(g) &= \operatorname{mult}_0 g^{-1}(l\mathbb{C}) \\ &= mu_0(l_n g_1 - l_1 g_n, \ldots, l_n g_{n-1} - l_{n-1} g_n, L) \ge \operatorname{ord} g_1 \cdots \operatorname{ord} g_{n-1}. \end{aligned}$$

If we assume additionally that $\bigcap_{i=1}^{n-1} (\mathit{in} g_i)^{-1}(0)$ is finite in $\mathbb{P}^{n-1}(\mathbb{C})$ we can replace above inequality by the equality. This completes the proof of (b).

COROLLARY 4.2. *If* $\bigcap_{i=1}^n (\mathit{in} g_i)^{-1}(0) = \emptyset$ *in* $\mathbb{P}^{n-1}(\mathbb{C})$ *and* $\operatorname{ord} g_n \ge \operatorname{ord} g_i$, $i = 1, \ldots, n-1$, *then* $T_0(g) = \operatorname{ord} g_1 \cdots \operatorname{ord} g_{n-1}$.

PROOF. $\bigcap_{i-1}^{n-1} (\mathit{in} g_i)^{-1}(0)$, is finite in $\mathbb{P}^{n-1}(\mathbb{C})$. By Proposition 4.1 (b) we obtain $T_0(g) = \operatorname{ord} g_1 \cdots \operatorname{ord} g_{n-1}$.

In the case $n = 2$ we know, that $\{\mathit{in} g_i = 0\}$ is finite for $i = 1, 2$ (if $g_i \not\equiv 0$). By Proposition 4.1 (b) we have

PROPOSITION 4.3. *If* $g = (g_1, g_2)$: $(\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$ *is a finite holomorphic mapping, then* $T_0(g) = \min\{\operatorname{ord} g_1, \operatorname{ord} g_2\}$.

We pass now to the estimation of the degree $\deg_0 f$ by the complex invariants.

PROOF OF THEOREM 1.3. According to Sard's theorem and the properties of the complexification of the real analytic sets (cf. [10], Ch. 5) we can find $l = (l_1, \ldots, l_n) \in \mathbb{R}^n$, $l_n \ne 0$ such that the mapping $f_l = (l_n f_1 - l_1 f_n, \ldots, l_n f_{n-1} - l_{n-1} f_n, f_n)$ fulfills the assumptions of Theorem 1.2, has the same degree as the mapping f and $T_0(f_\mathbb{C}) = \operatorname{mult}_0 S$, where

$S = \{ l_n f_{1,\mathbb{C}} - l_1 f_{n,\mathbb{C}} = 0, \ldots, l_n f_{n-1,\mathbb{C}} - l_{n-1} f_{n,\mathbb{C}} = 0 \}$. By Theorem 2.5 $|\deg_0 f| \leq k$, where k is the number of irreducible components of $S \cap \mathbb{R}^n$. If r is the number of irreducible components of S, then we have the following evaluation:

$$|\deg_0 f| \leq k \leq r \leq \operatorname{mult}_0 S = T_0(f_{\mathbb{C}}).$$

This ends the proof of Theorem 1.3.

By Theorem 1.3 and Proposition 4.3 we obtain immediately.

COROLLARY 4.4 (CF. [4],THEOREM 2.1I). *If* $f\colon (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0)$ *is a finite real analytic mapping, then* $|\deg_0 f|^2 \leq m_0(f_{\mathbb{C}})$.

By Theorem 1.3 and the following Teissier's inequality: $T_0(g)^n \leq m_0(g)^{n-1}$ if $g\colon (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}^n, 0)$ is a finite holomorphic mapping, (cf. [13],[12],[14]) we have

COROLLARY 4.6 (CF. [4] THEOREM 2.1I). *If* $f\colon (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ *is a finite real analytic mapping, then* $|\deg_0 f|^n \leq m_0(f_{\mathbb{C}})^{n-1}$.

We need now the following, easy to prove lemma (comp. [11] Lemma 3.11).

LEMMA 4.6. *Let* $f\colon (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ *be a real analytic mapping such that 0 is isolated in* $f^{-1}(0)$ *and* $k = \operatorname{rank} df(0) > 0$. *Then there exists a real analytic mapping* $\tilde{f}\colon (\mathbb{R}^k, 0) \longrightarrow (\mathbb{R}^k, 0)$ *such that 0 is isolated in* $\tilde{f}^{-1}(0)$, $|\deg_0 f| = |\deg_0 \tilde{f}|$ *and* $\operatorname{ord}_0 f \geq 2$. *Moreover if* f *is finite, then* $m_0(f_{\mathbb{C}}) = m_0(\tilde{f}_{\mathbb{C}})$.

By the above lemma, Theorem 1.3 and Proposition 4.3 (a) we obtain

COROLLARY 4.7 (CF. [4], THEOREM 2.1II). *If* $f\colon (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ *is a finite real analytic mapping singular at 0, then* $|\deg_0 f| \leq (1/2) m_0(f_{\mathbb{C}})$.

At the end of this paper let us mention the following problem: to find conditions on f to have equality $|\deg_0 f|^n = m_0(f_{\mathbb{C}})^{n-1}$. According to Theorem 1.3 we can divide this problem into two questions:

(1) when $|\deg_0 f| = T_0(f_{\mathbb{C}})$? (in the real-complex domain) and
(2) when $T_0(f_{\mathbb{C}})^n = m_0(f_{\mathbb{C}})^{n-1}$? (in the complex domain).

The answer in the case $n = 2$ was given by Teissier (cf. [14]). The key point in Teissier's proof is to check that the assumption $|\deg_0 f| = m_0(f_{\mathbb{C}})^2$ implies that the tangents cones at 0 of $f_{1,\mathbb{C}} = 0$ and of $f_{2,\mathbb{C}} = 0$ have the same degree and no common components. We can give another proof, which is a consequence of the following proposition.

PROPOSITION 4.7. *Let* $g = (g_1, g_2)\colon (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$ *be a finite holomorphic mapping. Then* $T_0(g) = m_0(g)^2$ *iff* $\operatorname{ord} g_1 = \operatorname{ord} g_2$ *and* $\{ in g_1 = 0 \} \cap \{ in g_2 = 0 \} = \emptyset$.

PROOF. It is known that $m_0(g) = \operatorname{ord} g_1 \operatorname{ord} g_2$ iff $\{ in g_1 = 0 \} \cap \{ in g_2 = 0 \} = \emptyset$. To complete the proof it is enough to make use of Proposition 4.3.

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