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THURSTON DISTANCE ON THE TEICHMÜLLER SPACE OF HYPERBOLIC 3-MANIFOLDS

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In this paper we show that the Thurston distance together with the critical exponent on the Teichmüller space of a convex cocompact hyperbolic 3-manifold distinguishes the different points.

1. INTRODUCTION

In [5] Thurston introduced a nonsymmetric Finsler metric K on the Teichmüller space $\mathcal{T}(S)$ of the closed surface S. It is defined by

$$K(g,h) = \sup_{\alpha \in C} \left\{ \log \left(\frac{l_h(\alpha)}{l_g(\alpha)} \right) \right\}$$

where C is the free homotopy classes of closed loops. He showed that $K(g,h)$ is equal to the minimum of global Lipschitz constants of homeomorphisms in a given homotopy class. From this it easily follows that $K(g,h) \geqslant 0$ and the equality holds if and only if $g = h$. To prove that K is the minimal Lipschitz constant, he uses the generalisation of an earthquake on a surface, namely the cataclysm. For more rigorous definitions and work, see [**1**].

But in the higher dimensional case, an appropriate notion like an earthquake does not exist. However, Thurston's theorem indicates that the comparison of closed geodesic lengths can be used as some kind of a distance in the Teichmüller space. Let $\mathcal{T}_{cc}(\Gamma)$ be the set of faithful, discrete convex cocompact representations from Γ into $Iso(H^3_{\mathbb{R}})$. Define the distance $K(g,h)$ as above between g and h in $\mathcal{T}_{cc}(\Gamma)$.

In this paper we show the following theorem.

THEOREM 1. *$\delta(g) = \delta(h)$ and $K(g,h) = 0$ if and only if g and h are the same points in $\mathcal{T}_{cc}(\Gamma)$, where δ is the critical exponent of the group.*

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Unfortunately, we cannot show the triangle inequality for this distance K, though we believe it is the case as in the 2-dimensional case.

To prove this theorem we shall use an ergodic approach like Patterson-Sullivan measure supported on the limit set of the group, together with the notion of geodesic stretch introduced by [**4**].

Another neat theorem about $\mathcal{T}_{cc}(\Gamma)$, more generally about the space $\mathcal{R}(\Gamma)$ of non-elementary representations from Γ into $Iso(H^3_{\mathbb{R}})$ is the following. See [**2, 3**].

THEOREM 2. *There is a finite set $\{g_1, \dots, g_N\}$ in Γ such that if $l(\phi(g_i)) = l(\xi(g_i))$ for all $i = 1, \cdots, N$, then ϕ and ξ are conjugate.*

So for two hyperbolic infinite volume 3-manifolds M and N with the same finitely generated fundamental group, if they have the same geodesic lengths on the finite set of closed loops, then they are isometric.

The method of proof of this theorem relies on the concept of the cross-ratio on the ideal boundary of negatively curved space, which agrees with the Furstenberg boundary in this case. Another ingredient of the proof is the simple observation that the polynomial ring over $\mathbb{R}$ is Noetherian. See [**2, 3**]. We should comment that these theorems are still valid for any rank one symmetric spaces.

2. PRELIMINARIES

Let $X = H^3_{\mathbb{R}}$ and $\partial X = S^2$ be the ideal boundary of X.

DEFINITION 1: The *cross ratio* of four points x, y, z, w in ∂X is defined by:

$$[x, y, z, w] = \frac{|x-z||y-w|}{|x-w||y-z|}$$

where x, y, z, w are in $\mathbb{R}^2$ when S^2 is viewed as the one point compactification of $\mathbf{R}^2$.

Note when we use the upper-half space model, the ideal boundary is naturally identified with the Riemann sphere and this cross ratio coincides with the usual cross ratio of four complex numbers.

DEFINITION 2: Let α be an isometry of X. The *translation length* $l(\alpha)$ of α is defined by:

$$l(\alpha) = \inf_{x \in X} \Big\{ d\big(x, \alpha(x)\big) \Big\}.$$

DEFINITION 3: Let Γ be a subgroup of $Iso(X)$. The limit set of Γ is defined as:

$$\Lambda_\Gamma = \overline{\Gamma x} \cap \partial X$$

where $\overline{\Gamma x}$ is the closure of the orbit Γx in $X \cup \partial X$ where $X \cup \partial X$ is equipped with the sphere topology. For a representation $\rho : G \to Iso(X)$, we denote the limit set of $\rho(G)$ by Λ_ρ.

3. Marked length spectrum as an invariant of Teichmüller space

In this section we prove:

Theorem 3. *Let $\rho, \phi : \Gamma \to Iso(X)$ be two representations with the same marked length spectrum. Then there is an element $\alpha \in Iso(X)$ such that $\rho = \alpha\phi\alpha^{-1}$.*

The proof consists of two simple lemmas.

Lemma 1. *(Marked length spectrum determines the cross ratio.) Let a and b be hyperbolic isometries with disjoint fixed points. Then*

$$\lim_{n\to\infty} e^{l(a^n b^n) - l(a^n) - l(b^n)} = [a^-, b^-, a^+, b^+]$$

where a^- is the repelling fixed point of a and a^+ is the attracting fixed point of a.

Proof: The distance between two points x and y in X when we view X as a unit ball in $\mathbb{R}^3$ is

$$\cosh\big(d(x,y)\big) = \frac{\big|1 - \langle x, y\rangle\big|}{(1-\langle x,x\rangle)^{1/2}(1-\langle y,y\rangle)^{1/2}}.$$

Then it is easy to see that the cross ratio of four points x, y, z, w can be written as:

$$[x,y,z,w] = \frac{\langle\langle z,x\rangle\rangle\langle\langle w,y\rangle\rangle}{\langle\langle w,x\rangle\rangle\langle\langle z,y\rangle\rangle}$$

where $\langle\langle x,y\rangle\rangle = \big|1 - \langle x,y\rangle\big|$. Then choose sequences $\{x_i^n\}$ such that $\lim_{n\to\infty} x_i^n = x_i$ for $i = 1,2,3,4$. If we put $d_{13}^n = d(x_1^n, x_3^n)$, $d_{24}^n = d(x_2^n, x_4^n)$, $d_{23}^n = d(x_2^n, x_3^n)$, $d_{14}^n = d(x_1^n, x_4^n)$ then

$$\begin{aligned}
\big|[x_1,x_2,x_3,x_4]\big| &= \frac{\langle\langle x_3,x_1\rangle\rangle\langle\langle x_4,x_2\rangle\rangle}{\langle\langle x_4,x_1\rangle\rangle\langle\langle x_3,x_2\rangle\rangle} \\
&= \lim \frac{\cosh d_{13}^n \cosh d_{24}^n}{\cosh d_{14}^n \cosh d_{23}^n} \\
&= \lim \frac{(e^{d_{13}^n} + e^{-d_{13}^n})(e^{d_{24}^n} + e^{-d_{24}^n})}{(e^{d_{14}^n} + e^{-d_{14}^n})(e^{d_{23}^n} + e^{-d_{23}^n})} \\
&= \lim \frac{e^{d_{13}^n} e^{d_{24}^n}}{e^{d_{14}^n} e^{d_{23}^n}} + \lim \frac{e^{d_{13}^n} e^{-d_{24}^n}}{e^{d_{14}^n} e^{d_{23}^n}} + \lim \frac{e^{d_{24}^n} e^{-d_{13}^n}}{e^{d_{14}^n} e^{d_{23}^n}} \\
&= \lim e^{d_{13}^n + d_{24}^n - d_{14}^n - d_{23}^n} + \lim e^{d_{13}^n - d_{14}^n - d_{23}^n - d_{24}^n} \\
&\qquad + \lim e^{d_{24}^n - d_{14}^n - d_{23}^n - d_{13}^n}.
\end{aligned}$$

Since $\lim_{n\to\infty} d_{13}^n + d_{24}^n - d_{14}^n - d_{23}^n$ always exists, both $d_{13}^n - d_{14}^n - d_{23}^n - d_{24}^n$ and $d_{24}^n - d_{14}^n - d_{23}^n - d_{13}^n$ go to $-\infty$. It is easy to see that $\lim_{n\to n} l(a^n b^n) - l(a^n) - l(b^n) = \lim_{n\to\infty} d_{13}^n + d_{24}^n - d_{14}^n - d_{23}^n$. For the detailed exposition we refer the reader to **[2]**. ∎

Lemma 2. *(Pairwise distance determines the rigid body.) Let x_i and x_i' be points in $\mathbb{R}^n$. If $|x_i - x_j| = |x_i' - x_j'|$ for each pair, then there is an isometry of $\mathbb{R}^n$ which sends x_i to x_i'.*

PROOF: This is an easy exercise in linear algebra. ∎

PROOF OF THEOREM 3 By the above lemma, if we define the map f from the limit set of ρ to the limit set of ϕ in the way that for each $a \in \Gamma$, $f(\rho(a)^{\pm}) = \phi(a)^{\pm}$, f preserves the cross ratio on the limit set. By conjugating representations if necessary, we may assume that 0 and ∞ are in the limit sets and $f(0) = 0, f(\infty) = \infty$. Then fix any point $y \in \Lambda_\rho$ and for any $x \in \Lambda_\rho$

$$\frac{|x|}{|y|} = \left|[x, y, 0, \infty]\right|^{1/2} = \left|[f(x), f(y), 0, \infty]\right|^{1/2} = \frac{|f(x)|}{|f(y)|}.$$

So $|x| = |y|/|f(y)| = C|f(x)|$ for $C = |y|/|f(y)|$. Similarly

$$\frac{d(x,z)}{|z|} = \left|[0, x, \infty, z]\right|^{1/2} = \left|[0, f(x), \infty, f(z)]\right|^{1/2} = \frac{d(f(x), f(z))}{|f(z)|}.$$

Hence $d(x,z) = Cd(f(x), f(z))$. By conjugating one of the representations, we can make $C = 1$. So we get the map from Λ_ρ to Λ_ϕ which preserves the pairwise distance in $\mathbb{R}^2$. Since the representations are not elementary, f is actually an isometry of $\mathbb{R}^2$ and it came from an isometry α of X. This shows that two representations are conjugate by α. ∎

4. GEODESIC STRETCH

In this section we introduce the general notion of the geodesic stretch and prove some useful connections with the marked length spectrum. Let $M = \widetilde{M}/G$ be a manifold, where G is a deck transformation group of the universal cover $\widetilde{M}$. Fix two Riemannian metrics g_1, g_2. For each vector v in $(SM)_{g_1}$ and for each $t \in \mathbb{R}$, consider the lift $\widetilde{\gamma}_v$ of the geodesic γ_v to the universal cover $\widetilde{M}$. Let $a(v,t) = d_{g_2}(\widetilde{\gamma}_v(0), \widetilde{\gamma}_v(t))$ be the distance of the endpoints of the segment $\widetilde{\gamma}_v(s)$, $0 \leqslant s \leqslant t$, with respect to the metric g_2 lifted to $\widetilde{M}$. Then

$$a(v, t_1 + t_2) \leqslant a(v, t_1) + a({g_1}^{t_1} v, t_2)$$

for all $v \in S\widetilde{M}$.

Define $I_\mu(g_1, g_2, v) = \lim\limits_{t\to\infty} (a(v,t)/t)$ for a geodesic flow invariant probability measure μ. Define $\Omega((SM)_{g_1})$ to be the set of non-wandering points in $(SM)_{g_1}$ that is, the unit vectors whose forward and backward end points belong to the limit set. Note this is compact when the manifold is convex cocompact. If M is convex cocompact, negatively curved and μ is the Bowen-Margulis measure, then $I_\mu(g_1, g_2, v)$ exists for μ almost all $v \in \Omega((SM)_{g_1})$ and it is a μ integrable function on $\Omega((SM)_{g_1})$ invariant under the geodesic flow ${g_1}^t$.

DEFINITION 4: *Geodesic stretch.* The geodesic stretch of the metric g_2 relative to g_1 and the measure μ is defined as:

$$I_\mu(g_1, g_2) = \int_{\Omega((SM)_{g_1})} I_\mu(g_1, g_2, v)\, d\mu.$$

LEMMA 3. *If two locally symmetric, negatively curved metrics g_1, g_2 define convex cocompact metrics on M, then they are quasi-isometric. So there is a constant R such that every geodesic in one metric is at most R Hausdorff distance away from the geodesic in the other metric.*

PROOF: This follows from the Morse Lemma saying that there is some constant C such that every quasi-geodesic has a geodesic within C-Hausdorff distance, and the fact that the geometry of M is completely determined by its convex core since the metric is locally symmetric. ∎

We want to prove the following theorem.

THEOREM 4. *Let M be a convex cocompact manifold with a metric g_1 which is a quotient of a hyperbolic 3-space. Let g_2 be another hyperbolic metric which makes M convex cocompact. Then $I_{\mu_{BM}}(g_1, g_2) \geqslant \delta(g_1)/\delta(g_2)$ where $I_{\mu_{BM}}(g_1, g_2)$ is the geodesic stretch of g_2 relative to g_1 and the Bowen-Margulis measure μ_{BM} of g_1. Furthermore the following are equivalent.*

1. $I_{\mu_{BM}}(g_1, g_2) = \delta(g_1)/\delta(g_2) = 1$.
2. *There is a time preserving conjugacy between $\Omega(g_1)$ and $\Omega(g_2)$.*
3. *The two manifolds have the same marked length spectrum.*
4. *g_1 and g_2 are isometric.*

PROOF: Let $h(g)$ denote the critical exponent $\delta(\Gamma)$ where Γ realises the metric g on M. Note that as in the closed manifold case, $\delta(g)$ is equal to the Hausdorff dimension of its limit set with respect to the Busemann metric and is realised by

$$\lim_{R\to\infty} \frac{\log\big(volume\big(B(x,R) \cap C(g)\big)\big)}{R}$$

where $C(g)$ is the convex hull of the group Γ realising the convex cocompact metric g and $B(x, R)$ is the geodesic ball of radius R in the universal cover. If g_1 is another metric which is convex cocompact, since the two metrics are equivalent on the quotient of the convex hull,

$$h(g) = \lim_{R\to\infty} \frac{\log\Big(volume_{g_1}\big(B^g\big(B(x,R) \cap C(g)\big)\big)\Big)}{R}.$$

First note that since the geodesic flow is ergodic with respect to the Bowen-Margulis measure μ_{BM} (since the quotient of the convex hull is compact, consider the geodesic flow defined on this compact set), we have

$$\lim_{t\to\infty} \frac{a(v,t)}{t} = I_{\mu_{BM}}(g_1, g_2)$$

for μ_{BM} almost all $v \in (SM)_{g_1}$. Secondly by Lemma 3, given any number $R > 0$, there is $C = C(R)$ such that $B^{g_1}(p, R) \subset B^{g_2}(p, C)$.

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Now we want to show that for large $t > 0$,

$$B^{g_1}(p,t) \subset B^{g_2}(p, C' + t(I_{\mu_{BM}}(g_1,g_2)+\varepsilon))$$

for some constant C'. Cover $S(p,t)$ by $B^{g_1}(x_i,R)$ balls with $x_i \in S(p,t)$. Since $\lim_{t\to\infty} a(v,t)/t = I_{\mu_{BM}}(g_1,g_2)$ for almost all v, for any $B^{g_1}(x_i,R)$, there is $y_i \in B^{g_1}(x_i,R) \cap S(p,t)$ so that $d_{g_2}(p,y_i) \leqslant t(I_{\mu_{BM}}(g_1,g_2)+\varepsilon)$. Since $B^{g_1}(x_i,R) \subset B^{g_2}(y_i,C)$ for some fixed constant C, $B^{g_1}(p,t) \subset B^{g_2}(p, C' + t(I_{\mu_{BM}}(g_1,g_2)+\varepsilon))$ for some C'. Then

$$\begin{aligned} h(g_1) - \varepsilon' &\leqslant \lim_{t\to\infty} \frac{\log(vol_{g_2}(B^{g_1}(p,t)\cap C(g_1)))}{t} \\ &\leqslant \lim_{t\to\infty} \frac{\log\Big(vol_{g_2}(B^{g_2}(p, C' + t(I_{\mu_{BM}}(g_1,g_2)+\varepsilon)) \cap C(g_1))\Big)}{C' + t(I_{\mu_{BM}}(g_1,g_2)+\varepsilon)} \\ &\qquad \cdot \frac{C' + t(I_{\mu_{BM}}(g_1,g_2)+\varepsilon)}{t} \\ &= h(g_2)\cdot I_{\mu_{BM}}(g_1,g_2). \end{aligned}$$

This gives the first part of the theorem.

For the second part of the theorem, it is shown in Proposition 3.7 in [**4**] that $I_{\mu_{BM}}(g_1,g_2) = h(g_1)/h(g_2)$ implies the Patterson-Sullivan measures are equivalent and in turn it implies that geodesic flows are conjugate. This implies that they have the same marked length spectrum and by Theorem 3 they are isometric. ∎

5. Proof of Theorem 1

It suffices to show that $\delta(g) = \delta(h), K(g,h) = 0$ implies that g and h are isometric. Then $K(g,h) = 0$ implies that $l_h(\alpha) \leqslant l_g(\alpha)$ for any closed loop α.

Let $\widetilde{\gamma}_v$ be the lift of the geodesic corresponding to a closed geodesic α. Denote the lift of the geodesic in the metric h corresponding to α by $\widetilde{\gamma}'_v$. Then by the Morse lemma, their Hausdorff distance measured in either metric is bounded by the universal constant. Then $a(v,t) = d_h(\widetilde{\gamma}_v(0), \widetilde{\gamma}_v(t)) \leqslant d_h(\widetilde{\gamma}'_v(0), \widetilde{\gamma}'_v(t)) + C \leqslant (l_h(\alpha))/(l_g(\alpha))t + M$ for some constant C by the Morse lemma. One should be careful that t is measured in g-distance. Then $\lim_{t\to\infty} a(v,t)/t \leqslant 1$. This shows that $I_{\mu_{BM}}(g,h,v) \leqslant 1$ for almost all $v \in \Omega(SM)$ since closed orbits are dense in $\Omega(SM)$. Then this will imply that $I_{\mu_{BM}}(g,h) \leqslant 1$. But by Theorem 4, $I_{\mu_{BM}}(g,h) \geqslant 1$ since $\delta(g) = \delta(h)$. This shows that $I_{\mu_{BM}}(g,h) = 1$ and again by the Theorem 4, this implies that g and h are isometric.

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