LINEAR SYSTEMS OF NONOSCILLATORY DIFFERENTIAL
EQUATIONS WITH DELAYED ARGUMENTS

K. GOPALSAMY

Sufficient conditions are obtained for a not necessarily
scalar system of the form

$$\frac{d^n x(t)}{dt^n} + (-1)^{n-1} A(t)X(t - \tau(t)) = 0$$

to be nonoscillatory.

1.   Introduction

The purpose of this paper is to obtain a set of sufficient
conditions under which the n-th order nonautonomous differential
difference equations of the form

(1.1) $$\frac{d^n x(t)}{dt^n} + (-1)^{n-1} \sum_{j=1}^{m} a_j(t)x(t - \tau_j(t)) = 0$$

where n and m are positive integers ($n \geq 1$, $m \geq 1$) will have a non-
oscillatory solution and subsequently generalise this result for non-
scalar systems. Although the literature concerned with oscillation of
differential equations with and without deviating arguments is quite
extensive (see Kartsatos [2]) published work dealing with the existence
of nonoscillatory solutions of differential difference equations is rather
scarce. Using the characteristic equation associated with an autonomous
majorant of (1.1) we will obtain first sufficient conditions for (1.1) to
have nonoscillatory solutions.

As it is customary we will say that (1.1) is oscillatory if and

only if all solutions of (1.1) defined on an interval of the form $[\alpha,\infty)$
for some real constant α have zeros on every interval of the form $[\beta,\infty)$
for $\beta \geq \alpha$. (1.1) will be called nonoscillatory if (1.1) has at least one
solution which has no zeros on an interval of the form $[\gamma,\infty)$ for some
$\gamma \geq \alpha$.

2.   Nonoscillatory scalar systems

We will examine (1.1) under the following assumptions:

(H_1) a_j, τ_j $(j = 1,2,\ldots,m)$ are bounded continuous functions

defined on $[0,\infty)$ such that the following hold;

$$(2.1) \qquad \left. \begin{array}{l} 0 \leq a_j(t) \leq a_j^* \\ 0 \leq \tau_j(t) \leq \tau_j^* \\ 0 \leq t - \tau_j(t) \end{array} \right\} j = 1,2,\ldots,n \, ; \quad t \geq 0$$

(2.2) (H_2) $\qquad \left(\sum_{j=1}^{n} a_j^* \right) \tau^n e^n / n^n \leq 1 \, ; \quad \tau = \max(\tau_1^*, \tau_2^*, \ldots, \tau_n^*)$

Let us now consider the autonomous "majorant" of (1.1) in the form

$$(2.3) \qquad \frac{d^n y(t)}{dt^n} + (-1)^{n-1} \sum_{j=1}^{m} a_j^* y\left(t - \tau_j^*\right) = 0 \, .$$

The characteristic equation associated with (2.3) is given by

$$(2.4) \qquad \lambda^n = (-1)^n \sum_{j=1}^{m} a_j^* \exp[-\lambda \tau_j^*]$$

which with $\lambda = -\mu$ reads

$$(2.5) \qquad \mu^n = \sum_{j=1}^{m} a_j^* \exp[\mu \tau_j] \, .$$

We will show that when (2.2) holds, (2.5) has a real positive root.

Suppose first $m = 1$ and consider $g : [0,\infty) \to (-\infty,\infty)$ defined by

$$g(\mu) = \mu^n - a_1^* \exp[\mu \tau_1^*] \, .$$

It will follow from

$$g(0) = -a_1^* < 0$$

and

$$g\left(\frac{n}{\tau_1^*}\right) = (n/\tau_1^*)^n - a_1^* \exp[\tau_1^*(n/\tau_1^*)]$$

$$= (n/\tau_1^*)^n[1 - a_1^* e^n (\tau_1^*)^n/n^n]$$

$$\geq 0$$

that g has a zero on $[0, n/\tau_1^*]$. The general case with $m \geq 2$ is treated as follows. We rewrite (2.5) in the form

(2.6)
$$\sigma = \sum_{j=1}^{m} a_j^* \exp[\tau_j^*(\sigma)^{1/n}]$$

$$= h(\sigma) \quad \text{(say)}.$$

Since

$$\frac{dh(\sigma)}{d\sigma} = \sum_{j=1}^{m} a_j^* \exp[\tau_j^*(\sigma)^{1/n}] (\tau_j^*/n)(\sigma)^{(1-n)/n} ,$$

we have for $\sigma \in [0, \min_{1 \leq j \leq m}(n/\tau_j^*)^n]$,

$$\frac{dh(\sigma)}{d\sigma} < \left[\sum_{j=1}^{m} a_j^*(\tau_j^*)^n\right] e^n/n^n$$

$$\leq \left\{ \left[\sum_{j=1}^{m} a_j^*(\tau_j^*)^n\right] e^n/n^n\right\} - \varepsilon$$

for some ε, $0 < \varepsilon < 1$

$$\leq 1 - \varepsilon \quad \text{for} \quad 0 < \sigma < \sigma^* = \min_{i \leq j \leq m} (n/\tau_j^*)^n .$$

One can now show that a sequence $\{\sigma_k\}$, $k = 0,1,2,\ldots$ defined by

$$\sigma_{k+1} = h(\sigma_k) , \quad k = 0,1,2,\ldots$$

for an arbitrarily chosen $\sigma_0 \in (0,\sigma^*)$ will converge to a limit say $\sigma_0 > 0$ which is a root of (2.6). Corresponding to this σ_0 we have a negative root of the characteristic equation given by $\lambda = -(\sigma_0)^{1/n}$.

We are now ready to formulate the following:

THEOREM 2.1. *Suppose the coefficients* a_j *and the delays* τ_j *(j = 1,2,\ldots,n) satisfy the hypotheses (H_1) and (H_2). Then (1.1) is nonoscillatory.*

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Proof. We consider a sequence $\{y_k(t), \ k = 0,1,2,\ldots\}$ of functions defined as follows:

$$(2.7) \qquad y_0(t) = \exp[-(\sigma_0)^{1/n}t] \ ; \quad t \geqq -\tau \ , \quad \tau = \sup_{1 \leqq j \leqq n} \tau_j(t) \ , \quad t \geqq 0.$$

$$(2.8) \qquad y_{k+1}(t) = \begin{cases} \exp[-(\sigma_0)^{1/n}t] \ ; & t \in [-\tau,0] \\[2mm] \displaystyle\sum_{j=1}^{m} \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \, a_j(s) y_k(s - \tau_j(s)) ds \ , & t > 0 \ . \end{cases}$$

A direct verification leads to

$$y_1(t) \leqq \sum_{j=1}^{m} \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \, a_j^* \, \exp[-(\sigma_0)^{1/n}(s - \tau_j(s))] ds$$

$$t > 0$$

$$\leqq y_0(t) \ ; \quad t > 0$$

from which one can derive that

$$y_2(t) \leqq y_1(t) \ ; \quad t > 0 \ .$$

In a similar way one derives

$$(2.9) \qquad y_0(t) \geqq y_1(t) \geqq \ldots \geqq y_k(t) \geqq y_{k+1}(t) \geqq \ldots \geqq 0$$

$$t > 0 \ .$$

It will follow from (2.9) that the following limit exists in a pointwise sense;

$$(2.10) \qquad \lim_{k \to \infty} y_k(t) = \begin{cases} y^*(t) & \text{(say) for} \quad t > 0 \\ y_0(t) & \text{for} \quad t \in [-\tau,0] \ . \end{cases}$$

By the Lebesgue's convergence theorem it follows from (2.10) that

$$(2.11) \qquad y^*(t) = \sum_{j=1}^{m} \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \, a_j(s) y^*(s - \tau_j(s)) ds \ .$$

The positivity of y^* on $[0,\infty)$ follows from that of (2.11). Since y^* in (2.11) is differentiable n-times for $t > 0$ the result follows and the proof is complete.

3. Nonoscillatory vector systems

Let us now generalise our previous analysis to systems of the form

(3.1) $$\frac{d^n X(t)}{dt^n} + (-1)^{n-1} A(t) X(t - \tau(t)) = 0, \quad t > 0$$

with the following assumptions:

(H_3) $X(t) = \{x_1(t), \ldots, x_m(t)\}$ is an m-vector

(H_4) $A(t) = \{a_{ij}(t)\}$ is an $m \times m$ matrix of bounded continuous

functions defined on $[0, \infty)$ such that

$$0 \leq a_{ij}(t) \leq \ell_{ij} \quad \text{for all} \quad t \geq 0,$$

ℓ_{ij} being positive constants for $i, j = 1, 2, \ldots, n$.

(H_5) $\tau(t)$ is bounded and continuous for $t \geq 0$ such that

$t - \tau(t) \geq 0$ for $t \geq 0$

$0 \leq \tau(t) \leq \sigma$ for $t \geq 0$ and $\sigma > 0$ is a constant.

Definitions of oscillatory and nonoscillatory systems of nonscalar equations have not become standard in the literature. In fact the literature concerned with oscillation and nonoscillation of vector matrix systems is quite scarce. We will adopt the following.

DEFINITION.   The system (3.1) is said to nonoscillatory if at least one component of the vector $X(t) = \{x_1(t), \ldots, x_n(t)\}$ is nonoscillatory on $[0, \infty)$.

We note that the above is not the only way of generalising the notion of nonoscillation of scalar systems. For some work on oscillation of linear systems we refer to [1].

We will first derive the following:

LEMMA 3.1.   *Let* $B = (\ell_{ij})$, $(i, j = 1, 2, \ldots, m)$ *be an $m \times m$ matrix of positive constants and let* σ *be a positive constant. Then the autonomous linear system*

(3.2) $$\frac{d^n Y(t)}{dt^n} + (-1)^{n-1} B Y(t - \sigma) = 0$$

is nonoscillatory if

(3.3) $$(e^n \sigma^n \alpha)/n^n \leq 1$$

where α *is a positive Perron eigenvalue of* B .

Proof.   A famous theorem of Perron says that B has a positive eigenvalue corresponding to which B has a positive (componentwise)

eigenvector. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be the eigenvalues of B of which at least one is positive; let it be denoted by α. Let the corresponding eigenvector be $Z = (z_1, z_2, \ldots, z_m)$. The characteristic equation associated with (3.2) is given by

$$(3.4) \qquad \det.[\lambda^n I + (-1)^{n-1} B e^{-\lambda\sigma}] = 0$$

(I being the $m \times m$ identity matrix). Since we can rewrite (3.4) in the form

$$(3.5) \qquad \det.[(-\lambda)^n I - B e^{-\lambda\sigma}] = 0$$

we have from (3.5) that

$$(3.6) \qquad \det.[\mu^n I - B e^{\mu\sigma}] = 0 \iff \prod_{j=1}^{m} [\mu^n - \alpha_j e^{\mu\sigma}] = 0 .$$

Let α denote a real eigenvalue from $(\alpha_1, \alpha_2, \ldots, \alpha_m)$. Then as in the case of our scalar analysis, it will follow from (3.3) and the equation

$$\mu^n - \alpha e^{\mu\sigma} = 0$$

that (3.4) has a real negative root say $-\lambda^*$. It is now immediate that

$$Y(t) \equiv Z e^{-\lambda^* t}, \quad t \geq -\sigma$$

is a nonoscillatory solution of (3.2) and the proof is complete.

We can now formulate our main result as follows:

THEOREM 3.1. *Suppose the non-negative matrix A in (3.1) is dominated (elementwise) by a positive constant matrix B and the t-dependent delay $\tau(t)$ is dominated by a positive constant σ as in $(H_3) - (H_5)$. Let α be a real eigenvalue of B. If (3.3) holds then (3.1) is nonoscillatory.*

Proof. Proof is surprisingly simple and is quite similar to the scalar case; we define a sequence $\{X_k(t)\}$ of non-negative vectors as follows:

$$(3.7) \qquad X_0(t) \equiv Z e^{-\lambda^* t}, \quad t \geq -\tau$$

$$(3.8) \qquad X_{k+1}(t) = \begin{cases} Z e^{-\lambda^* t}, & t \in [-\tau, 0] \\ \displaystyle\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} A(s) X_k(s - \tau(s))\,ds, & t > 0 \end{cases}$$

where Z is the positive eigenvector associated with the positive

eigenvalue α of B. By componentwise comparison one can now show as in the scalar case that the sequence $\{X_k(t)\}$ has a positive (componentwise) pointwise limit say $X^*(t)$ which is a solution of (3.1); we will omit the details since they are exactly similar to the scalar case.

4. Comments

It is interesting to ask what can be said if (2.2) does not hold. We can say that when (2.2) is violated all bounded solutions of (2.3) will be oscillatory; to verify this assertion let us suppose

$$(4.1) \qquad \left[\sum_{j=1}^{m} a_j^*(\tau_j^*)^n \right] e^n/n^n > 1$$

and let (2.3) have a bounded nonoscillatory solution. Then (2.4) will have a real nonpositive root say $-\gamma$ with $\gamma \geqq 0$ satisfying

$$(4.2) \qquad \gamma^n = \sum_{j=1}^{m} a_j^* \exp[\gamma\tau_j^*] \ .$$

Since γ cannot be zero, we have $\gamma > 0$ and hence (4.2) leads to

$$1 = \sum_{j=1}^{m} a_j^*(\tau_j^*)^n \{\exp[\gamma\tau_j^*]\}/(\tau_j^*\gamma)^n \geq \left\{ \sum_{j=1}^{m} a_j^*(\tau_j^*)^n \right\} e^n/n^n$$

which contradicts (4.1). Thus without (4.1), (2.3) cannot have a bounded nonoscillatory solution. Similar remarks and arguments also apply to our condition (3.3) for the system (3.2) since we can consider a relevant factor of the characteristic equation in (3.6) (see also [1]). We mention some possible generalisations of our results; theorem (2.1) can be generalised to scalar systems of the form

$$\frac{d^n x(t)}{dt^n} + (-1)^{n-1} \sum_{j=0}^{n-1} (-1)^j \ a_j \ \frac{d^j x(t-\tau_j)}{dt^j} = 0$$

with the assumption that a_j, τ_j are positive constants satisfying

$$\left[\sum_{j=0}^{n-1} a_j \ \frac{(n+j)}{n} \ (\tau_j/n)^{n-j} \right] e^n \leq 1 \ ;$$

theorem 3.1 can be generalised for nonlinear systems of the form

$$\frac{d^n x(t)}{dt^n} + (-1)^{n-1} A(t)f(t,X(t-\tau(t))) = 0$$

where f is componentwise non-negative and nondecreasing with respect to

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its vector argument such that

$$0 \leqq A(t)f(t,X(t)) \leqq BX(t) \quad \text{for} \quad X(t) \in \mathbb{R}_m^+ .$$

References

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School of Mathematical Sciences,
Flinders University of South Australia,
Bedford Park,
S.A. 5042,
Australia.