REMETRIZATION IN STRONGLY COUNTABLE-DIMENSIONAL SPACES

B. R. WENNER

Although the Lebesgue dimension function is topologically invariant, the dimension-theoretic properties of a metric space can sometimes be made clearer by the introduction of a new, topologically equivalent metric. A considerable amount of effort has been devoted to the problem of constructing such metrics; one example of the fruits of this research is the following theorem by Nagata (**2**, Theorem 5).

In order that dim $R \leqq n$ *for a metrizable space R it is necessary and sufficient to be able to define a metric* $\rho(x, y)$ *agreeing with the topology of R such that for every* $\epsilon > 0$ *and for every point x of R,*

$$\rho(S_{\epsilon/2}(x), y_i) < \epsilon \qquad (i = 1, \ldots, n+2)$$

imply

$$\rho(y_i, y_j) < \epsilon \quad \textit{for some } i, j \textit{ with } i \neq j.$$

A metric ρ which satisfies the condition of this theorem is called *Nagata's metric* (this term was introduced, to the best of the author's knowledge, by Nagami (**1**, Definition 9.3)). The question arises as to whether or not an equivalent metric can be introduced (on a metric space) which is Nagata's metric on each of countably many finite-dimensional closed subspaces; although the answer is not known, we shall show that it is always possible to introduce an equivalent metric which has a slightly weaker property. The following definition will be needed.

Definition. Let (X, ρ) be a metric space, $Y \subset X$, dim $Y = n$, and ρ_Y the induced metric on Y. Define $S_\alpha(x|Y) = \{y \in Y: \rho_Y(y, x) < \alpha\}$. Then we say that ρ *has Property* A *on* Y if and only if there exists a $\delta > 0$ such that for every positive $\epsilon < \delta$ and every $x \in Y$,

$$\rho_Y(S_{\epsilon/2}(x|Y), y_i) < \epsilon \qquad (i = 1, \ldots, n+2)$$

imply

$$\rho_Y(y_i, y_j) < \epsilon \quad \text{for some } i, j \text{ with } i \neq j.$$

We note that if Y is a subspace for which ρ_Y is Nagata's metric on Y, then ρ has Property A on Y. The converse is not true, as can be seen by a zero-dimensional example: let $X = Y = \{w, x, y, z\}$, and define ρ by the formulae $\rho(w, x) = \rho(w, y) = \rho(x, y) = 1$, $\rho(w, z) = 2$, and $\rho(x, z) = \rho(y, z) = 3$. Then

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for $2 < \epsilon \leqq 3$ we see that $\rho(S_{\epsilon/2}(w), x) = 2 < \epsilon$ and $\rho(S_{\epsilon/2}(w), z) = 2 < \epsilon$ but $\rho(x, z) = 3 \geqq \epsilon$; thus ρ is not Nagata's metric on Y. It is immediate, however, that ρ has Property A on Y (choose $\delta \leqq 2$).

We are now in a position to prove the following theorem.

THEOREM 1. *Let X be a metric space, and for each $k = 1, 2, \ldots$ let X_k be a non-void finite-dimensional closed subset of X. Then there exists an equivalent metric for X which has Property* A *on each X_k.*

Proof. The desired metric is the function ρ which we have constructed in a previous paper (**3**, proof of Theorem 1). We shall use the terminology and conclusions of this proof; of particular importance is the result that for any sequence of integers $1 \leqq m_1 < m_2 < \ldots$ and any open subset U of X,

$$(1) \qquad S_{m_2 m_3 \ldots}(U) \subset S^3(U, \mathfrak{U}_{m_1+1}).$$

For any positive integer k we let n be the dimension of $X_k = Y$ and choose $\epsilon < \delta = 1/2^k$; then ϵ has the non-terminating expansion

$$\epsilon = 1/2^{m_1} + 1/2^{m_2} + \ldots,$$

where $k < m_1 < m_2 < \ldots$. Let $x, y_1, \ldots, y_{n+2}$ be members of Y such that $\rho_Y(S_{\epsilon/2}(x|Y), y_i) < \epsilon$ for each $i = 1, \ldots, n+2$. Then there exist points $x_1, \ldots, x_{n+2}$ of Y such that $\rho(x_i, y_i) < \epsilon$ and $\rho(x, x_i) < \epsilon/2$ for all $i = 1, \ldots, n+2$; these imply, respectively, that $y_i \in S(x_i, \mathfrak{S}_{m_1 m_2 \ldots})$ and that $x_i \in S(x, \mathfrak{S}_{m_1+1, m_2+1, \ldots})$ (**3**, (1.2)). The former implies that for each $i = 1, \ldots, n+2$ there exists $U_i \in \mathfrak{U}_{m_1}$ such that $x_i, y_i \in S_{m_2 m_3 \ldots}(U_i)$; hence, $x_i \in S^3(U_i, \mathfrak{U}_{m_1+1})$ by (1). The latter implies that there exists a $U_i' \in \mathfrak{U}_{m_1+1}$ such that $x, x_i \in S_{m_2+1, m_3+1, \ldots}(U_i') \subset S^3(U_i', \mathfrak{U}_{m_1+2}) = S(U_i', \mathfrak{U}_{m_1+2}^*) \subset S(U_i', \mathfrak{U}_{m_1+1})$; thus, $x \in S^3(x_i, \mathfrak{U}_{m_1+1})$.

Putting these two facts together, we see that $x \in S^6(U_i, \mathfrak{U}_{m_1+1})$; thus, $U_i \cap S^6(x, \mathfrak{U}_{m_1+1}) \neq \emptyset$ for each $i = 1, \ldots, n+2$; this implies that $U_i = U_j$ for some $i \neq j$ (**3**, condition 3 on the uniformity $\{\mathfrak{U}_i\}$). But then $y_i, y_j \in S_{m_2 m_3 \ldots}(U_i)$, therefore $y_i \in S(y_j, \mathfrak{S}_{m_1 m_2 \ldots})$ and $\rho(y_i, y_j) < \epsilon$, which proves the theorem.

Nagami has shown that a completion of an n-dimensional space with respect to Nagata's metric is n-dimensional (**1**, Theorem 9.4). An analogous result for Property A emerges from the following theorem.

THEOREM 2. *Let (X, ρ) be a metric space. For each $k = 1, 2, \ldots$ let X_k be a non-void finite-dimensional closed subset of X, and let ρ have Property* A *on each X_k. If X^* is a completion of X with respect to ρ, then for all $k = 1, 2, \ldots$,* $\dim [\text{cl}_{X^*}(X_k)] = \dim X_k$.

Proof. It suffices to show that for each $k = 1, 2, \ldots, \dim X_k^* = \dim X_k$, where by X_k^* we mean the completion of X_k with respect to the metric induced by ρ on X_k. Let $Y = X_k$ for some k, $n = \dim Y$, δ the number referred to in the definition of Property A (for ρ_Y), and m an integer such that $1/2^m < \delta$. Then for all $i \geqq m$ we define $\mathfrak{U}_i$ as in Nagami's proof (**1**, proof of Theorem 9.4) with respect to ρ_Y; Nagami's proof then shows that a completion Z of Y with respect to $\{\mathfrak{U}_i: i = m, m + 1, \ldots\}$ is essentially the same as Y^* and that $\dim Z \leqq n$; thus, $\dim Y^* \leqq n$ and the theorem is proved.

We might note one aspect of this work in the event that X is a strongly countable-dimensional metric space; i.e., $X = \bigcup_{i=1}^{\infty} X_i$, where each X_i is closed and finite-dimensional. Then an application of Theorems 1 and 2 provides an equivalent metric and a completion (with respect to this metric) in which the closure of each X_i is dimension-preserving.

The following analogue to the previously mentioned theorem of Nagami follows directly from Theorem 2.

COROLLARY 3. *Let* (X, ρ) *be a metric space,* $\dim X \leqq n$, *and* ρ *have Property* A *on* X. *If* X^* *is a completion with respect to* ρ, *then* $\dim X^* \leqq n$.

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University of Missouri,
Kansas City, Missouri