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Several Hardy Type Inequalities with Weights Related to Generalized Greiner Operator

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Abstract. In this paper, we establish several weighted $L^p(1 < p < \infty)$ Hardy type inequalities related to the generalized Greiner operator by improving the method of Kombe. Then the best constants in inequalities are discussed by introducing new polar coordinates.

1 Introduction

The generalized Greiner operator is of the form

$$\Delta_{\mathbb{L}} = \sum_{j=1}^{n}(X_j^2 + Y_j^2), \tag{1.1}$$

where

$$X_j = \frac{\partial}{\partial x_j} + 2ky_j|z|^{2k-2}\frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2kx_j|z|^{2k-2}\frac{\partial}{\partial t},$$

$z = x + iy \in \mathbb{C}^n, t \in \mathbb{R}, j = 1, \dots, n, k \geq 1$. When k is a positive integer, $\Delta_{\mathbb{L}}$ was proposed by Greiner [12] in his study of pseudo-convex domains in the space $\mathbb{C}^n$. In particular, if $k = 1$, then $\Delta_{\mathbb{L}}$ is just the sub-Laplacian on the Heisenberg group. These operators also appear as mathematical models of electromagnetic fields and quantum mechanics.

Let us describe some useful notions and properties about the generalized Greiner operator. The gradient associated with $\Delta_{\mathbb{L}}$ is as follows

$$\nabla_{\mathbb{L}} = (X_1, \dots, X_n, Y_1, \dots, Y_n),$$

and the family of dilations is $\delta_r(z, t) = (rz, r^{2k}t), r > 0$. The relevant homogeneous dimension is $Q = 2n + 2k$.

Let $\xi = (x_1, \dots, x_n, y_1, \dots, y_n, t) = (x, y, t) = (z, t) \in \mathbb{R}^{2n+1}$ with $n \geq 1$. We define the norm $|\xi| = (|z|^{4k} + t^2)^{\frac{1}{4k}}$. Throughout this paper we shall use the notation

$$N = |\xi|.$$

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The open ball of radius R centered at 0 will be denoted by

$$B_R = B_R(0) = \left\{ \xi \in \mathbb{R}^{2n+1} \;\middle|\; |\xi| < R \right\}.$$

The p-degenerate subelliptic operator is $\triangle_{\mathbb{L},p} u = \nabla_{\mathbb{L}}(|\nabla_{\mathbb{L}} u|^{p-2} \nabla_{\mathbb{L}} u)$. If $p = 2$, it coincides with $\triangle_{\mathbb{L}}$ in (1.1). Zhang and Niu [20] inferred that the fundamental solution of the $\triangle_{\mathbb{L},p}$ is given by

$$u_p := \begin{cases} N^{\frac{p-Q}{p-1}} & \text{if } p \neq Q, \\ -\log N & \text{if } p = Q. \end{cases} \tag{1.2}$$

In the same paper, they proved Hardy type inequalities

$$\int_{\mathbb{R}^{2n+1}} |\nabla_{\mathbb{L}} \phi|^p \geq \Big(\frac{Q-p}{p} \Big)^p \int_{\mathbb{R}^{2n+1}} \Big(\frac{|z|}{N} \Big)^{p(2k-1)} \Big(\frac{|\phi|}{N} \Big)^p,$$

where $1 < p < Q$, $\phi \in C_0^\infty(\mathbb{R}^{2n+1} \setminus \{0\})$. Their proof is realized by extending the classical Picone identity to the vector fields. Later, D'Ambrosio [6] considered such class of inequalities in the domain with an interesting extension of the divergence theorem. We notice that when $p = 2$ and k is integer, (1.2) is a particular version of formulas for fundamental solutions by Beals, Gaveau and Greiner in [3] and [4], and the method obtaining (1.2) in [20] is different from that in [3, 4].

We have known that Hardy type inequalities play important roles in partial differential equations with singular potentials (see [1, 8–11, 19]). It impels us continuously to study the inequalities with singular weights.

Kombe [15, 16] used the fundamental solutions of the sub-Laplacian on the Carnot group, and p-sub-Laplacian $\Delta_{\mathbb{G},p}$ on the polarizable Carnot group respectively, and the inequalities given in [2, 17] to establish Hardy inequalities for the case $1 < p < Q$, where Q is the homogeneous dimension related to the associated groups. Kombe's work puts forward a new way to research Hardy type inequalities.

In this paper we try to give some Hardy type inequalities with weights related to the operator, thereby improving Kombe's method. The common Hardy type inequalities are investigated for the case $1 < p < Q$ (see [14–16, 18, 20]). However, our results include the case $p \geq Q$.

The plan of this paper is as follows. In Section 2 we introduce the polar coordinates corresponding to the generalized Greiner operator. The polar coordinates on the Heisenberg group were checked by Greiner [13], also see D'Ambrosio [5], and the ones for the generalized Baouendi–Grushin operator by D'Ambrosio and Lucente [7].

In Section 3 we prove the weighted Hardy type inequalities for $p = 2$. In Section 4 several Hardy type inequalities for $p \neq Q$ are examined. We point out that the constants in inequalities are sharp by using the polar coordinates from Section 2. Finally, we provide a Hardy type inequality for $p = Q$ by choosing appropriate auxiliary functions associated with the fundamental solution in this case.

2 The Polar Coordinates

Let $\Omega \subset \mathbb{R}^{2n+1}$ be an open set and let

$$r = \sqrt{\sum_{j=1}^{n} x_j^2 + \sum_{j=1}^{n} y_j^2}.$$

A function $u\colon \Omega \subset \mathbb{R}^{2n+1} \to \mathbb{R}$ such that $u(\xi) = u(r,t)$ (*i.e.*, u depends only on r and t) is said to be *cylindrical*; and in particular if $u(\xi) = u(|\xi|)$, that is, u depends only on $|\xi|$, then u is said to be *radial*.

Let $\Omega = B_{R_2} \backslash \overline{B_{R_1}}$ with $0 \le R_1 < R_2 \le +\infty$, and let $u \in C(\Omega)$ be cylindrical. For the sake of computing $\int_\Omega u d\xi$, we consider the transformation

$$\xi = (x, y, t) = \phi(N, \theta, \theta_1, \dots, \theta_{2n-1})$$

defined by

$$\begin{aligned}
x_1 &= N(\sin\theta)^{\frac{1}{2k}} \cos\theta_1,\\
y_1 &= N(\sin\theta)^{\frac{1}{2k}} \sin\theta_1 \cos\theta_2,\\
x_2 &= N(\sin\theta)^{\frac{1}{2k}} \sin\theta_1 \sin\theta_2 \cos\theta_3,\\
y_2 &= N(\sin\theta)^{\frac{1}{2k}} \sin\theta_1 \sin\theta_2 \sin\theta_3 \cos\theta_4,\\
&\ \vdots\\
x_n &= N(\sin\theta)^{\frac{1}{2k}} \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{2n-2} \cos\theta_{2n-1},\\
y_n &= N(\sin\theta)^{\frac{1}{2k}} \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{2n-2} \sin\theta_{2n-1},\\
t &= N^{2k} \cos\theta,
\end{aligned}$$

where $R_1 < N < R_2, \theta \in (0,\pi), \theta_i \in (0,\pi)$, $i = 1, 2, \dots, 2n-2$, and $\theta_{2n-1} \in (0, 2\pi)$.

Let $J(\phi)$ be the Jacobian of ϕ. A direct computation shows that

$$|J(\phi)| = N^{Q-1}(\sin\theta)^{\frac{n}{k}-1} \sin^{2n-2}\theta_1 \cdots \sin^2\theta_{2n-3} \sin\theta_{2n-2}.$$

Therefore

$$\int_\Omega u(r,t) d\xi = \omega_n \int_0^\pi d\theta \int_{R_1}^{R_2} N^{Q-1} (\sin\theta)^{\frac{n}{k}-1} u(N(\sin\theta)^{\frac{1}{2k}}, N^{2k} cos\theta))\, dN,$$

where

$$\omega_n = \int_0^\pi d\theta_1 \int_0^\pi d\theta_2 \cdots \int_0^\pi d\theta_{2n-2} \int_0^{2\pi} d\theta_{2n-1} \sin^{2n-2}\theta_1 \cdots \sin^2\theta_{2n-3} \sin\theta_{2n-2}$$

is the $2n$-Lebesgue measure of the unitary Euclidean sphere in $\mathbb{R}^{2n}$. In particular, if u has the form $u(\xi) = \psi_p \nu(|\xi|)$, $\psi_p = |z|^{p(2k-1)}/|\xi|^{p(2k-1)}$, then

$$\int_\Omega \psi_p \nu(|\xi|)\, d\xi = \omega_n \int_0^\pi d\theta \int_{R_1}^{R_2} N^{Q-1} (\sin\theta)^{\frac{n}{k}-1} \frac{[N(\sin\theta)^{\frac{1}{2k}}]^{4k-2}}{N^{4k-2}} \nu(N)\, dN$$
$$= s_n \int_{R_1}^{R_2} N^{Q-1} \nu(N)\, dN,$$

where we have set $s_n = \omega_n \int_0^\pi (\sin\theta)^{\frac{Q-2}{2k}}\, d\theta$.

3 The Weighted Hardy Inequality for $p = 2$

Theorem 3.1 *Let $\alpha \in \mathbb{R}$ and $\varphi \in C_0^\infty(\mathbb{R}^{2n+1} \backslash \{0,0\})$. Then we have*

$$\int_{\mathbb{R}^{2n+1}} N^\alpha |\nabla_{\mathbb{L}} \varphi|^2\, dzdt \geq \Big(\frac{\alpha + 2k + 2n - 2}{2} \Big)^2 \int_{\mathbb{R}^{2n+1}} N^\alpha \frac{|z|^{4k-2}}{N^{4k}} \varphi^2\, dzdt,$$

where $N = (|z|^{4k} + t^2)^{\frac{1}{4k}}$. Moreover, $\big(\frac{\alpha+2k+2n-2}{2}\big)^2$ is the best constant.

Proof Let $\varphi = N^\beta \psi$, where $\psi \in C_0^\infty(\mathbb{R}^{2n+1} \backslash \{0,0\})$ and $\beta \in \mathbb{R} \backslash \{0,0\}$ which will be determined later. So

$$N^\alpha |\nabla_{\mathbb{L}} \varphi|^2 = \beta^2 N^{\alpha+2\beta-2} \psi^2 |\nabla_{\mathbb{L}} N|^2 + 2\beta N^{\alpha+2\beta-1} \psi \nabla_{\mathbb{L}} N \nabla_{\mathbb{L}} \psi + N^{\alpha+2\beta} |\nabla_{\mathbb{L}} \psi|^2,$$

and integrating over $\mathbb{R}^{2n+1}$, we obtain

$$\text{(3.1)} \qquad \int_{\mathbb{R}^{2n+1}} N^\alpha |\nabla_{\mathbb{L}} \varphi|^2\, dzdt = \int_{\mathbb{R}^{2n+1}} \beta^2 N^{\alpha+2\beta-2} \psi^2 |\nabla_{\mathbb{L}} N|^2\, dzdt + \int_{\mathbb{R}^{2n+1}} 2\beta N^{\alpha+2\beta-1} \psi \nabla_{\mathbb{L}} N \nabla_{\mathbb{L}} \psi\, dzdt + \int_{\mathbb{R}^{2n+1}} N^{\alpha+2\beta} |\nabla_{\mathbb{L}} \psi|^2\, dzdt.$$

By computation we have

$$|\nabla_{\mathbb{L}} N|^2 = \sum_{j=1}^n [(X_j N)^2 + (Y_j N)^2] = N^{2-4k} |z|^{4k-4} \sum_{j=1}^n (x_j^2 + y_j^2) = \frac{|z|^{4k-2}}{N^{4k-2}}.$$

Applying integration by parts to the second term on the right-hand side of (3.1) yields

$$\int_{\mathbb{R}^{2n+1}} 2\beta N^{\alpha+2\beta-1} \psi \nabla_{\mathbb{L}} N \nabla_{\mathbb{L}} \psi\, dzdt = -\frac{\beta}{\alpha+2\beta} \int_{\mathbb{R}^{2n+1}} \nabla_{\mathbb{L}} [\nabla_{\mathbb{L}} (N^{\alpha+2\beta})] \psi^2\, dzdt$$
$$= -\frac{\beta}{\alpha+2\beta} \int_{\mathbb{R}^{2n+1}} \triangle_{\mathbb{L}} (N^{\alpha+2\beta}) \psi^2\, dzdt.$$

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Then

$$(3.2)\qquad \int_{\mathbb{R}^{2n+1}} N^{\alpha}|\nabla_{\mathbb{L}}\varphi|^2 dzdt = \int_{\mathbb{R}^{2n+1}} \beta^2 N^{\alpha+2\beta-2}\psi^2|\nabla_{\mathbb{L}}N|^2\,dzdt - \frac{\beta}{\alpha+2\beta}\int_{\mathbb{R}^{2n+1}} \triangle_{\mathbb{L}}(N^{\alpha+2\beta})\psi^2\,dzdt + \int_{\mathbb{R}^{2n+1}} N^{\alpha+2\beta}|\nabla_{\mathbb{L}}\psi|^2\,dzdt.$$

Note that

$$(3.3)\qquad \triangle_{\mathbb{L}}N^{\alpha+2\beta} = (\alpha+2\beta)(\alpha+2\beta+2n+2k-2)N^{\alpha+2\beta-4k}|z|^{4k-2}.$$

Inserting (3.3) into (3.2), we have

$$\begin{aligned}\int_{\mathbb{R}^{2n+1}} N^{\alpha}|\nabla_{\mathbb{L}}\varphi|^2\,dzdt &= [\beta^2-\beta(\alpha+2\beta+2n+2k-2)]\int_{\mathbb{R}^{2n+1}} N^{\alpha+2\beta-4k}|z|^{4k-2}\psi^2\,dzdt \\ &\quad + \int_{\mathbb{R}^{2n+1}} N^{\alpha+2\beta}|\nabla_{\mathbb{L}}\psi|^2\,dzdt \\ &\geq [-\beta^2-\beta(\alpha+2n+2k-2)]\int_{\mathbb{R}^{2n+1}} N^{\alpha+2\beta-4k}|z|^{4k-2}\psi^2\,dzdt.\end{aligned}$$

Since $-\beta^2-\beta(\alpha+2n+2k-2)$ attains the maximum at $\beta=\frac{2-\alpha-2n-2k}{2}$, we have

$$\begin{aligned}\int_{\mathbb{R}^{2n+1}} N^{\alpha}|\nabla_{\mathbb{L}}\varphi|^2\,dzdt &\geq \Big(\frac{2-\alpha-2n-2k}{2}\Big)^2\int_{\mathbb{R}^{2n+1}} N^{\alpha-4k}|z|^{4k-2}\varphi^2\,dzdt \\ &= \Big(\frac{\alpha+2n+2k-2}{2}\Big)^2\int_{\mathbb{R}^{2n+1}} N^{\alpha}\frac{|z|^{4k-2}}{N^{4k}}\varphi^2\,dzdt.\end{aligned}$$

The result is obtained. ∎

4 The Weighted Hardy Inequality for $p \neq Q$

Theorem 4.1 *Let $\varphi \in C_0^{\infty}(\mathbb{R}^{2n+1}\setminus\{0\})$, $Q \geq 3$, $p \neq Q$, and $\alpha > \frac{p-Q}{p}$. Then the following inequality holds:*

$$(4.1)\qquad \int_{\mathbb{R}^{2n+1}} [N^{\alpha}|\nabla_{\mathbb{L}}\varphi|]^p\,dx \geq \left|\frac{Q-p+\alpha p}{p}\right|^p \int_{\mathbb{R}^{2n+1}} N^{(\alpha-1)p}|\nabla_{\mathbb{L}}N|^p|\varphi|^p\,dx.$$

Futhermore, the constant $\left|\frac{Q-p+\alpha p}{p}\right|^p$ is sharp.

Proof Let $\varphi = N^r\psi$, where $\psi \in C_0^\infty(\mathbb{R}^{2n+1} \setminus \{0\})$ and $r \in \mathbb{R} \setminus \{0\}$ which will be chosen later. So

$$N^\alpha|\nabla_{\mathbb{L}}(N^r\psi)| = |rN^{\alpha+r-1}\psi\nabla_{\mathbb{L}}N + N^{\alpha+r}\nabla_{\mathbb{L}}\psi|. \tag{4.2}$$

We now use the following inequality: for any $a, b \in \mathbb{R}^n$ and $1 < p < 2$,

$$|a+b|^p - |a|^p \geq c(p)\frac{|b|^2}{(|a|+|b|)^{2-p}} + p|a|^{p-2}a \cdot b, \tag{4.3}$$

where $c(p) > 0$ (see [2, 17]). From (4.3) we get

$$\begin{aligned}
&\int_{\mathbb{R}^{2n+1}} |rN^{\alpha+r-1}\psi\nabla_{\mathbb{L}}N + N^{\alpha+r}\nabla_{\mathbb{L}}\psi|^p\,dx \\
&\geq \int_{\mathbb{R}^{2n+1}} |rN^{\alpha+r-1}\psi\nabla_{\mathbb{L}}N|^p\,dx \\
&\quad + c(p)\int_{\mathbb{R}^{2n+1}} \frac{|N^{\alpha+r}\nabla_{\mathbb{L}}\psi|^2}{(|rN^{\alpha+r-1}\psi\nabla_{\mathbb{L}}N| + |N^{\alpha+r}\nabla_{\mathbb{L}}\psi|)^{2-p}}\,dx \\
&\quad + p\int_{\mathbb{R}^{2n+1}} |rN^{\alpha+r-1}\psi\nabla_{\mathbb{L}}N|^{p-2}(rN^{\alpha+r-1}\psi\nabla_{\mathbb{L}}N)(N^{\alpha+r}\nabla_{\mathbb{L}}\psi)\,dx \\
&\geq |r|^p\int_{\mathbb{R}^{2n+1}} N^{\alpha p+rp-p}|\psi|^p|\nabla_{\mathbb{L}}N|^p dx \\
&\quad + |r|^{p-2}r\int_{\mathbb{R}^{2n+1}} N^{\alpha p+rp-p+1}|\nabla_{\mathbb{L}}N|^{p-2}\nabla_{\mathbb{L}}N \cdot \nabla_{\mathbb{L}}(|\psi|^p)\,dx.
\end{aligned} \tag{4.4}$$

Applying integration by parts to the second term in the right-hand side of (4.4) leads to

$$\begin{aligned}
&\int_{\mathbb{R}^{2n+1}} N^{\alpha p+rp-p+1}|\nabla_{\mathbb{L}}N|^{p-2}\nabla_{\mathbb{L}}N \cdot \nabla_{\mathbb{L}}(|\psi|^p)\,dx \\
&\qquad = -\int_{\mathbb{R}^{2n+1}} |\psi|^p\nabla_{\mathbb{L}}(N^{\alpha p+rp-p+1}|\nabla_{\mathbb{L}}N|^{p-2}\nabla_{\mathbb{L}}N)\,dx
\end{aligned}$$

and so from (4.2) and (4.4),

$$\begin{aligned}
\int_{\mathbb{R}^{2n+1}} [N^\alpha|\nabla_{\mathbb{L}}\varphi|]^p\,dx &\geq |r|^p\int_{\mathbb{R}^{2n+1}} N^{\alpha p+rp-p}|\psi|^p|\nabla_{\mathbb{L}}N|^p\,dx \\
&\quad - |r|^{p-2}r\int_{\mathbb{R}^{2n+1}} |\psi|^p\nabla_{\mathbb{L}}(N^{\alpha p+rp-p+1}|\nabla_{\mathbb{L}}N|^{p-2}\nabla_{\mathbb{L}}N)\,dx.
\end{aligned} \tag{4.5}$$

For treating the second term in the right-hand side of (4.5), we choose $r = \frac{p-Q-\alpha p}{p}$ and have

$$\begin{aligned}
&\int_{\mathbb{R}^{2n+1}} |\psi|^p\nabla_{\mathbb{L}}(N^{\alpha p+rp-p+1}|\nabla_{\mathbb{L}}N|^{p-2}\nabla_{\mathbb{L}}N)\,dx \\
&\qquad = \int_{\mathbb{R}^{2n+1}} |\psi|^p\nabla_{\mathbb{L}}(N^{1-Q}|\nabla_{\mathbb{L}}N|^{p-2}\nabla_{\mathbb{L}}N)\,dx.
\end{aligned} \tag{4.6}$$

Since $u_p = N^{\frac{p-Q}{p-1}}$ is the fundamental solution of the operator $-\triangle_{\mathbb{L},p}$ for $p \neq Q$,

$$\begin{aligned}\triangle_{\mathbb{L},p} u_p &= \nabla_{\mathbb{L}}\Big[\Big|\nabla_{\mathbb{L}} N^{\frac{p-Q}{p-1}}\Big|^{p-2}\nabla_{\mathbb{L}}(N^{\frac{p-Q}{p-1}})\Big]\\ &= \nabla_{\mathbb{L}}\Big[\Big|\frac{p-Q}{p-1}N^{\frac{p-Q}{p-1}-1}\nabla_{\mathbb{L}}N\Big|^{p-2}(\frac{p-Q}{p-1}N^{\frac{p-Q}{p-1}-1}\nabla_{\mathbb{L}}N)\Big]\\ &= \Big|\frac{p-Q}{p-1}\Big|^{p-2}\Big(\frac{p-Q}{p-1}\Big)\nabla_{\mathbb{L}}((N)^{\frac{1-Q}{p-1}})^{p-2}|\nabla_{\mathbb{L}}N|^{p-2}N^{\frac{1-Q}{p-1}}\nabla_{\mathbb{L}}N)\\ &= \Big|\frac{p-Q}{p-1}\Big|^{p-2}\Big(\frac{p-Q}{p-1}\Big)\nabla_{\mathbb{L}}(N^{1-Q}|\nabla_{\mathbb{L}}N|^{p-2}\nabla_{\mathbb{L}}N),\end{aligned}$$

and using (4.6) yields

$$\begin{aligned}\int_{\mathbb{R}^{2n+1}} &|\psi|^p \nabla_{\mathbb{L}}(N^{\alpha p+rp-p+1}|\nabla_{\mathbb{L}}N|^{p-2}\nabla_{\mathbb{L}}N)\,dx\\ &= \Big(\frac{p-1}{p-Q}\Big)\Big|\frac{p-1}{p-Q}\Big|^{p-2}\int_{\mathbb{R}^{2n+1}}|\psi|^p(\triangle_{\mathbb{L},p}u_p)dx\\ &= \Big(\frac{p-1}{p-Q}\Big)\Big|\frac{p-1}{p-Q}\Big|^{p-2}(-|\varphi(0)|^p N^{Q-p+\alpha p}(0))\\ &= 0.\end{aligned}$$

Therefore

$$\begin{aligned}\int_{\mathbb{R}^{2n+1}}[N^\alpha|\nabla_{\mathbb{L}}\varphi|]^p\,dx &\geq |r|^p\int_{\mathbb{R}^{2n+1}}N^{\alpha p+rp-p}|\psi|^p|\nabla_{\mathbb{L}}N|^p\,dx\\ &= \Big|\frac{Q-p+\alpha p}{p}\Big|^p\int_{\mathbb{R}^{2n+1}}N^{\alpha p+rp-p}\frac{|\varphi|^p}{N^{rp}}|\nabla_{\mathbb{L}}N|^p\,dx\\ &= \Big|\frac{Q-p+\alpha p}{p}\Big|^p\int_{\mathbb{R}^{2n+1}}N^{(\alpha-1)p}|\varphi|^p|\nabla_{\mathbb{L}}N|^p\,dx.\end{aligned}$$

The inequality (4.1) for $1 < p < 2$ is obtained.

In the case $p > 2$, we use the inequality

$$|a+b|^p - |a|^p \geq c(p)|b|^p + p|a|^{p-2}a\cdot b, \tag{4.7}$$

where $c(p) > 0$ and $a, b \in \mathbb{R}^n$ (see [2,17]), and prove (4.1) with a similar discussion.

If $p = 2$, then it is just the result in Theorem 3.1.

To illustrate that the constant $\big|\frac{Q-p+\alpha p}{p}\big|^p$ is sharp, we consider the following family of functions

$$\varphi_\varepsilon(x) = \begin{cases} 1 & \text{if } N(x) \in [0,1],\\ N^{-(\frac{Q-p+\alpha p}{p}+\varepsilon)} & \text{if } N(x) > 1.\end{cases}$$

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Note that evidently

$$\varphi_\varepsilon'(x) = \begin{cases} 0 & \text{if } N(x) \in [0,1], \\ -(\frac{Q-p+\alpha p}{p} + \varepsilon) N^{-\frac{Q+\alpha p}{p} - \varepsilon} \nabla_{\mathbb{L}} N & \text{if } N(x) > 1. \end{cases}$$

A direct calculation shows that

$$\int_{\mathbb{R}^{2n+1}} N^{(\alpha-1)p} |\nabla_{\mathbb{L}} N|^p |\varphi_\varepsilon|^p \, dx = \int_{N(x)\le 1} N^{(\alpha-1)p} |\nabla_{\mathbb{L}} N|^p \, dx + \int_{N(x)>1} N^{-Q-\varepsilon p} |\nabla_{\mathbb{L}} N|^p \, dx,$$

$$\int_{\mathbb{R}^{2n+1}} [N^\alpha |\nabla_{\mathbb{L}} (\varphi_\varepsilon)|]^p \, dx = \int_{N(x)>1} \Big| \frac{Q-p+\alpha p}{p} + \varepsilon \Big|^p N^{-Q-\varepsilon p} |\nabla_{\mathbb{L}} N|^p \, dx.$$

By the polar coordinates in Section 2, we have

$$\int_{N(x)\le 1} N^{(\alpha-1)p} |\nabla_{\mathbb{L}} N|^p \, dx = s_n \int_0^1 \rho^{(\alpha-1)p} \cdot \rho^{Q-1} d\rho = \frac{1}{(\alpha-1)p+Q} s_n,$$

$$\int_{N(x)>1} N^{-Q-\varepsilon p} |\nabla_{\mathbb{L}} N|^p \, dx = s_n \int_1^\infty \rho^{-Q-\varepsilon p} \cdot \rho^{Q-1} d\rho = \frac{1}{\varepsilon p} s_n.$$

Letting $\varepsilon \longrightarrow 0$, it follows that

$$\begin{aligned} &\frac{\int_{\mathbb{R}^{2n+1}} N^{(\alpha-1)p} |\nabla_{\mathbb{L}} N|^p |\varphi|^p dx}{\int_{\mathbb{R}^{2n+1}} [N^\alpha |\nabla_{\mathbb{L}} \varphi|]^p dx} \\ &\quad = \Big| \frac{p}{Q-p+\alpha p + \varepsilon p} \Big|^p \Big(\frac{\int_{N(x)\le 1} N^{(\alpha-1)p} |\nabla_{\mathbb{L}} N|^p dx}{\int_{N(x)>1} N^{-Q-\varepsilon p} |\nabla_{\mathbb{L}} N|^p dx} + 1 \Big) \\ &\quad = \Big| \frac{p}{Q-p+\alpha p + \varepsilon p} \Big|^p \Big(\frac{\varepsilon p}{(\alpha-1)p} + 1 \Big) \\ &\quad \longrightarrow \Big| \frac{p}{Q-p+\alpha p} \Big|^p, \end{aligned}$$

and then the constant $\left| \frac{Q-p+\alpha p}{p} \right|^p$ is sharp. ∎

5 The Hardy Inequality for $p = Q$

In this section, we investigate the case $p = Q$ for the generalized Greiner operator.

Theorem 5.1 *Let $\varphi \in C_0^\infty(\mathbb{R}^{2n+1} \setminus \{0\})$, $p = Q$, $Q \ge 3$. Then the following inequality is valid.*

$$\int_{\mathbb{R}^{2n+1}} |\nabla_{\mathbb{L}} \varphi|^p \, dx \ge \Big(\frac{p-1}{p} \Big)^p \int_{\mathbb{R}^{2n+1}} |\log N|^{-p} \frac{|\nabla_{\mathbb{L}} N|^p}{N^p} |\varphi|^p \, dx. \tag{5.1}$$

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Proof Let $\varphi = |\log N|^r \psi$ where $\psi \in C_0^\infty(\mathbb{R}^{2n+1} \setminus \{0\})$ and $r \in \mathbb{R} \setminus \{0\}$, which will be further specified in the following. Noting

$$|\nabla_{\mathbb{L}}\varphi| = |\nabla_{\mathbb{L}}[|logN|^r\psi]| = |r|\log N|^{r-1}N^{-1}\psi\nabla_{\mathbb{L}}N + |\log N|^r\nabla_{\mathbb{L}}\psi|,$$

and using the inequality (4.7), we get

$$\begin{aligned}(5.2)\quad \int_{\mathbb{R}^{2n+1}} &|r|\log N|^{r-1}N^{-1}\psi\nabla_{\mathbb{L}}N + |\log N|^r\nabla_{\mathbb{L}}\psi|^p\,dx \\ &\geq |r|^p \int_{\mathbb{R}^{2n+1}} N^{-p}|\log N|^{(r-1)p}|\psi|^p|\nabla_{\mathbb{L}}N|^p\,dx \\ &\quad + |r|^{p-2}r\int_{\mathbb{R}^{2n+1}} N^{1-p}|\log N|^{(r-1)(p-1)+r}|\nabla_{\mathbb{L}}N|^{p-2}\nabla_{\mathbb{L}}N\nabla_{\mathbb{L}}(|\psi|^p)\,dx.\end{aligned}$$

Now the integration by parts gives

$$\begin{aligned}\int_{\mathbb{R}^{2n+1}} &N^{1-p}|\log N|^{(r-1)(p-1)+r}|\nabla_{\mathbb{L}}N|^{p-2}\nabla_{\mathbb{L}}N\nabla_{\mathbb{L}}(|\psi|^p)\,dx \\ &= -\int_{\mathbb{R}^{2n+1}} |\psi|^p\nabla_{\mathbb{L}}(N^{1-p}|\log N|^{(r-1)(p-1)+r}|\nabla_{\mathbb{L}}N|^{p-2}\nabla_{\mathbb{L}}N)\,dx.\end{aligned}$$

Recalling that $u_p = -\log N$ is the fundamental solution of $-\triangle_{\mathbb{L},p}$ for $p = Q$ such that

$$\begin{aligned}\Delta_{\mathbb{L},p}u_p &= \nabla_{\mathbb{L}}[|\nabla_{\mathbb{L}}(-\log N)|^{p-2}\nabla_{\mathbb{L}}(-\log N)] \\ &= \nabla_{\mathbb{L}}[|N^{-1}\nabla_{\mathbb{L}}N|^{p-2}(-N^{-1}\nabla_{\mathbb{L}}N)] \\ &= -\nabla_{\mathbb{L}}(N^{1-p}|\nabla_{\mathbb{L}}N|^{p-2}\nabla_{\mathbb{L}}N),\end{aligned}$$

and choosing $r = \frac{p-1}{p}$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^{2n+1}} &|\psi|^p\nabla_{\mathbb{L}}(N^{1-p}|\log N|^{(r-1)(p-1)+r}|\nabla_{\mathbb{L}}N|^{p-2}\nabla_{\mathbb{L}}N)\,dx \\ &= \int_{\mathbb{R}^{2n+1}} |\psi|^p\nabla_{\mathbb{L}}(N^{1-p}|\nabla_{\mathbb{L}}N|^{p-2}\nabla_{\mathbb{L}}N)\,dx \\ &= \int_{\mathbb{R}^{2n+1}} (\Delta_{\mathbb{L},p}u_p)|\psi|^p\,dx \\ &= 0.\end{aligned}$$

Now the inequality (5.1) is proved from (5.2). ■

References

[1] P. Baras, and J. A. Goldstein, *The heat equation with a singular potential*. Trans. Amer. Math. Soc. **284**(1984), no. 1, 121–139. doi:10.2307/1999277

[2] G. Barbatis, S. Filippas, and A. Tertikas, *A unified approach to improved L^p Hardy inequalities with best constants*. Trans. Amer. Math. Soc. **356**(2004), no. 6, 2169–2196. doi:10.1090/S0002-9947-03-03389-0

[3] R. Beals, B. Gaveau, and P. Greiner, *On a geometric formula for the fundamental solution of subelliptic Laplacians*. Math. Nachr. **181**(1996), 81–163.

[4] ———, *Uniform hypoelliptic Green's functions.* J. Math. Pures Appl. **77**(1998), no. 3, 209–248.

[5] L. D'Ambrosio, *Critical degenerate inequalities on the Heisenberg group*. Manuscripta Math. **106**(2001), no. 4, 519–536. doi:10.1007/s229-001-8031-2

[6] ———, *Hardy type inequalities related to degenerate elliptic differential operators*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **4**(2005), no. 3, 451–486.

[7] L. D'Ambrosio and S. Lucente, *Nonlinear Liouville theorems for Grushin and Tricomi operators*. J. Differential Equations **193**(2003), no. 2, 511–541. doi:10.1016/S0022-0396(03)00138-4

[8] J. Garcia Azorero, and I. Peral, *Hardy inequalities and some critical elliptic and parabolic problems*. J. Differential Equations **144**(1998), no. 2, 441–476. doi:10.1006/jdeq.1997.3375

[9] N. Garofalo and E. Lanconelli, *Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation*. Ann. Inst. Fourier (Grenoble) **40**(1990), no. 2, 313–356.

[10] J. A. Goldstein and I. Kombe, *Nonlinear degenerate parabolic equations on the Heisenberg group*. Inter. J. Evol. Equ. **1**(2005), no. 1, 1–22.

[11] ———, *Nonlinear degenerate parabolic differential equations with singular lower order term*. Adv. Differential Equations **8**(2003), no. 10, 1153–1192.

[12] P. C. Greiner, *A fundamental solution for a nonelliptic partial differential operator*. Canad. J. Math. **31**(1979), no. 5, 1107–1120.

[13] ———, *Spherical harmonics on the Heisenberg group*. Canad. Math. Bull. **23**(1980), no. 4, 383–396.

[14] G. Hardy, *Note on a theorem of Hilbert*. Math. Z. **6**(1920), no. 3-4, 314–317. doi:10.1007/BF01199965

[15] I. Kombe, *The Hardy inequality and nonlinear parabolic equations on Carnot groups*. arXiv:math/0504057.

[16] ———, *Sharp Hardy type inequalities on the Carnot groups*. arXiv:math/0501522.

[17] P. Lindqvist, *On the equation* $\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$. Proc. Amer. Math. Soc. **109**(1990), no. 1, 157–164. doi:10.2307/2048375

[18] P. Niu, H. Zhang, and Y. Wang, *Hardy type and Rellich type inequalities on the Heisenberg group*. Proc. Amer. Math. Soc. **129**(2001), no. 12, 3623–3630 (electronic). doi:10.1090/S0002-9939-01-06011-7

[19] B. Opic, and A. Kufner, *Hardy-type inequalities.* Pitman Research Notes in Mathematics Series 219, Longman Scientific & Technical, Harlow, 1990.

[20] H. Zhang and P. Niu, *Hardy-type inequalities and Pohozaev-type identities for a class of p-degenerate subelliptic operators and applications*. Nonlinear Anal. **54**(2003), no. 1, 165–186. doi:10.1016/S0362-546X(03)00062-2

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