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ON THE CONSECUTIVE EIGENVALUES OF THE LAPLACIAN OF A COMPACT MINIMAL SUBMANIFOLD IN A SPHERE

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Abstract

Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$ denote the sequence of eigenvalues of the Laplacian of a compact minimal submanifold in a unit sphere. Yang and Yau obtained an upper bound on λ_{n+1} in terms of λ_n and the sum $\lambda_1 + \cdots + \lambda_n$. In this note we shall prove an improved version of this upper bound by using the method of Hile and Protter.

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1. Introduction

Let M^m be a m-dimensional compact (without boundary) minimal submanifold in an N-dimensional unit sphere S^N which lies in the $(N+1)$-dimensional Euclidean space $\mathbb{R}^{N+1}$. Let Δ denote the Laplacian acting on smooth functions defined on M^m. Then Δ has a discrete set of eigenvalues and we list them counting multiplicity as $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$. In [4], Yang and Yau, using the Payne-Polya-Weinberger method [3], proved that this sequence of eigenvalues satisfies a universal inequality, that is, an inequality which depends only on the dimension m and the fact that M^m is minimal in S^N and otherwise does not depend on the geometry of M^m. Before we can state their result we need to point out that there is a mistake in their calculation. The last term in [4, (3.10)] should be $4\Lambda_k/mA$ and not $2\Lambda_k/mA$ as stated there. Using this corrected term in the rest of their calculation we found that the corrected statement of their result should be the following

THEOREM (Yang and Yau [4]). *Let M^m be a compact minimal submanifold in S^N and let $S = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. Then*

$$\lambda_{n+1} \leq \lambda_n + m + \frac{2\sqrt{S^2 + m^2(n+1)S} + 2S}{m(n+1)} \tag{1.1}$$

for $n = 1, 2, \ldots$.

In [2], Hile and Protter introduced a technical improvement into the Payne-Polya-Weinberger method. In this note we shall combine the method of Hile and Protter together with the method of Yang and Yau to obtain improved bounds of (1.1).

For each $t > 0$, let σ_t denote the unique solution on (λ_n, ∞) of the equation

$$\sum_{i=1}^{n} \frac{\lambda_i}{x - \lambda_i} = \frac{m(n+1)t}{(1+t)^2}. \tag{1.2}$$

The fact that this equation has a unique solution on (λ_n, ∞) is clear because the left hand side is a decreasing function in x, approaches to ∞ as $x \to \lambda_n^+$ and approaches to 0 as $x \to \infty$. Note that σ_t is a decreasing function of t in $(0, 1)$, reaching a minimum when $t = 1$ and is an increasing function of t in $(1, \infty)$. Therefore the expression $\sigma_t + (1+t)m$ will attain a minimum at some $t \in (0, 1)$. Let σ denote this minimum value, that is,

$$\sigma = \min_{t>0}[\sigma_t + (1+t)m].$$

MAIN THEOREM. *Let M^m be a compact minimal submanifold in S^N. Then*

$$\lambda_{n+1} \leq \sigma \tag{1.3}$$

for $n = 1, 2, \ldots$.

In order to get explicit bounds we shall approximate (1.2) by a quadratic equation and then estimate the corresponding minimum value σ. We shall show

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THEOREM 1. *Let M^m be a compact minimal submanifold in S^N and let $S = \lambda_1 + \cdots + \lambda_n$. Then for $l = 1, 2, \ldots, n$, we have*

$$\begin{aligned}\lambda_{n+1} \le \lambda_n + m + \frac{2\sqrt{S^2 + m^2(n+1)S} + 2S}{m(n+1)} \\ - \left[\frac{S(\sqrt{S + m^2(n+1)} + \sqrt{S})^2}{2m(n+1)\sqrt{S^2 + m^2(n+1)S}} + \frac{\lambda_n - \lambda_l}{2}\right] \\ + \left\{\left[\frac{S(\sqrt{S + m^2(n+1)} + \sqrt{S})^2}{2m(n+1)\sqrt{S^2 + m^2(n+1)S}} + \frac{\lambda_n - \lambda_l}{2}\right]^2 \right. \\ \left. - \sum_{i=1}^{l} \frac{\lambda_i(\lambda_n - \lambda_l)(\sqrt{S + m^2(n+1)} + \sqrt{S})^2}{m(n+1)\sqrt{S^2 + m^2(n+1)S}}\right\}^{1/2}.\end{aligned} \tag{1.4}$$

Using $\sqrt{a^2 - b} \le a - b/2a$ to estimate the last term in (1.4) we get the following result, which is less complicated looking but weaker than (1.4).

COROLLARY 1. *With the same assumption as in Theorem 1, we have*
(1.5)

$$\begin{aligned}\lambda_{n+1} \le \lambda_n + m + \frac{2\sqrt{S^2 + m^2(n+1)S} + 2S}{m(n+1)} \\ - \sum_{i=1}^{l} \frac{\lambda_i(\lambda_n - \lambda_l)(\sqrt{S + m^2(n+1)} + \sqrt{S})^2}{S(\sqrt{S + m^2(n+1)} + \sqrt{S})^2 + (\lambda_n - \lambda_l)m(n+1)\sqrt{S^2 + m^2(n+1)S}}.\end{aligned}$$

Clearly (1.5) is stronger than (1.1). Both (1.4) and (1.5) reduce to (1.1) when $l = n$ and so can be considered as generalizations of (1.1). Of course the implicit bound (1.3) is the best among the four bounds we have discussed so far.

2. Proof of the Main Theorem

In this section we shall use the following ranges for indices: $0 \le i, j, k \le n$; $1 \le \alpha \le N + 1$. Let x_α be the coordinate functions of the minimal immersion. Therefore $\sum_\alpha x_\alpha^2 = 1$ and it is a standard fact that $\Delta x_\alpha + m x_\alpha = 0$ [1, page 312]. We shall now assume that the first $n+1$ eigenvalues $0 = \lambda_0 < \lambda_1 \le \cdots \le \lambda_n$ together with the corresponding normalized eigenfunctions

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$u_0, \dots, u_n$ are given. So we have $\Delta u_i + \lambda_i u_i = 0$ and $\int u_i u_j = \delta_{ij}$ where as in the rest of this section the integral is taken over M^m and δ_{ij} is the Kronecker delta. Let

$$a_{ij}^{\alpha} = \int x_\alpha u_i u_j \tag{2.1}$$

and so we have

$$a_{ij}^{\alpha} = a_{ji}^{\alpha}. \tag{2.2}$$

Define the functions

$$\phi_i^\alpha = x_\alpha u_i - \sum_j a_{ij}^\alpha u_j \tag{2.3}$$

and we have by (2.1) that $\int \phi_i^\alpha u_j = 0$ for all α, i, j. Therefore by Rayleigh's Theorem [1, page 16] and Stokes' Theorem we have

$$\lambda_{n+1} \int (\phi_i^\alpha)^2 \le -\int \phi_i^\alpha \Delta \phi_i^\alpha. \tag{2.4}$$

From (2.3) we have

$$\Delta \phi_i^\alpha = -(m + \lambda_i) x_\alpha u_i + 2\langle \nabla x_\alpha, \nabla u_i \rangle + \sum_j a_{ij}^\alpha \lambda_j u_j$$

and hence

$$-\int \phi_i^\alpha \Delta \phi_i^\alpha = (m + \lambda_i) \int x_\alpha u_i \phi_i^\alpha - 2 \int \langle \nabla x_\alpha, \nabla u_i \rangle \phi_i^\alpha. \tag{2.5}$$

Again from (2.3) we have

$$(\phi_i^\alpha)^2 = x_\alpha u_i \phi_i^\alpha - \sum_j a_{ij}^\alpha u_j \phi_i^\alpha$$

and this implies

$$\int (\phi_i^\alpha)^2 = \int x_\alpha u_i \phi_i^\alpha. \tag{2.6}$$

From (2.4), (2.5) and (2.6) we obtain, after summing over α and i, that

$$(\lambda_{n+1} - m) \sum_{\alpha, i} \int (\phi_i^\alpha)^2 \le \sum_{\alpha, i} \lambda_i \int (\phi_i^\alpha)^2 - \sum_{\alpha, i} 2 \int \langle \nabla x_\alpha, \nabla u_i \rangle \phi_i^\alpha. \tag{2.7}$$

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Using (2.2) and the fact that $\sum_\alpha x_\alpha^2 = 1$, we find that

$$\begin{aligned}
&-\sum_{\alpha,i} 2\int \langle \nabla x_\alpha, \nabla u_i\rangle \phi_i^\alpha \\
&\quad = -\sum_{\alpha,i} 2\int \langle \nabla x_\alpha, \nabla u_i\rangle x_\alpha u_i + \sum_{\alpha,i,j} 2a_{ij}^\alpha \int \langle \nabla x_\alpha, \nabla u_i\rangle u_j \\
&\quad = -\sum_{\alpha,i} \frac{1}{2}\int \langle \nabla x_\alpha^2, \nabla u_i^2\rangle + \sum_{\alpha,i,j} a_{ij}^\alpha \int \langle \nabla x_\alpha, \nabla(u_iu_j)\rangle \\
&\quad = -\sum_{\alpha,i,j} a_{ij}^\alpha \int (\Delta x_\alpha) u_i u_j \\
&\quad = \sum_{\alpha,i,j} a_{ij}^\alpha \int m x_\alpha u_i u_j \\
&\quad = mA \quad \text{where } A = \sum_{\alpha,i,j} (a_{ij}^\alpha)^2.
\end{aligned} \tag{2.8}$$

From (2.7) and (2.8), we have, for any real number $t > 0$, that

$$\begin{aligned}
&(\lambda_{n+1} - m)\sum_{\alpha,i}\int (\phi_i^\alpha)^2 \\
&\quad \le \sum_{\alpha,i}\lambda_i \int (\phi_i^\alpha)^2 - \sum_{\alpha,i} 2(1+t)\int \langle \nabla x_\alpha, \nabla u_i\rangle \phi_i^\alpha - tmA.
\end{aligned} \tag{2.9}$$

On the other hand, using the Cauchy-Schwarz inequality we find, for any $C_i > 0$, that we have

$$\begin{aligned}
&\sum_{\alpha,i} -2(1+t)\int \langle \nabla x_\alpha, \nabla u_i\rangle \phi_i^\alpha \\
&\quad \le \sum_{\alpha,i} 2\left\{\left[\int (1+t)^2 \langle \nabla x_\alpha, \nabla u_i\rangle^2\right]\left[\int (\phi_i^\alpha)^2\right]\right\}^{1/2} \\
&\quad = \sum_{\alpha,i} 2\left\{C_i \int (\phi_i^\alpha)^2\right\}^{1/2}\left\{\frac{1}{C_i}\int (1+t)^2\langle \nabla x_\alpha, \nabla u_i\rangle^2\right\}^{1/2} \\
&\quad \le \sum_{\alpha,i}\left\{C_i\int (\phi_i^\alpha)^2 + \frac{1}{C_i}\int (1+t)^2\langle \nabla x_\alpha, \nabla u_i\rangle^2\right\} \\
&\quad = \sum_{\alpha,i} C_i \int (\phi_i^\alpha)^2 + \sum_i \frac{1}{C_i}\int (1+t)^2 \|\nabla u_i\|^2 \\
&\quad = \sum_{\alpha,i} C_i \int (\phi_i^\alpha)^2 + (1+t)^2 \sum_i \frac{\lambda_i}{C_i}.
\end{aligned} \tag{2.10}$$

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From (2.9) and (2.10) we obtain

$$(2.11)\qquad \begin{aligned}&(\lambda_{n+1} - m)\sum_{\alpha,i}\int(\phi_i^\alpha)^2\\ &\quad\le \sum_{\alpha,i}(\lambda_i + C_i)\int(\phi_i^\alpha)^2 + (1+t)^2\sum_i \frac{\lambda_i}{C_i} - tmA.\end{aligned}$$

Now we set $C_n = C > 0$ and choose $C_i = C + \lambda_n - \lambda_i$. Then, substituting this into (2.11), we obtain

$$(2.12)\qquad (\lambda_{n+1} - m - C - \lambda_n)\sum_{\alpha,i}\int(\phi_i^\alpha)^2 \le (1+t)^2\sum_i\frac{\lambda_i}{C_i} - tmA.$$

From (2.3) and $\sum_\alpha x_\alpha^2 = 1$, we find that

$$\begin{aligned}\sum_{\alpha,i}\int(\phi_i^\alpha)^2 &= \sum_{\alpha,i}\int\left\{x_\alpha^2u_i^2 - 2x_\alpha u_i\sum_j a_{ij}^\alpha u_j + \sum_{j,k}a_{ij}^\alpha u_j u_k\right\}\\ &= \sum_i\int u_i^2 - 2\sum_{\alpha,i,j}a_{ij}^\alpha\int x_\alpha u_i u_j + \sum_{\alpha,i,j}(a_{ij}^\alpha)^2\\ &= n+1-A\end{aligned}$$

and so $-tm\sum_{\alpha,i}\int(\phi_i^\alpha)^2 = -tm(n+1) + tmA$ and, adding this to (2.12), we obtain, after putting $x = C + \lambda_n$, that

$$(2.13)\qquad \begin{aligned}&(\lambda_{n+1} - x - (1+t)m)\sum_{\alpha,i}\int(\phi_i^\alpha)^2\\ &\quad\le (1+t)^2\sum_i\frac{\lambda_i}{x-\lambda_i} - tm(n+1).\end{aligned}$$

Therefore, if σ_t is a solution of (1.2) on (λ_n, ∞) then from (2.13) we have

$$(2.14)\qquad \lambda_{n+1} \le \sigma_t + (1+t)m.$$

Since (2.14) holds for all $t > 0$, we have

$$\lambda_{n+1} \le \sigma = \min_{t>0}\{\sigma_t + (1+t)m\}.$$

3. Some explicit bounds

In order to obtain explicit bounds on λ_{n+1}, we shall approximate the equation (1.2). Since $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$, we have

$$\sum_{i=1}^n\frac{\lambda_i}{x-\lambda_i} \le \frac{1}{x-\lambda_n}\sum_{i=1}^n\lambda_i$$

and so the solution on (λ_n, ∞) of (1.2) is bounded above by the solution on (λ_n, ∞) of

$$\frac{1}{x-\lambda_n}\sum_{i=1}^{n}\lambda_i = \frac{m(n+1)t}{(1+t)^2}. \tag{3.1}$$

Solving (3.1) we have $x = \lambda_n + (1+t^2)S/m(n+1)t$ where $S = \sum_{i=1}^{n}\lambda_i$ and from this we obtain

$$\lambda_{n+1} \le \lambda_n + \frac{(1+t^2)}{m(n+1)t}S + (1+t)m. \tag{3.2}$$

From direct calculation, the right hand side of (3.2) is minimum when

$$t = \sqrt{\frac{S}{S+m^2(n+1)}}. \tag{3.3}$$

Substituting (3.3) into (3.2), we get back (1.1).

In order to get better results than (1.1) we shall now approximate (1.2) by a quadratic equation. For each $1 \le l \le n$, we have

$$\sum_{i=1}^{n}\frac{\lambda_i}{x-\lambda_i} \le \frac{1}{x-\lambda_l}\sum_{i=1}^{l}\lambda_i + \frac{1}{x-\lambda_n}\sum_{i=l+1}^{n}\lambda_i$$

and so the solution on (λ_n, ∞) of (1.2) is bounded above by the solution on (λ_n, ∞) of

$$\frac{1}{x-\lambda_l}\sum_{i=1}^{l}\lambda_i + \frac{1}{x-\lambda_n}\sum_{i=l+1}^{n}\lambda_i = \frac{m(n+1)t}{(1+t)^2}. \tag{3.4}$$

If x_t denotes the solution on (λ_n, ∞) of (3.4), then once again we have

$$\lambda_{n+1} \le x_t + (1+t)m \tag{3.5}$$

and the minimum for $t \in (0, \infty)$ of the right hand side of (3.5) will give us a better bound than (1.1). In practice it is quite complicated to locate exactly this minimum and the idea is to approximate this minimum by (3.3). Substituting (3.3) into (3.4) and (3.5) and then by a direct calculation we obtain (1.4) and this proves Theorem 1.

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